Geometric Continuity of Parametric Curves: Three Equivalent Characterizations

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Geometric continuity of curves has received a good deal of research attention in recent years. The purpose of this article is to distill some of the important basic results into a self-contained presentation. In the January 1990 issue of *CG&A* we will continue the discussion by offering applications of the theoretical background provided here.

Parametric spline curves are typically constructed so that the first n parametric derivatives agree where the curve segments abut. This type of continuity condition has become known as C^n or *nth-order parametric continuity.* As has previously been shown, the use of parametric continuity disallows many parameterizations that generate geometrically smooth curves.

A relaxed form of nth-order parametric continuity has been developed and dubbed *nth-order geometric continuity* or G^n. This article explores three equivalent characterizations of geometric continuity. First, the concept of *equivalent parameterizations* is used to view geometric continuity as a measure of continuity that is *parameterization independent,* that is, a measure *invariant under reparameterization.* The second characterization develops necessary and sufficient conditions, called *Beta-constraints,* for geometric continuity of curves. Finally, the third characterization shows that two curves meet with G^n continuity if and only if their arc length parameterizations meet with C^n continuity.

G^n continuity provides for the introduction of n quantities known as *shape parameters*, which can be made available to a designer in a CAD environment to modify the shape of curves without moving control vertices.

Several applications of geometric continuity will be presented in our companion article in the January 1990 issue of *CG&A*.[1] First, composite Bezier curves will be stitched together with G^1 and G^2 continuity using geometric constructions. Then, a subclass of the Catmull-Rom splines that are based on geometric continuity and possess shape parameters will be discussed. Finally, quadratic G^1 and cubic G^2 Beta-splines will be developed using the geometric constructions for the geometrically continuous Bezier segments.

Parametric representation

A *parametric* function defines a mapping from a *domain parameter space* into geometric or Euclidean

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space. The definition of a curve involves functions of a single parameter, whereas for a surface it uses functions of a pair of parameters. Specifically, in the case of curves, the parametric function defines a mapping from u into Euclidean two-space as $\mathbf{Q}(u) = [x(u), y(u)]$ or into Euclidean three-space as $\mathbf{Q}(u) = [x(u), y(u), z(u)]$. This function can be used to define a curve by letting u range over some interval of the u axis. For a surface, the parametric function is a mapping from u,v into three-space as $\mathbf{Q}(u,v) = [x(u,v), y(u,v), z(u,v)]$. A surface is then defined using this function by letting u and v range over some rectangle in the u,v plane. Note the use of boldface is to demonstrate that the function is vector valued.

In the case of curves, if the domain parameter is thought of as time, the parametric function is used to locate the particle in space at a given instant. As time passes, the particle sweeps out a path, thereby tracing the curve. A parametric function therefore defines more than just a path; there is also information about the direction and speed of the particle as it moves along the path.

Piecewise representation and smoothness

In recent years computer-aided geometric design (CAGD) has relied heavily on mathematical descriptions of objects based on a special kind of parametric function known as a *parametric spline function.* A parametric spline function is a *piecewise* function where each of the pieces is a parametric function. The pieces of a curve are known as *segments,* while those of a surface are called *patches.* An important aspect of these functions is the way the segments are joined together. The locations where the pieces of the function abut are called *joints* in the case of curves, and *borders* in the case of surfaces. The equations that govern this joining are called *continuity constraints.* In CAGD, the continuity constraints are typically chosen to impart a given order of smoothness to the spline. The order of smoothness chosen will naturally depend on the application. For some applications, such as architectural drawing, it is sufficient for the curves to be continuous only in position. Other applications, such as the design of mechanical parts, require first- or second-order smoothness.

We have been intentionally vague about what we mean by "smoothness." In fact, there is more than one type of smoothness; the type used should be application dependent. For instance, if parametric splines are being used to define the path of an object in an animation system, the object must move smoothly. It is therefore not enough for the path of the object to be smooth; the speed of the object as it moves along the path must also be continuous. This type of motion can be guaranteed by requiring continuity of position and the *first parametric derivative vector,* also known as the *velocity vector.* If higher order continuity is required, we can demand continuity of the *second parametric derivative,* or *acceleration vector.* However, in many CAGD applications, only the resulting path is important; the rate at which the points along the curve are swept out is irrelevant. This second notion of smoothness allows discontinuities in speed, as long as the resulting path is geometrically smooth. We shall refer to the first kind of smoothness as *parametric continuity* and to the second kind as *geometric continuity.*

