# Hierarchical B-Splines

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**Abstract.** In this paper we develop a hierarchical spline space as a linear span of tensor product B-splines. We give a selection mechanism for the B-splines, which ensures the linear independence while allowing complete local control of the refinement. Further, we define a quasi interpolant which achieves the optimal local approximation order.

**Key words.** hierarchical B-splines, subdivision, linear independence, quasi-interpolant, approximation, stability

# 1 Introduction

Spline functions play a fundamental role in many areas of numerical analysis and geometric modeling. While the univariate theory has reached a state of perfection [6],[7], in several variables two important issues are not sufficiently well understood: adaptive refinement of smooth approximations on bounded domains $\Omega \subset \mathbb{R}^d$ and modeling of objects of arbitrary topological shape. This paper addresses the first problem, the construction of hierarchical spline spaces on a bounded domain $\Omega$ which allow local subdivision of the parameter domain.

The hierarchical spline spaces are defined as linear span of tensor product B-splines of different grid levels. This obvious idea is similar to previous approaches in particular to the construction of spline wavelets [13], [10]. Moreover, with standard techniques of multiresolution analysis in [5] orthonormal bases and in [3] biorthonormal bases of compactly supported wavelets in $L_2(\mathbb{R})$ were developed. Chui and Quak [2] studied B-splines and mulitresolution analysis. They can give an explicit but complicated spline-wavelet for their wavelet space. Furthermore, an adaptive approximation scheme was proposed by de Boor and Rice [9] and modified by Dahmen in [4].

The basic new feature of our approach is a simple selection mechanism for B-splines which ensures linear independence on a bounded domain $\Omega \subset \mathbb{R}^d$ while allowing complete local control of the refinement. The use of B-splines in our approach leads to an explicit correlation between the mesh of B-spline knots and the surface represented. The resulting smooth spline space has the same approximation properties as discontinuous polynomial approximations. Moreover, the hierarchical B-spline basis is weakly stable, that is, the stability constant grows like $O(n)$, where $n$ is the number of grid levels.

Our paper is organized as follows. We describe in the first section the hierarchical spline space and show the linear independence of the defining collection of B-splines. In the second section, we define a quasi interpolant which is based on an adaptive selection principle and achieves the optimal local approximation order. In the third section, we discuss the stability of the hierarchical B-spline basis. Finally, in the fourth section we discuss some examples of approximation with hierarchical biquadratic B-splines.

# 2 Hierarchical B-spline bases

We consider a dyadic sequence of grids determined by the scaled lattices $(k_1/2^p, k_2/2^p)$.

On the grid of the dyadic level $p$ we denote by

$$B_k^p(x) = B_{k_1,k_2}^p(x_1, x_2) = B_\alpha(2^p x_1 - k_1) B_\beta(2^p x_2 - k_2) \tag{2.1}$$

the cardinal B-splines with degree $(\alpha, \beta)$ and **open** support

$$\text{supp } B_k^p = (k_1/2^p, (k_1 + \alpha + 1)/2^p) \times (k_2/2^p, (k_2 + \beta + 1)/2^p) \,. \tag{2.2}$$

Let

$$\Omega := \Omega^0 \supseteq \Omega^1 \supseteq \dots \tag{2.3}$$

be a nested sequence of domains with the following properties:

$$\Omega^{p+1} = \bigcup_{j \in J} \text{supp } B_j^p \quad \text{for some } J \subseteq \mathbb{Z}^2, \tag{2.4}$$

$$\partial\Omega^p \cap \partial\Omega^{p+1} = \emptyset. \tag{2.5}$$

From (2.4) it follows with (2.1) and (2.2) that, for an appropriate set $D^p$,

$$\Omega^p = \bigcup_{k \in D^p} \text{supp } B_k^p. \tag{2.6}$$

From the B-splines $B_k^p$, $k \in D^p$, we select those with support not contained in $\Omega^{p+1}$ as basis-functions for the hierarchical spline space. In other words, we set

$$I^p := \{k \in \mathbb{Z}^2 | \text{ supp } B_k^p \subseteq \Omega^p \text{ and supp } B_k^p \not\subseteq \Omega^{p+1}\} \tag{2.7}$$

and define the linear hierarchical spline space

$$S_\Omega := S_\Omega^{\alpha,\beta} := \mathrm{span}\{B_j^p \mid p \geq 0 \text{ and } j \in I^p\}$$

of the degree $(\alpha, \beta)$ as the linear span of the B-splines

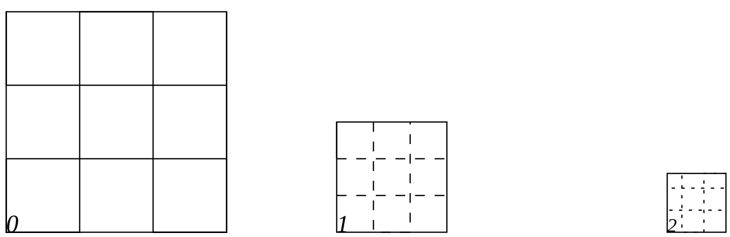
$$B_j^p, \quad p \geq 0,\, j \in I^p.$$

As is apparent from this definition, the B-splines $B_k^p$, $k \in D^p$ whose support is $\Omega^{p+1}$ are replaced by B-splines $B_k^{p+1}$ on the finer grid and thus can also be represented in the space $S$.

A spline function in $S$ has the representation

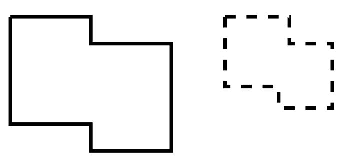
$$\begin{aligned} S :& \mathbb{R}^2 \to \mathbb{R}, \\ & x \mapsto S(x) = \sum_{p \geq 0} \sum_{k \in I^p} b_k^p B_k^p(x) \quad \text{with } b_k^p \in \mathbb{R}. \end{aligned}$$

To clarify the definitions, we discuss some examples. For these examples, we use biquadratic B-splines and the domain $\Omega^0 = (0,9) \times (0,9)$. As can be seen in the next figure, the support of a B-spline at level $p$ consists of nine meshes of the grid with gridsize $1/2^p$.




Support of B-spline $B_j^0$ Support of B-spline $B_j^1$ Support of B-spline $B_j^2$

As described in the figure above, we indicate this by writing in the lower left corner of the B-spline support the level of the corresponding B-spline.



domain of $\Omega^1$ domain of $\Omega^2$

The domains $\Omega^1$ and $\Omega^2$ will described with thicker lines.

In the first example (figure 2.1), we set $\Omega^1 = (2,5) \times (3,6)$. Because of (2.4), this is a smallest domain which we can use for $\Omega^1$. To fulfill the conditions (2.7) of the index set $I^0$ , we have to remove the B-spline $B_{2,3}^0$ and we have to insert nine B-splines at the level 1 (These are the B-splines with the number 1 in the lower left corner of their support).

