Homework #1

Complete the following problems and be sure to show all of your work. Please type-up your solutions and submit as a PDF file. This homework assignment is worth 100 points in total.

Problem #1 (20 points)

The technical staff of a hospital wishes to develop a computerized menu-planning system. To start with, a lunch menu is sought. The menu is divided into three major categories: vegetables, meat, and dessert. At least one equivalent serving of each category is desired. The cost per serving of some suggested items as well as their content of carbohydrates, vitamins, protein, and fats are summarized below.

Categories	Items	Carbohydrates	Vitamins	Protein	Fats	Cost in $/serving
Vegetables	Peas	1	3	1	0	0.10
	Green beans	1	5	2	0	0.12
	Okra	1	5	1	0	0.13
	Corn	2	6	1	2	0.09
	Macaroni	4	2	1	1	0.10
	Rice	5	1	1	1	0.07
Meat	Chicken	2	1	3	1	0.70
	Beef	3	8	5	2	1.20
	Fish	3	6	6	1	0.63
Dessert	Orange	1	3	1	0	0.28
	Apple	1	2	0	0	0.42
	Pudding	1	0	0	0	0.15
	Jello	1	0	0	0	0.12

Suppose that the minimal requirement of carbohydrates, vitamins, protein, and fats per meal are respectively 5, 10, 10, and 2. Formulate the menu-planning problem (minimizing cost) as a linear program: clearly define the variables, and state the objective function and constraints with proper justification. Be sure that your formulation is written in explicit form.

Problem #2 (20 points)

The quality of air in an industrial region largely depends on the effluent emission from n plants. Each plant can use m different types of fuel. Suppose that the total energy needed at plant j is b_j British thermal units per day and that e_{ij} is the effluent emission per ton of fuel type i at plant j. Further suppose that fuel type i costs c_i dollars per ton and that each ton of this fuel type generates α_{ij} British thermal units at plant j. The level of air pollution in the region is not to extend b micrograms per cubic meter. Finally, let γ_j be a meteorological parameter relating emissions at plant j to air quality at the region. Formulate the problem of determining the optimal (minimum cost) mix of fuels to be used at each plant as a linear program: clearly define the variables, and state the objective function and constraints with proper justification. Be sure that your formulation is written in implicit form.

Problem #3 (20 points)

A small foundry needs to schedule the production of four different castings during the next week. The production requirements of each casting are summarized in the following table:

Product	Unit production times (minutes)				
	Pouring	Cleaning	Grinding	Inspection	Packing
A	3	8	10	1	3
B	1	12	6	1	5
C	2	6	9	1	3
D	1	7	7	1	2

The unit profit for Products A, B, C, and D are \$18, \$15, \$13, and \$14, respectively. Current demands indicate that all castings that are made can be sold; however, contracts dictate that at least 200 units of Product A and 300 units of Product D must be produced. The estimated times available for each of the operations during the next week are:

Pouting 40 hours,
Cleaning 80 hours,
Grinding 80 hours,
Inspection 20 hours,
Packing 40 hours.

The decision problem is to determine how much of each of product should be produced next week to maximize the total profit. Formulate explicitly the linear program for this problem: clearly define the variables, and state the objective function and constraints with proper justification.

Problem #4 (20 points)

(a) Reformulate the following problem as a linear program:

$$\text{maximize} \quad z = \frac{4x_1 + x_2 - 3x_4 + 1}{2x_1 + x_3 + 4x_4 + 3},$$

$$\begin{aligned} \text{subject to} \quad & x_1 - 2x_2 + x_3 + 2x_4 \le 10, \\ & x_2 - x_3 + 5x_4 \le 12, \\ & x_i \ge 0, \quad i=1,\ldots 4. \end{aligned}$$

(b) Solve the following linear programming problem graphically:

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 4x_2, \\ \text{subject to} \quad & x_1 + 2x_2 \le 6, \\ & -2x_1 + x_2 \le 4, \\ & 5x_1 + 3x_2 \le 15, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Problem #5 (20 points)

An approach to portfolio optimization is the scenario approach. In this, we identify a few scenarios (less than 10, say) that might occur during the next year. For each scenario we estimate the return on each investment. Then, we estimate the expected return and risk of the portfolio.

The Sentinel Finance Company, a small firm, wishes to invest in four stocks. The cost of each stock **($ per share**) and the forecasts of the return **($ per share**) for each stock made by the company's five analysts are given in the following table:

	Stock 1	**Stock 2**	**Stock 3**	**Stock 4**
Cost	$30.00	$45.00	$27.00	$53.00
Forecast 1	3.00	13.00	4.00	25.00
Forecast 2	1.00	4.50	.60	15.00
Forecast 3	2.75	1.75	2.75	20.00
Forecast 4	4.50	5.00	1.90	5.00
Forecast 5	3.25	2.75	3.75	35.00
Expected return ($/share)	2.90	5.40	2.60	20.00

Assume that all forecasts are equally likely. Additionally, the finance company would like to invest no more than $100,000. Sentinel has the following requirements for its investment portfolio:

1. Achieve an expected return of at least 10% of total amount invested.
2. Achieve a minimum risk [as measured by the mean absolute deviation from the expected return (a surrogate for variance)].
3. Invest at least 10% of the total investment in stock 4.

Formulate the above portfolio optimization problem as a linear program.

Extra Credit Problem (5 points)

Consider the linear program: maximize $c\,x$, subject to $A\,x \le b$, $x \ge \bar{0}$, where c is a nonzero vector. Suppose that the point x_0 is such that $A\,x_0 < b$ and $x_0 > \bar{0}$. Show that x_0 cannot be an optimal solution.