Most parametric spline formulations in CAGD are based on parametric continuity. However, the parametric constraints are frequently too strict for our purposes in CAGD, since they require conservation of derivatives in the parametric realm, and thus need very special parameterizations. In contrast, the constraints imposed by geometric continuity accommodate the differences between parameterizations of adjacent curve segments. A practical consequence is that when developing a spline formulation, geometric continuity can be used in place of parametric continuity. This endows the formulation with *shape parameters,* each of which can be made available to a designer in a CAD environment to modify the shape of curves without moving any of the control vertices that define the curve.

For example, the overconstraining parametric conditions are the ones that form the foundation for the B-spline formulation.[2] In the case of the Beta-spline representation, however, we do not deal directly with the parameterization, but rather with the geometry resulting from the parameterization.[3,4] Similarly, the nonuniform rational B-spline curve (sometimes called a NURB), which is the ratio of two nonuniform B-splines,[5,6] is parametrically continuous, whereas its geometrically continuous analogue is the rational Beta-spline.[7-11]

The shape parameters provide an intuitive and natural means to specify shape. The extra freedom provided by the shape parameters means that the design procedure using Beta-splines is somewhat more flexible than the procedure for B-splines. A typical design application using B-splines involves the definition of a control polygon, generation of the corresponding curve, then movement of existing vertices or insertion

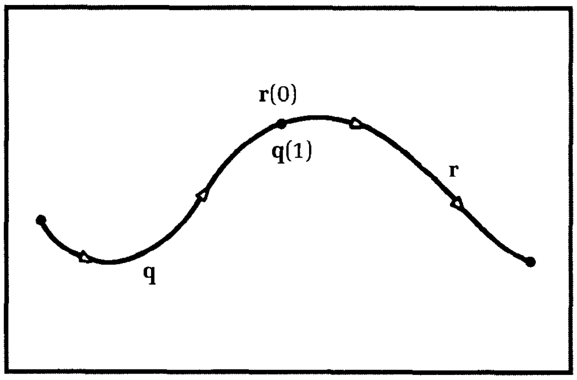

Figure 1. Two parameterized curves q and r meeting at a common point.

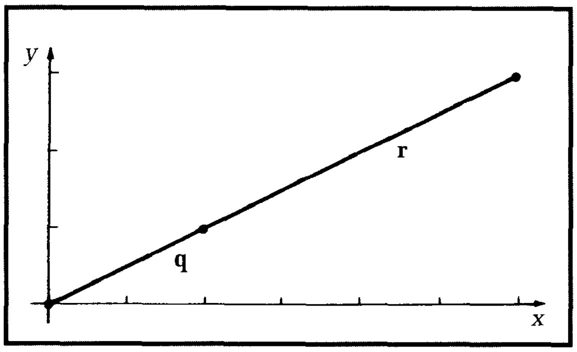

Figure 2. A discontinuous first derivative with a continuous unit tangent vector.

of new vertices to refine the curve in regions of particular interest. A design process exploiting Beta-splines does not always require that vertices be moved or added; the shape parameters can also be used to refine the curve.

Background on geometric continuity

The roots of nth-order geometric continuity are first- and second-order geometric continuity. This refers to continuity of the unit tangent and curvature vectors; various forms of this idea were proposed in the CAGD literature.[3,4,12-17] Motivated by the desire to construct higher order Beta-splines with more than the two shape parameters of the cubic case and with geometric continuity beyond that of the unit tangent and curvature vectors, we developed definitions of nth-order *geometric continuity,* denoted G^n, for CAGD. G^n continuity provides for the introduction of n shape parameters. This article is an abbreviated presentation of geometric continuity; more complete treatments can be found elsewhere.[18-21] On a historical note, we have recently learned of similar work in the German geometry literature called *contact of order n.*[22,23]

We explore here three equivalent characterizations of geometric continuity. First, the concept of *equivalent parameterizations* is used to view geometric continuity as a measure of continuity that is *parameterization independent,* that is, a measure *invariant under reparameterization.* The second characterization develops necessary and sufficient conditions, called *Beta-constraints,* for geometric continuity of curves. Finally, the third characterization shows that two curves meet with G^n continuity if and only if their arc length parameterizations meet with C^n continuity.