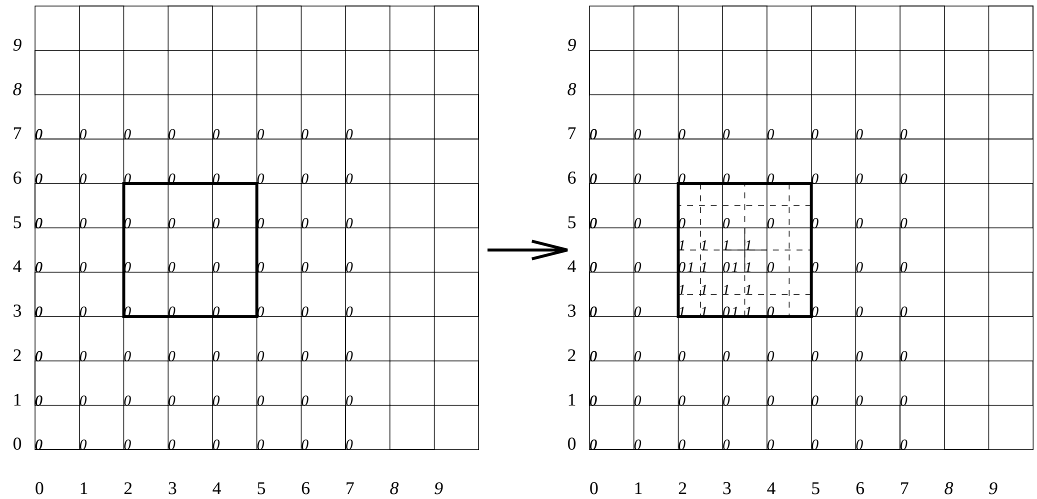


Figure 2.1: Smallest domain $\Omega^1$.

The support of the B-spline $B^0_{2,4}$ for example, overlaps $\Omega^1$ but is not a subset of it. So the index of this B-spline has to remain in the indexset $I^0$.

The solid lines at the level 1 in the middle of the domain $\Omega^1$ mark the four meshes from the grid at level 1, where all level 1 B-splines whose support overlaps this domain are a member of the indexset $I^1$.

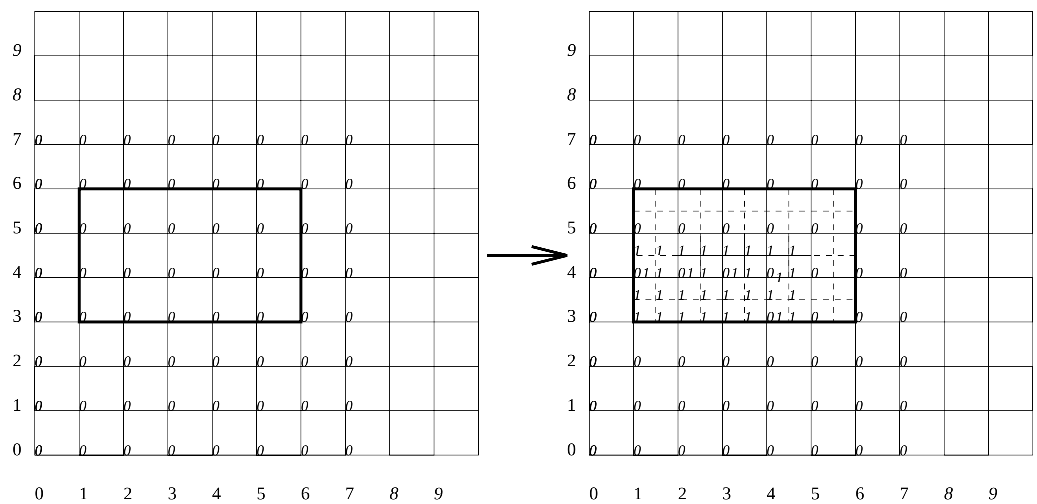


Figure 2.2: $\Omega^1$ defined as union of the support of four B-splines.

In the second example (figure 2.2) $\Omega^1$ is defined as the domain $(1,6)\times(3,6)$. Here, we have to remove the B-splines $B^0_{1,3}$, $B^0_{2,3}$ and $B^0_{3,3}$ to meet the conditions of (2.7). On the other hand, we have to insert a few B-splines $B^1_j$.

If we want to approximate a function with a given error tolerance, we choose some B-splines, which we want to subdivide to obtain a better local approximation order. In this case, we can define $\Omega^1$ by (2.6). But, as we see from the example, if we want to subdivide only the B-splines $B^0_{1,3}$ and $B^0_{3,3}$, we must subdivide the B-spline $B^0_{2,3}$, too.

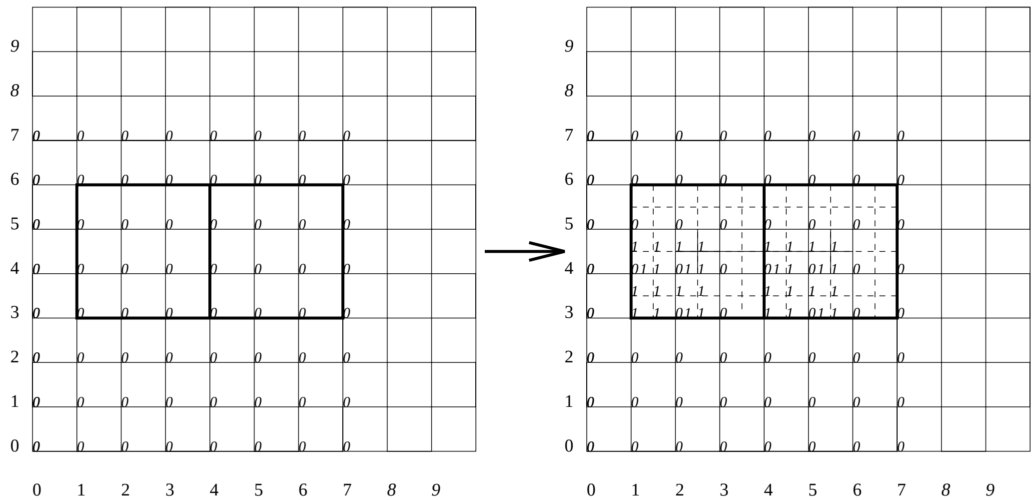


Figure 2.3: Borderline case definition of $\Omega^1$ with two disjunct domains.

The third example (figure 2.3) is a borderline case. Because of the open support of the B-splines, we don't have to remove the B-splines $B^0_{2,3}$ and $B^0_{3,3}$. The line between the points $(4,3)$ and $(6,3)$ is neither in the support of $B^0_{1,3}$ nor in the support of $B^0_{4,3}$. So $\Omega^0$ consists of the disjunct domains $(1,4)\times(3,6)$ and $(4,7)\times(3,6)$. The only points in the support of $B^0_{2,3}$ which are not included in $\Omega^1$ lie on this line.