We should note that other parameterization-independent forms of continuity can be developed, the foremost of which is referred to as *Frenet frame continuity.* (The term "geometric continuity" was originally coined in the computer graphics and modeling community[3,4] and was used to refer to Beta-constraints,[18] but subsequently it has been used to indicate "Frenet frame continuity."[24-28]) Although geometric continuity and Frenet frame continuity are identical for first and second order, they differ for $n \geq 3$. This is currently an active area of research, but beyond the scope of this article; more complete discussions can be found in the literature.[11,24-27,29-32]

Geometric continuity has become an important topic of research, and recent work has been reported.[10,11,28,31-41] In the following, it will be important to distinguish between *parameterizations* and *curves.* A parameterization is a function that describes a curve. A curve is the image of a parameterization. There can be many different parameterizations that describe the same curve.

Parametric continuity

Let us now examine how continuity has classically been specified for parametric functions in CAGD. We refer to this measure of continuity as parametric continuity. As mentioned in the previous section, it is typical to stitch pieces of parametric functions together to obtain a parametric spline. Borrowing concepts from fields such as numerical analysis and approximation theory, it seems reasonable to require that the derivatives of the pieces agree at the joint.

Consider the situation shown in Figure 1, where two C^∞ parameterizations (a parameterization is C^∞ if it is infinitely differentiable) $\mathbf{q}(u)$, $u \in [0,1]$ and $\mathbf{r}(t)$, $t \in [0,1]$ meet at a common point such that

$$\mathbf{r}(0) = \mathbf{q}(1)$$

These parameterizations are said to meet with *nth-order parametric continuity*, denoted C^n, if the first n parametric derivatives match at the common point; that is, if

$$\mathbf{r}^{(i)}(0) = \mathbf{q}^{(i)}(1), \quad i = 1, \cdots, n \tag{1}$$

where superscript (i) denotes the ith derivative. Unfortunately, parametric continuity does not capture our intuitive notion of smoothness, as demonstrated by the following example. This shows that it is possible for the first derivative vector to be discontinuous, even though the curve possesses a physically continuous unit tangent vector throughout its length.

Example 1. Figure 2 shows the two parameterizations $\mathbf{q}(u)$ and $\mathbf{r}(u)$ defined by

$$\mathbf{q}(u) = (2u, u), \quad u \in [0, 1]$$
$$\mathbf{r}(t) = (4t + 2, 2t + 1), \quad t \in [0, \frac{1}{2}]$$

These parameterizations meet with positional continuity at the point (2,1). Note, however, that their first derivative vectors don't match:

$$\mathbf{q}^{(1)}(1) = (2, 1)$$
$$\mathbf{r}^{(1)}(0) = (4, 2)$$

implying that

$$\mathbf{r}^{(1)}(0) \neq \mathbf{q}^{(1)}(1)$$

These line segments are collinear and have a continuous unit tangent vector, namely $(2/\sqrt{5}, 1/\sqrt{5})$, even though there is a jump in the first derivative vector at the joint. In other words, these parameterizations do not meet with first-order parametric continuity, even though the curve segments appear to meet very smoothly. □

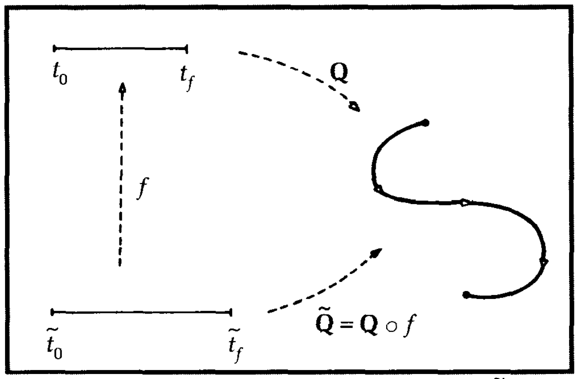

Figure 3. Q is reparameterized by *f* to obtain $\tilde{\mathbf{Q}}$.

This example provides insight into the shortcomings of parametric continuity. Parametric continuity places too much emphasis on the particulars of the parameterizations. It does not necessarily reflect the smoothness of the resulting composite curve; rather, it is a measure of smoothness for parameterizations. Also, parametric continuity disallows many parameterizations that would generate visually smooth curves. To understand fully how to avoid situations like the one in Example 1, we turn now to the ideas of *reparameterization* and *equivalent parameterizations.*

Reparameterization and equivalent parameterizations

Recall that a parameterization $\mathbf{Q}(u)$ is said to be C^n if it is n times continuously differentiable. Let $\mathbf{Q}(u), u \in [u_0, u_f]$, and $\tilde{\mathbf{Q}}(\tilde{u})$, $\tilde{u} \in [\tilde{u}_0, \tilde{u}_f]$, be two *regular C^n* parameterizations. (A parameterization is regular if its first derivative vector never vanishes.) These parameterizations are said to be equivalent; that is, they describe the same oriented curve, if there exists a regular C^n function $f: [\tilde{u}_0, \tilde{u}_f] \Rightarrow [u_0, u_f]$ such that

(i) $\tilde{\mathbf{Q}}(\tilde{u}) = \mathbf{Q}(f(\tilde{u}))$. That is, $\tilde{\mathbf{Q}} = \mathbf{Q} \circ f$
(ii) $f([\tilde{u}_0, \tilde{u}_f]) = [u_0, u_f]$
(iii) $f^{(1)} > 0$

Intuitively, $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ trace out the same set of points in the same order. We also say that $\mathbf{Q}$ has been reparameterized to obtain $\tilde{\mathbf{Q}}$, and we call f an *orientation-preserving change of variables* (see Figure 3). The following example illustrates a concrete example of equivalent parameterizations.

Example 2. Let $\mathbf{q}$ be as in Figure 2, and let $\tilde{\mathbf{q}}$ be defined by

$$\tilde{\mathbf{q}}(\tilde{u}) = (4\tilde{u}, 2\tilde{u}), \quad \tilde{u} \in [0, \frac{1}{2}]$$

To show that $\mathbf{q}(u) = (2u, u)$ and $\tilde{\mathbf{q}}(\tilde{u}) = (4\tilde{u}, 2\tilde{u})$ are equivalent parameterizations, we observe that

$$\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(2\tilde{u}), \quad \text{for all } \tilde{u} \in [0, \frac{1}{2}]$$

Thus, we have found a mapping $f{:}[0, 1/2] \Rightarrow [0, 1]$ defined by $f(\tilde{u}) = 2\tilde{u}$ that satisfies property (i) of equivalent parameterizations. It is easily verified that f satisfies the other two properties as well. We therefore conclude that $\mathbf{q}$ and $\tilde{\mathbf{q}}$ describe the same oriented curve, which in this case is the oriented line segment from (0,0) to (2,1). □

The existence of equivalent parameterizations means that there are many distinct parameterizations that describe the same oriented curve. Differential geometers are therefore careful to distinguish between properties of a particular parameterization and properties of the oriented curve it describes. This distinction can be made more precise by separating out the properties of parameterizations that remain invariant under reparameterization. Mathematically, let $\mathbf{q}$ and $\tilde{\mathbf{q}}$ be equivalent parameterizations, and let Property ($\mathbf{q}$) represent some property of $\mathbf{q}$ (that is, some statement about $\mathbf{q}$). Property ($\mathbf{q}$) is *intrinsic* if and only if

$$\text{Property}(\mathbf{q}) = \text{Property}(\tilde{\mathbf{q}})$$

Intrinsic properties are shared by all equivalent parameterizations, and can therefore be interpreted as fundamental properties of the curve being described. As an example, we ask, Is the first derivative vector an intrinsic property? By definition, the first derivative vector is intrinsic if and only if

$$\mathbf{q}^{(1)} = \tilde{\mathbf{q}}^{(1)} \tag{2}$$

where $\mathbf{q}$ and $\tilde{\mathbf{q}}$ are arbitrary equivalent parameterizations. To determine if Equation 2 does in fact hold, we use the chain rule from calculus:

$$\begin{aligned} \mathbf{q}^{(1)} &= (\tilde{\mathbf{q}} \circ f)^{(1)} && \text{(by definition of equivalence)} \\ &= \tilde{\mathbf{q}}^{(1)} f^{(1)} && \text{(by the chain rule)} \\ &\neq \tilde{\mathbf{q}}^{(1)} && \text{(since } f^{(1)} \text{ is not necessarily 1)} \end{aligned}$$