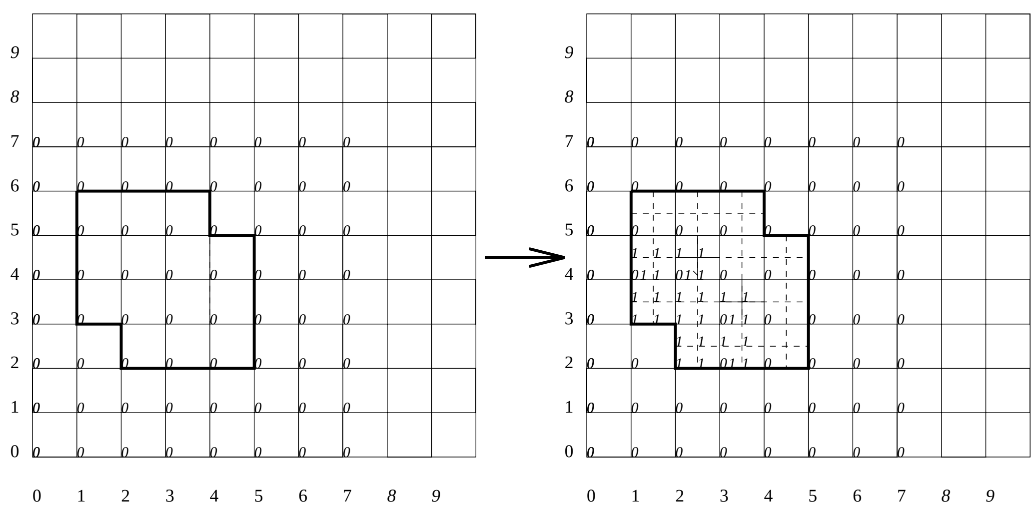


Figure 2.4: $\Omega^1$ defined as union of the overlapping support of two B-splines.

In the fourth example (figure 2.4), we define a slightly more complicated domain $\Omega^1$. We take two B-splines ($B^0_{1,3}$ and $B^0_{2,2}$) and define $\Omega^1$ to be the union of their support. In this case, we do not have to remove more B-splines to satisfy (2.7).

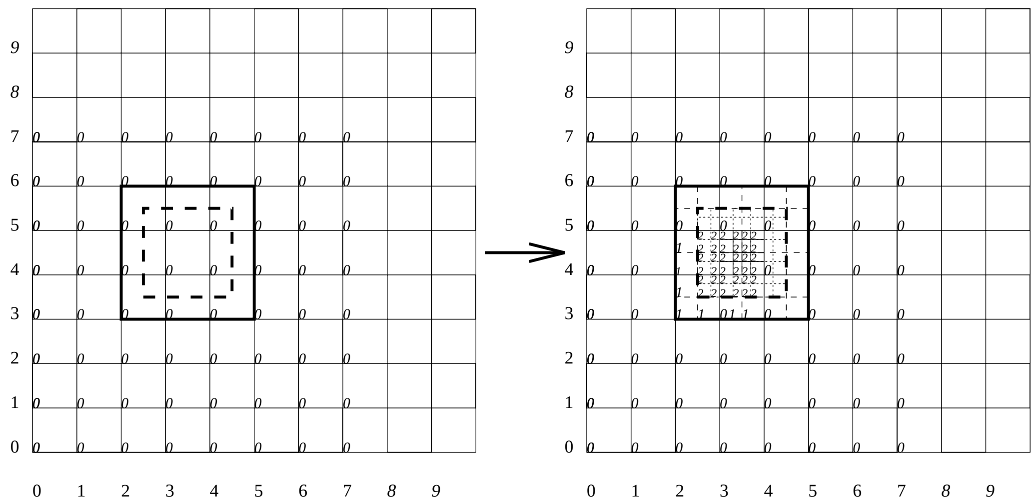


Figure 2.5: Maximum domain for $\Omega^2$.

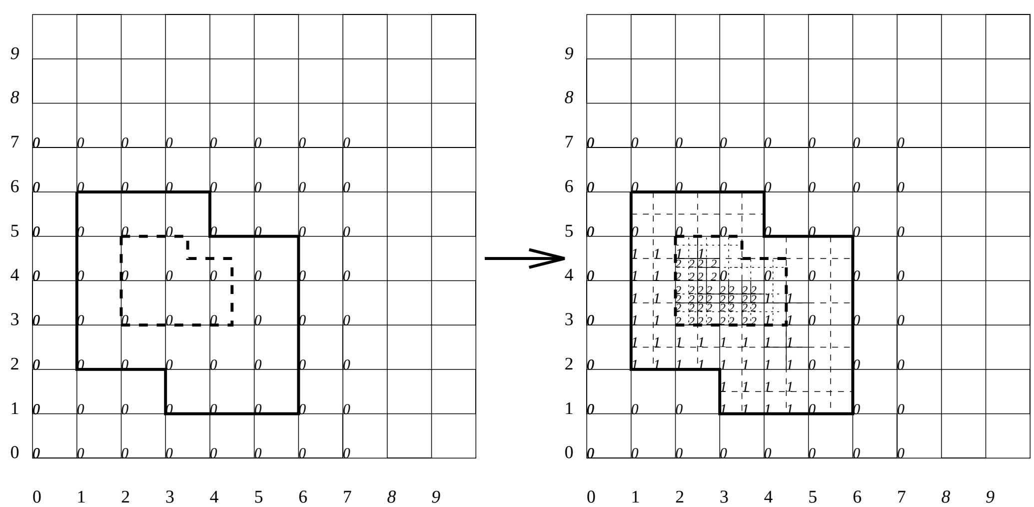


Figure 2.6: Hierarchical spline space with three dyadic levels.

The fifth and sixth example (figure 2.5 and 2.6) show three nested domains $\Omega^0 \supseteq \Omega^1 \supseteq \Omega^2$ and the resulting B-spline space. For the clarity of the figure, we wrote at points with several level numbers only the number of the lowest level. Because of (2.5), $\Omega^2$ in figure 2.5 describe the largest domain which we can define in this example.

The figure 2.7 shows a more complex example than the previous examples with four levels.