Thus, the first derivative vector is not an intrinsic property, and is therefore not a fundamental property of an oriented curve. There is, however, a closely related property that is intrinsic, namely, the *unit tangent vector.* If $\mathbf{T}(\mathbf{q})$ denotes the unit tangent vector of $\mathbf{q}$, then

$$\begin{aligned} \mathbf{T}(\mathbf{q}) &\equiv \frac{\mathbf{q}^{(1)}}{|\mathbf{q}^{(1)}|} && \text{(definition of } \mathbf{T}(\mathbf{q})\text{)} \\ &= \frac{(\tilde{\mathbf{q}} \circ f)^{(1)}}{|(\tilde{\mathbf{q}} \circ f)^{(1)}|} && \text{(by definition of equivalence)} \\ &= \frac{\tilde{\mathbf{q}}^{(1)} f^{(1)}}{|\tilde{\mathbf{q}}^{(1)} f^{(1)}|} && \text{(by the chain rule)} \\ &= \frac{\tilde{\mathbf{q}}^{(1)} f^{(1)}}{|\tilde{\mathbf{q}}^{(1)}| f^{(1)}} && \text{(since } f^{(1)} > 0\text{)} \\ &= \frac{\tilde{\mathbf{q}}^{(1)}}{|\tilde{\mathbf{q}}^{(1)}|} \\ &= \mathbf{T}(\tilde{\mathbf{q}}) \end{aligned}$$

From the point of view of differential geometry, Example 1 can now be explained as follows: C^n continuity requires equality of derivative vectors, and since derivative vectors are not intrinsic, C^n continuity can be destroyed simply by reparameterizing one of the curves. What is needed then is a measure of continuity that is parameterization independent. In other words, we would like a measure of continuity that is invariant under reparameterization—one which remains valid after arbitrary reparameterization. The following definition of geometric continuity provides such a measure.

Definition 1. Let $\mathbf{q}(u)$ and $\mathbf{r}(t)$ be two regular C^n parameterizations meeting at a point $\mathbf{J}$. They meet with nth-order geometric continuity, denoted G^n, if there exists a parameterization $\tilde{\mathbf{q}}$ equivalent to $\mathbf{q}$ such that $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^n continuity at the joint $\mathbf{J}$.

We can readily verify that geometric continuity is intrinsic in the sense that if $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity, and if $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ are any pair of equivalent parameterizations for $\mathbf{q}$ and $\mathbf{r}$, respectively, then $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{r}}$ also meet with G^n continuity.

To develop some familiarity with the definition, let us apply it to the parameterizations of Example 1. In particular, if we choose $\tilde{\mathbf{q}}$ to be the equivalent parameterization constructed in Example 2, then we see that

$$\tilde{\mathbf{q}}^{(1)}(\tfrac{1}{2}) = (4, 2)$$
$$\mathbf{r}^{(1)}(0) = (4, 2)$$

implying that

$$\mathbf{r}^{(1)}(0) = \tilde{\mathbf{q}}^{(1)}(\tfrac{1}{2})$$

Thus, $\tilde{\mathbf{q}}$ and $\mathbf{r}$ meet with C^1 continuity at $\mathbf{J} = (2,1)$; hence, $\mathbf{q}$ and $\mathbf{r}$ meet with G^1 continuity.

The characterization of geometric continuity based on the existence of equivalent parameterizations can be summarized as follows: Don't base continuity on the parameterizations at hand; reparameterize if necessary to find ones that meet with C^n continuity. Although this is a useful theoretical tool for probing the intricacies of geometric continuity, there are other characterizations of practical significance. We now briefly present two such equivalent characterizations.

Beta-constraints

Let $\mathbf{q}(u)$, $u \in [0,1]$ and $r(t)$, $t \in [0,1]$ be two regular C^n parameterizations meeting with G^n continuity at $\mathbf{r}(0) = \mathbf{q}(1)$, as was shown in Figure 1. According to Definition 1, there must exist an orientation-preserving change of variables $u : [\tilde{u}_0, 1] \Rightarrow [0,1]$ such that

$$\mathbf{r}^{(i)}(0) = \tilde{\mathbf{q}}^{(i)}(1), \quad i = 1, \cdots, n \tag{3}$$

where

$$\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u})), \quad \tilde{u} \in [\tilde{u}_0, 1]$$