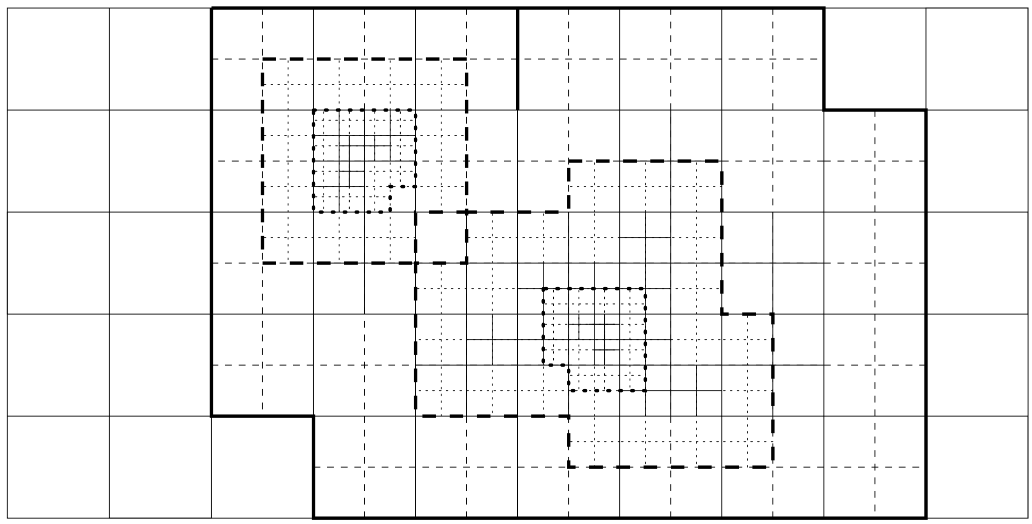


Figure 2.7: Hierarchical spline space with four dyadic levels.

# 3 Properties of the Hierarchical B-Splines

In this section we show that the B-splines $B_j^p$ are linearly independent.

**Theorem 3.1** *The set* $\{B_j^p \mid p \geq 0,\, j \in I^p\}$ *is a basis for its span.*

**Proof:** We have to show that all B-spline coefficients vanish if the spline function does so on $\Omega^0$:

$$S(x) = \sum_{p \geq 0} \sum_{k \in I^p} b_k^p B_k^p(x) \equiv 0 \Rightarrow b_k^p = 0 \quad \forall k \in I^p \; \forall p \geq 0.$$

This theorem can be proved by induction on the level $p$. We begin by showing that all $b_k^0$ with $k \in I^0$ vanish.

Let $p = 0$ and consider one coefficient $b_k^p \in S_\Omega$. Because of the property (2.4) there exists at least one $x \in \operatorname{supp} B_k^0$, which for all $m > 0$ and for all $l \in I^m$ is not in the support of the B-splines $B_l^m$. In addition we can choose this point as a grid point $x = t_{(k_1+\nu, k_2+\mu)} := (t_{k_1+\nu}, t_{k_2+\mu}) := (k_1 + \nu, k_2 + \mu)$ with $1 \leq \nu \leq \alpha$ and $1 \leq \mu \leq \beta$, because the border of $\Omega^1$ lies in the grid-lines of $\Omega^0$.

We claim

$$P_k^0 \left( \sum_{p \geq 0} \sum_{j \in I^p} b_j^p B_j^p(x) \right) = b_k^0$$

for an appropriate linear function $P_k^0$. We define [8], pages 178ff

$$P^0 f = \sum_k P_k^0 f B_k^0$$

with the linear functional

$$P_k^0 f = \sum_{\substack{0 \le j_1 \le \alpha \\ 0 \le j_2 \le \beta}} \frac{(-1)^{\alpha-j_1}\Psi_{k_1}^{(\alpha-j_1)}(\tau_{k_1})}{\alpha!} \frac{(-1)^{\beta-j_2}\Psi_{k_2}^{(\beta-j_2)}(\tau_{k_2})}{\beta!} f^{(j_1,j_2)}(\tau_{k_1},\tau_{k_2}), \tag{3.1}$$

$$\Psi_{k_1}(t) = (t - t_{k_1+1}) \cdots (t - t_{k_1+\alpha}), \tag{3.2}$$

$$\Psi_{k_2}(t) = (t - t_{k_2+1}) \cdots (t - t_{k_2+\beta}). \tag{3.3}$$

If we choose for some $\nu$ and $\mu$

$$\tau_{(k_1,k_2)} := x = t_{(k_1+\nu,k_2+\mu)} \tag{3.4}$$

we know with $P_j^0 B_k^0 = \delta_{j,k}$ [8] that

$$P_k^0 \left( \sum_{j \in I^0} b_j^0 B_j^0(x) \right) = b_k^0. \tag{3.5}$$

The fact that for $p > 0$ the point $x$ can only lie on the border of the B-spline $B_j^p$ leads to

$$(B_j^p)^{(j_1,j_2)}(\tau_{(k_1,k_2)}) = 0, \quad \text{for all } 0 \le j_1 \le \alpha - 1 \text{ and } 0 \le j_2 \le \beta - 1 \tag{3.6}$$

and for the indices $k_1 = \alpha$ and $k_2 = \beta$ it follows with (3.2), (3.3) and (3.4) that

$$\begin{aligned} \Psi_{k_1}^{(\alpha-\alpha)}(\tau_{k_1}) &= 0, \\ \Psi_{k_2}^{(0)}(\tau_{k_2}) &= 0. \end{aligned} \tag{3.7}$$

With $f = B_j^p$, (3.6) and (3.7) the linear functional (3.1) has the value $P_k^0 B_j^p = 0$ for all $p > 0$. This yields directly to

$$P_k^0 \left( \sum_{j \in I^p} b_j^p B_j^p(x) \right) = 0. \tag{3.8}$$

With (3.5), (3.8) and the assumption $S \equiv 0$ we achieve the desired result for the level $p = 0$:

$$b_k^0 = P_k^0 \left( \sum_{p \ge 0} \sum_{j \in I^p} b_j^p B_j^p \right) = P_k^0(S) = P_k^0(0) = 0, \quad \forall k \in I^0.$$

Assume that for all $p \le p_0$ and for all $k \in I^p$ the coefficients $b_k^p = 0$. For $p = p_0 + 1$ we find a point $x = t_{(k_1+\nu,k_2+\mu)}^{p_0+1} := ((k_1 + \nu)/2^{p_0+1}, (k_2 + \mu)/2^{p_0+1})$ with analogous properties. Proceeding as above we can also define a projection

$P^{p_0+1}$ with the same properties for the levels $p \geq p_0 + 1$ as above. Therefore and because of the assumption $b_k^p = 0$ for all $p \leq p_0$ we get

$$P^{p_0+1}\left(\sum_{j \in I_p} b_j^p B_j^p\right) = P^{p_0+1}(0) = 0$$

and with this we obtain the result

$$b_k^{p_0+1} = P_k^{p_0+1}\left(\sum_{p \geq 0}\sum_{j \in I^p} b_j^p B_j^p\right) = P_k^{p_0+1}(S) = P_k^{p_0+1}(0) = 0, \quad \forall k \in I^{p_0+1}.$$

$\square$

# 4 Approximation with Hierarchical B-Splines

Due to the B-spline representation, spline approximation methods can be implemented almost as efficiently as non-smooth piecewise polynomial approximations. Optimal order of accuracy can already be obtained by computing the B-spline coefficients as weighted average of a small number of function values. Such quasi-interpolation schemes are discussed in this section. The quasi-interpolation for hierarchical B-splines is based on the quasi interpolation of tensor product splines.