For simplicity (and without loss of generality) we have chosen $\tilde{u}_f = 1$. Using the chain rule, derivatives of $\tilde{\mathbf{q}}$ can be expanded in terms of derivatives of $\mathbf{q}$ and $\mathbf{u}$. If the chain rule is applied i times, $\tilde{\mathbf{q}}^{(i)}$ can be expressed as a function — call it CR_i — of the first i derivatives of $\mathbf{q}$ and the first i derivatives of u:

$$\tilde{\mathbf{q}}^{(i)}(\tilde{u}) = CR_i(\mathbf{q}^{(1)}(u(\tilde{u})), \cdots, \mathbf{q}^{(i)}(u(\tilde{u})), u^{(1)}(\tilde{u}), \cdots, u^{(i)}(\tilde{u})), \quad i = 1, \cdots, n \tag{4}$$

Evaluating this expression at $\tilde{u} = 1$ and using the fact that $u(1) = 1$, we find that

$$\tilde{\mathbf{q}}^{(i)}(1) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), u^{(1)}(1), \cdots, u^{(i)}(1)), \quad i = 1, \cdots, n$$

This can be rewritten as

$$\tilde{\mathbf{q}}^{(i)}(1) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), \beta 1, \cdots, \beta i), \quad i = 1, \cdots, n \tag{5}$$

by performing the substitutions

$$\beta j = u^{(j)}(1), \quad j = 1, \cdots, i$$

The quantities $\beta 1, \cdots, \beta i$ are real numbers, and since $u^{(1)}(1) > 0$ (property (iii) of an orientation-preserving change of variables), we can conclude that $\beta 1 > 0$. Substituting Equation 5 into Equation 3 yields the so-called *Beta-constraints:*

$$\mathbf{r}^{(i)}(0) = CR_i(\mathbf{q}^{(1)}(1), \cdots, \mathbf{q}^{(i)}(1), \beta 1, \cdots, \beta i), \quad i = 1, \cdots, n \tag{6}$$

This argument shows that if $\mathbf{q}$ and $\mathbf{r}$ meet with G^n continuity, then there exist real parameters $\beta 1, \cdots, \beta n$ (with $\beta 1 > 0$), commonly called *shape parameters,* satisfying the Beta-constraints. More important for applications, the converse is also true. To be precise, the parameterizations $\mathbf{q}(u)$, $u \in [0,1]$ and $r(t)$, $t \in [0,1]$ meet with G^n continuity at $\mathbf{r}(0) = \mathbf{q}(1)$, if and only if there exist real numbers $\beta 1, \cdots, \beta n$ (with $\beta 1 > 0$) such that Equations 6 are satisfied. For a formal proof, see the literature.[18,19,21]

As an example of the form of the Beta-constraints, the constraints for G^4 continuity are

$$\mathbf{r}^{(1)}(0) = \beta 1 \mathbf{q}^{(1)}(1) \tag{7a}$$

$$\mathbf{r}^{(2)}(0) = \beta 1^2 \mathbf{q}^{(2)}(1) + \beta 2 \mathbf{q}^{(1)}(1) \tag{7b}$$

$$\mathbf{r}^{(3)}(0) = \beta 1^3 \mathbf{q}^{(3)}(1) + 3\beta 1 \beta 2 \mathbf{q}^{(2)}(1) + \beta 3 \mathbf{q}^{(1)}(1) \tag{7c}$$

$$\mathbf{r}^{(4)}(0) = \beta 1^4 \mathbf{q}^{(4)}(1) + 6\beta 1^2 \beta 2 \mathbf{q}^{(3)}(1) + (4\beta 1 \beta 3 + 3\beta 2^2)\mathbf{q}^{(2)}(1) + \beta 4 \mathbf{q}^{(1)}(1) \tag{7d}$$

where $\beta 2$, $\beta 3$, and $\beta 4$ are arbitrary, but $\beta 1$ is constrained to be positive.