On the domain $\Omega^p$, a general quasi interpolant $P^p$ has the form:

$$P^p f = \sum_k (P_k^p f) B_k^p,$$

where $P_k^p$ are linear functionals which depend only on the restriction of $f$ to some set

$$U_k^p \subseteq \operatorname{supp} B_k^p \tag{4.1}$$

and which are bounded uniformly in $k$:

$$|P_k^p f| \leq \|P^p\| \|f\|_{U_k^p} = \|P\| \|f\|_{U_k^p}, \quad \text{with } \|f\|_{U_k^p} := \sup_{x \in U_k^p} |f(x)|. \tag{4.2}$$

The quasi interpolant is of order $(\alpha', \beta') \leq (\alpha, \beta)$ if $P^p$ reproduces polynomials of coordinate degree $\leq (\alpha', \beta')$, i.e.,

$$P^p g = g, \quad \text{for all polynomials } g \text{ with degree} \leq (\alpha', \beta').$$

For example, we may choose $P_k^p$ as a linear combination of function values,

$$P_k^p f = \sum_j P_{k,j} f(u_{k,j}), \quad P_{k,j} \in \mathbb{R}^n,\ u_{k,j} \in U_k^p := [t_{k_1+1}^p, t_{k_1+\alpha}^p] \times [t_{k_2}^p, t_{k_2+\beta-1}^p]$$

in which case $\|P\| = \max_k \sum_j |P_{k,j}|$.

If we start on the domain $\Omega^0$ with the gridsize $h_{x_i}(x_i)$ in the direction $x_i$, $i \in \{1,2\}$ at the point $(x_1, x_2)$, the approximation order for a bounded quasi interpolant is well known [11]:

**Lemma 4.1** *Let $P^p$ be a quasi interpolant of coordinate order $(\alpha', \beta') \le (\alpha, \beta)$ and $f$ a function $f \in C^{(\alpha'+1,\beta'+1)}$. Let $h^p_{x_i}(x_i) = h_{x_i}(x_i)/2^p$ the gridsize at the level $p$. Then*

$$
\begin{aligned}
&|f(x_1,x_2) - (P^p f)(x_1,x_2)| \le \\
&\qquad \mathrm{const}\, \|P\| \left( |h^p_{x_1}(x_1)|^{\alpha'+1} \|f^{(\alpha'+1,0)}\|_{U^p(x)} + |h^p_{x_2}(x_2)|^{\beta'+1} \|f^{(0,\beta'+1)}\|_{U^p(x)} \right), \\
&\mathit{with} \quad U^p(x) = \bigcup_{\substack{k \\ x \in \operatorname{supp} B^p_k}} U^p_k,
\end{aligned}
$$

*where the constant* const *does not depend on $f$, the knot sequences and $h^p_{x_1}$ and $h^p_{x_2}$.*

For the approximation with hierarchical B-splines we use the quasi interpolant for tensor product splines on the set $\omega^p$ defined as

$$
\omega^p := \{x \in \Omega^p \mid \forall_{k \in \mathbb{Z}^2}\, x \in \operatorname{supp} B^p_k \Rightarrow B^p_k \in S_\Omega\}.
$$

This means that those $x$ belong to $\omega^p$ for which all B-splines $B^p_k$, that do not vanish at $x$, can be represented in $S_\Omega$.

We define the quasi interpolant $P^p$ on $\Omega^p$ as above with the restriction that $U^p_k$ in (4.1) is restricted to subsets of $\omega^p$:

$$
P^p : C^{(\alpha'+1,\beta'+1)} \mid \omega^p \to \operatorname{span}\{B^p_k \mid k \in \mathbb{Z}^2\}, \tag{4.3}
$$

$$
f|_{\omega^p} \mapsto P^p f = \sum_{\substack{k \in \mathbb{Z}^2 \\ \operatorname{supp} B^p_k \cap \omega^p \neq \emptyset}} P^p_k f B^p_k, \tag{4.4}
$$

where $P^p_k$ are linear functionals, which depend only on the restriction of $f$ to

$$
U^p_{k,\omega} \subseteq [t^p_{k_1}, t^p_{k_1+\alpha+1}] \times [t^p_{k_2}, t^p_{k_2+\beta+1}] \cap \omega^p. \tag{4.5}
$$

The general quasi interpolant for hierarchical B-splines is defined as:

$$
Q = \lim_{p \to \infty} Q^p = \sum_{p \ge 0} \sum_{k \in I^p} Q^p_k f B^p_k, \tag{4.6}
$$

where

$$S^p f = \sum_{\substack{k \\ \operatorname{supp} B_k^p \subseteq \Omega^{p+1}}} Q_k^p f B_k^p, \tag{4.7}$$

$$\begin{aligned} Q^p &= Q^{p-1} - S^{p-1} + P^p(id - Q^{p-1} + S^{p-1}), \quad Q^0 = P^0 \\ &= \sum_{q \le p} \sum_{k \in I^p} Q_k^p f B_k^p. \end{aligned} \tag{4.8}$$

The linear functional $S^p$ eliminates only these coefficients $Q_k^p f$ which are generated from $P^p$ but which are not included in $I^p$. The corresponding B-splines $B_k^p$ can be represented with $S_\Omega$.

We note that because of $P^p S^{p-1} = S^{p-1}$ we have

$$Q^p = Q^{p-1} + P^p(id - Q^{p-1}). \tag{4.9}$$

With this quasi interpolant for hierarchical B-splines we can give an error estimate for the approximation of a function $f \in C^{(\alpha'+1,\beta'+1)}$.

**Theorem 4.1** *With $Q$ defined as above, let $x \in \Omega^0$ and $m$ be the integer with $x \in \omega^m \setminus \omega^{m+1}$. Then, for any function $f \in C^{(\alpha'+1,\beta'+1)}$ with $(\alpha', \beta') \le (\alpha, \beta)$:*

$$\begin{aligned} |f(x_1,x_2) - &(Qf)(x_1,x_2)| \\ &\le \text{const}\, \|P\| \Bigg[ \left(1 + \frac{\|P\|}{1 - 2^{-(\alpha'+1)}}\right) \frac{(h_{x_1}^0(x_1))^{\alpha'+1}}{2^{m(\alpha'+1)}} \|f^{(\alpha'+1,0)}\|_{U^m(x)} + \\ &\qquad \left(1 + \frac{\|P\|}{1 - 2^{-(\beta'+1)}}\right) \frac{(h_{x_2}^0(x_2))^{\beta'+1}}{2^{m(\beta'+1)}} \|f^{(0,\beta'+1)}\|_{U^m(x)} \Bigg] \\ &\le \widehat{\text{const}} \Bigg[ \left(\frac{(h_{x_1}^0(x_1))}{2^m}\right)^{\alpha'+1} \|f^{(\alpha'+1,0)}\|_{U^m(x)} + \\ &\qquad \left(\frac{(h_{x_2}^0(x_2))}{2^m}\right)^{\beta'+1} \|f^{(0,\beta'+1)}\|_{U^m(x)} \Bigg] \\ &\textit{with} \quad U^m(x) = \bigcup_{\substack{k \\ x \in \operatorname{supp} B_k^m}} (U_{k,\omega}^m) \cup \bigcup_{\substack{k \\ x \in \operatorname{supp} B_j^{m+1} \\ U_{j,\omega}^{m+1} \subseteq \operatorname{supp} B_k^m}} (U_{k,\omega}^m), \end{aligned} \tag{4.10}$$

*where the constants* const *and* $\widehat{\text{const}}$ *do not depend on $f$, the knot sequences, $h_{x_1}^0$ and $h_{x_2}^0$.*

Before we can prove this theorem, we have to look at a simplification of the construction of $Q^p$.

**Lemma 4.2** *Let $Q$, $Q^p$, $P^p$ and $\omega^p$ be defined as above. Then*

$$(Q^p f)|_{\omega^p} = (P^p f)|_{\omega^p}\ . \tag{4.11}$$

**Proof** We can prove this lemma by induction. First we have to show that

$$(Q^0 f)|_{\omega^0} = (P^0 f)|_{\omega^0} \ . \tag{4.12}$$

This follows directly from $Q^0 = P^0$. Assume that for all $p \leq p_0$ (4.11) will be true. For $p = p_0 + 1$ and with (4.8) we can expand $(Q^p f)|_{\omega^p}$ to

$$(Q^p f)|_{\omega^p} = (Q^{p-1} f)|_{\omega^p} + (P^p (f - Q^{p-1} f))|_{\omega^p} \ . \tag{4.13}$$

Because $\omega^p \subseteq \omega^{p-1}$ and (4.12) we know $(Q^{p-1}f)|_{\omega^p} = (P^{p-1}f)|_{\omega^p}$. $P^p$ is a projection onto the tensor product spline space. So it is clear that $P^p P^{p-1} f = P^{n-1} f$. With this and (4.13), we deduce

$$(Q^p f)|_{\omega^p} = (P^{p-1} f)|_{\omega^p} + (P^p f)|_{\omega^p} - (P^{p-1} f)|_{\omega^p} = (P^p f)|_{\omega^p} \ . \tag{4.14}$$

□

Now we can prove the theorem 4.1.

**Proof of theorem 4.1** We can expand the formula (4.6) with (4.9) to

$$(Qf)(x) = (Q^m f)(x) + \sum_{l>m} \left(P^l \left(f - Q^{l-1} f\right)\right)(x). \tag{4.15}$$

The projection $P^l$ needs only values from $f - Q^{l-1}f$ in $\omega^l \subseteq \omega^{l-1}$. So we can apply lemma (4.2) to $Q^{l-1}f$ in (4.15)

$$P^l (f - Q^{l-1} f) = P^l (f - P^{l-1} f). \tag{4.16}$$

With (4.2), (4.15) and (4.16) we can estimate the error to

$$\begin{aligned} |f(x) - (Qf)(x)| \leq |f(x) - (Q^m f)(x)| &+ \\ \sum_{l>m} \|P\| \left\| f - P^{l-1} f \right\|_{U_\omega^l(x)} & \\ \text{with} \quad U_\omega^l(x) = \bigcup_{\substack{k \\ x \in \operatorname{supp} B_k^l}} U_{k,\omega}^l . & \end{aligned} \tag{4.17}$$

Inserting the results of the error estimate for tensor product splines (lemma 4.1) in (4.17) yields with $\omega^l \subseteq \omega^{l+1} \subseteq \Omega^{l+1}$ and $\omega^l \subseteq \Omega^l \subseteq \Omega^{l+1}$ to

$$\begin{aligned} &|f(x) - (Qf)(x)| \leq \\ &\quad \text{const} \, \|P\| \Bigg[ \left( 1 + \|P\| \sum_{l \geq 0} \left( \frac{1}{2^{\alpha'+1}} \right)^l \right) \frac{(h_{x_1}^0 (x_1))^{\alpha'+1}}{2^{m(\alpha'+1)}} \| f^{(\alpha'+1,0)} \|_{U^m(x)} + \\ &\qquad \left( 1 + \|P\| \sum_{l \geq 0} \left( \frac{1}{2^{\beta'+1}} \right)^l \right) \frac{(h_{x_2}^0 (x_2))^{\beta'+1}}{2^{m(\beta'+1)}} \| f^{(0,\beta'+1)} \|_{U^m(x)} \Bigg]. \end{aligned} \tag{4.18}$$

The use of the geometric series yields to the desired result. $\square$

To approximate a given surface $f$ with a predefined tolerance $\epsilon$, we proceed in the following way.

First, we have to approximate $f$ on $\Omega^0$ with $\omega^0 := \Omega^0$ with the well known tensor product approximation. Because of theorem 4.1 we have to look in the second step at areas $V_k^1 := (t_{k_1}^0, t_{k_1}^0 + 1) \times (t_{k_2}^0, t_{k_2}^0 + 1)$ and $x_k^1 \in V_k^1$, where the second side of (4.10) with $U^m(x) = V_k^1$ is bigger than $\epsilon$. Now, we have to define $\Omega^1$ so that

$$\bigcup_k U^1(x_k^1) \subseteq \Omega^1$$

and $\Omega^0$ and $\Omega^1$ meets the definition 2.6. To do this, we can set $\Omega^1$ to

$$\Omega^1 = \bigcup_{k} \bigcup_{\substack{l \\ V_k^1 \subseteq \operatorname{supp} B_l^0}} \operatorname{supp} B_l^0.$$

To adjust $S_\Omega$ we have to remove the B-splines $B_k^0$ with supp $B_k^0 \subseteq \Omega^1$ and then, to compensate for this we have to insert B-splines $B_k^1$ with $B_k^1 \subseteq \Omega^1$. The coefficients of the newly included B-splines $B_k^1$ will be calculated with the quasi interpolant $Q^1$ and the coefficients of the remaining B-splines $B_k^0$ will remain unchanged (cf. (4.6), (4.7) and (4.8)).

In the third step, we have to examine only newly approximated areas, therefore we have to look at $\Omega^1$. Here we proceed as in the second step. We search for areas $V_k^2 := (t_{k_1}^1, t_{k_1}^1 + 1) \times (t_{k_2}^1, t_{k_2}^1 + 1)$ and $x_k^2 \in V_k^2$, where the second side of (4.10) with $U^m(x) = V_k^2$ is bigger than $\epsilon$ and define $\Omega^2$ as above. The redefinition of $S_\Omega$ and the calculation of the coefficients is similar as in the second step.

We repeat this until the approximation error is for all $x$ less than the given tolerance.

The advantage of the described algorithm is that we can adapt our algorithm to the given function and we have to calculate only few coefficients on any given level. If we reach an area where the approximation error is small enough, we leave the coefficients of the overlapping B-splines for the rest of the algorithm unchanged. Moreover, we don't have to examine this area any further.