The Beta-constraints can be used in several ways. First, if the curve segments $\mathbf{q}$ and $\mathbf{r}$ are given, the Beta-constraints can be used to determine whether the segments meet with G^n continuity. According to Equation 7a, the segments meet with G^1 continuity if the first derivative vectors of the segments at their common joint are a positive multiple of one another. If this is the case, then the ratio of each component of $\mathbf{r}^{(1)}(0)$ to the corresponding component of $\mathbf{q}^{(1)}(1)$ will be the same, and this value is $\beta 1$. Since $\beta 1$ is constrained to be positive, this is the ratio of the length of $\mathbf{r}^{(1)}(0)$ to that of $\mathbf{q}^{(1)}(1)$. This value of $\beta 1$ can then be substituted into Equation 7b. Continuing, the segments will then meet with G^2 continuity if the vector $\mathbf{r}^{(2)}(0) - \beta 1^2 \mathbf{q}^{(2)}(1)$ is a multiple (positive or negative) of $\mathbf{q}^{(2)}(1)$. Similar to G^1 continuity, for G^2 continuity, the ratio of each corresponding component of these vectors is the same, and this value is $\beta 2$. Consequently, the absolute value of $\beta 2$ is the ratio of the length of the vector $\mathbf{r}^{(2)}(0) - \beta 1^2 \mathbf{q}^{(2)}(1)$ to that of the vector $\mathbf{q}^{(2)}(1)$. To detect geometric continuity of arbitrary order, this process can be continued indefinitely.

The second and more common use of the Beta-constraints proceeds by allowing the designer to adjust the values of the shape parameters. This concept will be elaborated on in our companion article.[1] Using this approach, the curve design system then constructs the curve segments so that the Beta-constraints are satisfied, thereby guaranteeing that the segments join smoothly in a geometric sense. Changing the value of a shape parameter changes the shapes of the curve, but always so as to preserve geometric smoothness.

Arc-length parameterization

The next characterization of geometric continuity is based on arc-length parameterizations. It can be shown that two parameterizations meet with G^n continuity if and only if the corresponding arc-length parameterizations meet with C^n continuity.[19]

To gain a better understanding of this characterization, consider the cases of $n = 1$ and $n = 2$ in more detail. The case $n = 1$ requires that the first derivatives with respect to arc length agree. But the first derivative with respect to arc length is the unit tangent vector. Thus, the case $n = 1$ is equivalent to requiring that the unit tangent vectors agree at the joint $\mathbf{J}$. Similarly, for $n = 2$, the second derivative with respect to arc length is required to be continuous. The second derivative with respect to arc length is the curvature vector. Hence, for $n = 2$, the unit tangent and curvature vectors must match at the joint. This can be stated more formally as a theorem for G^1 and G^2 continuity.

Theorem 1. Two parameterizations meet with G^1 continuity if and only if they have a common unit tangent vector; they meet with G^2 continuity if and only if they have common unit tangent and curvature vectors. □

These are exactly the requirements of G^1 and G^2 continuity as originally developed for the Beta-spline representation[3,4] and which appeared in various forms in the literature.[12-17] This characterization therefore has the appeal that it represents a generalization of previous definitions.

Conclusion

Geometric continuity is an intrinsic measure of continuity appropriate for spline development. Geometric continuity has been shown to be a relaxed form of parametric continuity independent of the parameterizations of the curve segments under consideration, but still sufficient for geometric smoothness of the resulting curve. However, geometric continuity is appropriate only for applications where the particular parameterization used is unimportant, since parametric discontinuities are allowed.

Three characterizations of geometric continuity were developed. First, the concept of equivalent parameterizations was used to view geometric continuity as a measure of continuity that is parameterization independent—in other words, a measure that is invariant under reparameterization. The second characterization developed the Beta-constraints, which are necessary and sufficient conditions for geometric continuity of curves. Finally, the third characterization showed that two curves meet with G^n continuity if and only if their arc-length parameterizations meet with C^n continuity.

Using the Beta-constraints instead of requiring continuous parametric derivatives introduces n degrees of freedom called shape parameters. The shape parameters can be made available to a designer in a CAGD environment as a convenient method for changing the shape of the curve without altering the control polygon.

In the next issue of *CG&A* we will continue to explore the notion of geometric continuity and will address several applications of this concept. We will investigate the construction of geometrically continuous Bezier curves, the development of a subclass of the Catmull-Rom splines based on geometric continuity and possessing shape parameters, and the geometric construction of quadratic G^1 and cubic G^2 Beta-spline curves. ■

Acknowledgments

This work was supported in part by the National Science Foundation under grant numbers CCR-8451997, DMC-8602141, and DMC-8802949; and the Control Data Corporation, AT&T Bell Laboratories, an IBM Faculty Development Award, and a Digital Equipment Corporation Faculty Program Award.

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