# 5 Stability

For practical use of the hierarchical B-splines it is very useful to know something about the stability. In the theory of tensor product splines, the stability estimate

$$\operatorname{const}(\alpha, \beta) \max_k |b_k| \leq \| \sum_k b_k B_k^p \|_\infty \leq \max_k |b_k| \tag{5.1}$$

is well known [8].

In the theory of hierarchical B-splines we have 'only' a weaker estimate:

**Theorem 5.1** *Let $S_\Omega$, $\Omega^p$ and $I^p$ with $p \geq 0$ as above. Then*

$$C_0(p) \max_{k \in I^p} |b_k^p| \leq \| \sum_{k \in I^p} b_k^p B_k^p \|_\infty \leq \max_{k \in I^p} |b_k^p|, \quad \forall p \geq 0 \tag{5.2}$$

*and*

$$\| \sum_{p=p_{min}}^{p_{max}} \sum_{k \in I^p} b_k^p B_k^p \|_\infty \leq C(p_{min}, p_{max}) \max_{\substack{p \geq 0 \\ k \in I^p}} |b_k^p| \tag{5.3}$$

*with $C(p_{min}, p_{max}) = (1 + p_{max} - p_{min})$.*

**Proof:** The stability (5.2) of the basis at each level $p$ follows directly from (5.1). The right inequality in (5.3) follows directly from the fact that for each $p$ the $B_k^p$ are a partition of unity.

We can't find a better constant $C$ in (5.3) so that the inequality gets stronger because of the following remark.

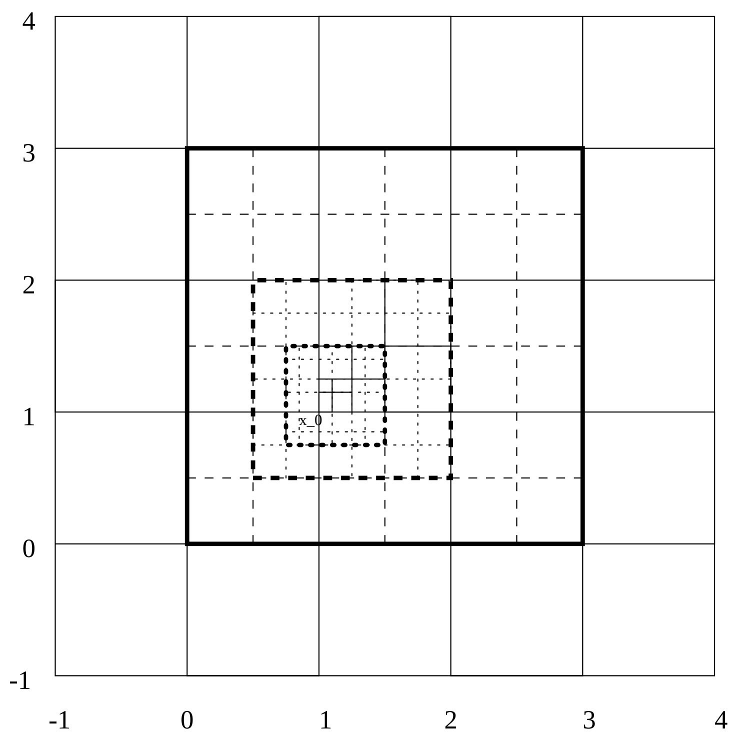



Figure 5.1: Hierarchical spline space for remark 5.1.

**Remark 5.1** *The constant $C(p_{min}, p_{max})$ in (5.3) depends linearly on $p_{min}$ and $p_{max}$.*

To verify this remark, we will construct a spline surface $S$ of degree $(2,2)$ which will show the linear dependence. We define $\Omega^0$ as $\mathbb{R}^2$ and take the grid with meshsize 1 on $\Omega^0$. In each dyadic level $p$ we remove one B-spline $B_k^p$ and insert the 16 little B-splines at the next level $p+1$, which are generated by the projection $P^{p+1}$:

$$
\begin{aligned}
\alpha &= \beta = 2, \\
\Omega^0 &= \mathbb{R}^2, \\
\Omega^1 &= [0,3]^2, \\
\Omega^p &= \left[1 - \frac{1}{2^{p-1}}, 1 + \frac{1}{2^{p-2}}\right]^2, \quad \text{with } p \geq 1, \\
h^0 &= 1, \quad \text{this means, we define on } \Omega^0 \text{ a grid with meshsize 1}, \\
I^0 &= \mathbb{N}^2 \setminus \{(1,1)\}, \\
I^p &= \{(k_1,k_2) \mid 0 \leq k_1, k_2 \leq 3\} \setminus \{(1,1)\} \quad \text{with } p \geq 1, \\
x_0 &= (1,1).
\end{aligned}
$$

If we evaluate the B-spline $B_k^p$ on $x_0$ we get for all $p \geq 0$

$$
B_k^p(x_0) = \begin{cases} 1/4 & \text{if } k \in \{(0,0),(0,1),(1,0),(1,1)\} \\ 0 & \text{other } k. \end{cases} \tag{5.4}
$$

Then the B-spline surface $S^{p_0}$, which is refined to the maximum level $p_0$

$$
S^{p_0} = \sum_{p=0}^{p_0} \sum_{k \in I^p} B_k^p \; + \; B_{(1,1)}^{p_0}
$$

has, evaluated on $x_0$ with (5.4), the value

$$
S(x_0) = \frac{3}{4}(p_0 + 1) + \frac{1}{4} = 1 + \frac{3}{4} p_0.
$$

This implies directly that

$$
\|S^{p_0}\|_\infty \geq 1 + \frac{3}{4} p_0 = \left(1 + \frac{3}{4} p_0\right) \max_{\substack{p \geq 0 \\ k \in I^p}} |b_k^p| \quad \text{because } b_k^p = 1.
$$

□

# 6 Examples

In this section, we will discuss the approximation of a function from $\mathbb{R}^2$ to $\mathbb{R}$ defined on the domain $[-1.3, 1.3]^2$. We use the error estimation of the theorem

4.1 with $(\alpha', \beta') = (\alpha, \beta) = (2, 2)$. So we have an approximation with $O(h^3)$, which is optimal for biquadratic B-splines.

As quasi interpolant for tensor product splines, we use with the terms of theorem 4.1:

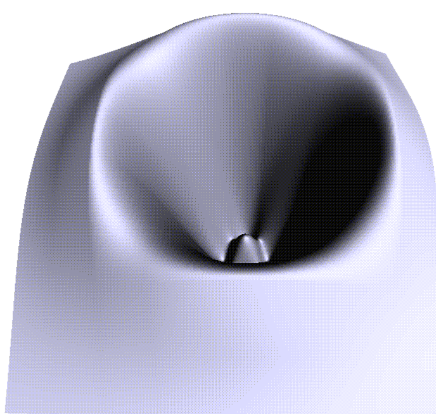
$$\begin{aligned} P^p :& C^{(2,2)} \mid \omega^p \to \operatorname{span}\{B_k^p \mid k \in \mathbb{Z}^2\}, \\ & f|_{w^p} \mapsto P^p f = \sum_{\substack{k \in \mathbb{Z}^2 \\ \operatorname{supp} B_k^p \cap \omega^p \neq \emptyset}} P_k^p f B_k^p. \end{aligned}$$

$$\begin{aligned} P_k^p f = & \sum_{(0,0) \le j \le (2,2)} P_{k,j}^p f(u_{k,j}), \\ & \text{with } u_{k,j} \in U_k^p \subseteq \operatorname{supp} B_k^p \cap \omega^p. \end{aligned}$$

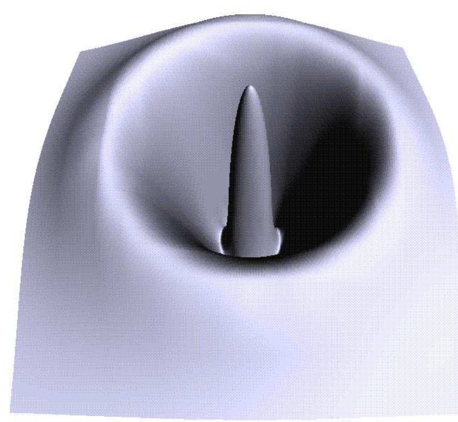
In our example, we approximate the function

$$f(x) = \sin\left(\frac{1}{x_1^2 + x_2^2 + \frac{2}{5\pi}}\right) \tag{6.1}$$

to several given error tolerances. This function is smooth outside the circle with radius 1. Therefore, we can approximate the function with large patches to reach the error tolerance. We do not have to use any patches with a small meshsize



(a) Large error tolerance.

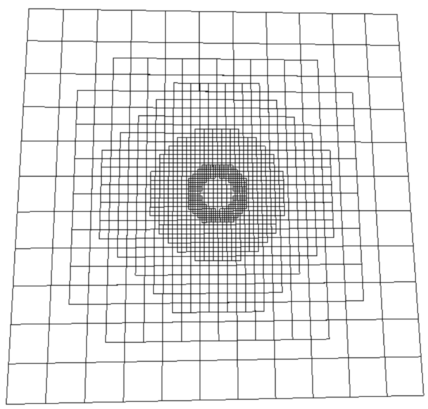


(b) Medium error tolerance.

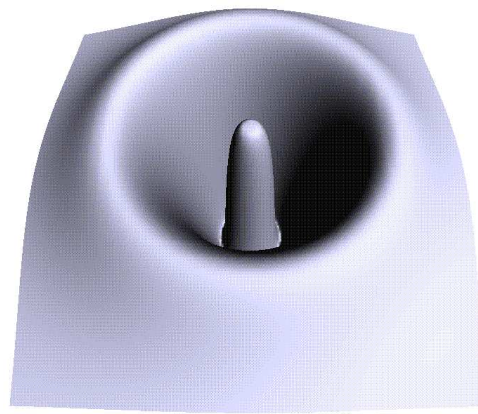
Figure 6.1: Spline function generated with different error tolerance.

to achieve a better approximation order outside the circle. Inside this circle, we need patches with a smaller meshsize, because the function falls and rises very fast, the more, as we get closer to the middle of the circle. In the center we have a 'plateau' where we do not need patches with such a small grid size than in the areas with a large gradient to reach the given error tolerance.

If we approximate this function with a large error tolerance the approximation process will approximate the inside of the circle not very well. The figures (a) and (b) in figure 6.1 show this. In the first example the error tolerance is so large that the middle peak is not noticed by the approximation process. In the second example with a slightly smaller error tolerance, the middle peak is noticed but the approximation is slightly worse.



(a) Hierarchical Net.



(b) Small error tolerance.

Figure 6.2: Spline function generated with a small error tolerance.

The figure 6.2 shows the approximation with a small error tolerance. Here, the middle peak is approximated well due to the fine grid of the hierarchical net in this area. The subfigure (a) shows the hierarchical net for the figure (b), where one can see the adaption of the hierarchical net to the given function.

In this example we need only **2342** patches to approximate the function to the given error tolerance. If we use only tensor product splines, we need for this approximation **30976** patches. So we have a reduction factor of ca. **14**.

The hierarchical spline spaces introduced in this paper can also be used to approximate surfaces in $\mathbb{R}^n$. In this case, the B-spline coefficients are vectors in $\mathbb{R}^n$ and the quasi-interpolants have to be modified accordingly. While this generalization of the theory is straight forward the modelling of surfaces of arbitrary topological shape is more difficult. This is the subject of a forthcoming paper.

# References


[1] Heinz Bauer. *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*, volume 3. Walter de Gruyter, Berlin, New York, 1978.

[2] C. K. Chui and E. Quak. Wavelets on a bounded interval. In D. Braess and L. L. Schumaker, editors, *International Series of Numerical Mathematics*, volume 105, pages 53–75, Basel, 1992. Birkhäuser.

[3] A. Cohen, I. Daubechies, and J.-C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. of Pure and Appl. Math*, (45):485–560, 1992.

[4] Wolfgang Dahmen. Adaptive approximation by multivariate smooth splines. *J. of Approx. Theory*, (36):119–140, 1981.

[5] Ingrid Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. on Pure and Appl. Mathematics*, XLI:909–996, 1988.

[6] Carl de Boor. The quasi-interpolant as a tool in elementary polynomial spline theory. In G. G. Lorentz, editor, *Approximation Theory*, pages 269–276. Academic Press, INC, New York, London, 1973.

[7] Carl de Boor. Splines as linear combinations of B-splines. In G. G. Lorentz, C. K. Chui, and L. L. Schumaker, editors, *Approximation Theory II*, pages 1–48. Academic Press, INC, New York, San Francisco, London, 1976.

[8] Carl de Boor. *A Practical Guide to Splines*. Applied Mathematical Sciences. Springer–Verlag, 1978.

[9] Carl de Boor and John R. Rice. An adaptive algorithm for multivariate approximation giving optimal convergence rates. *J. of. Approx. Theory*, (25):337–359, 1979.

[10] R.A. DeVore, B. Jawerth, and B.J. Lucier. Surface compression. *CAGD*, 9(3):219–239, 1992.

[11] Klaus Höllig. Skript zur Vorlesung 'Höhere Numerik'. Math. Inst. A, Univ. Stuttgart, 1994.

[12] Josef Hoschek and Dieter Lasser. *Grundlagen der geometrischen Datenverarbeitung*. B. G. Teubner Stuttgart, 2 edition, 1992.

[13] Tom Lyche and Knut Mørken. Knot removal for parametric B-spline curves and surfaces. *CAGD*, 4:217–230, 1987.

[14] Tom Lyche and Knut Mørken. A data-reduction strategy for splines with applications to the approximation for functions and data. *J.Num.Anal.*, 8:185–208, 1988.