# 315 Assignment 2

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## 1

Let $\mathcal{A}$ be the statement form composed of connectives $\rightarrow$, and true statement variables, all equivalently denoted $p$, such that $\nu(p) = 1$. Then we can show by induction that no matter the ordering of $\rightarrow$ and $p$, $\nu(\mathcal{A}) = 1$;
First, see the trivial base case, $\nu(p) = 1$.
Now, assume $\mathcal{B}$ is composed of $p$ and connectives $\rightarrow$, and $\nu(\mathcal{B}) = 1$.
Then we have $\nu((\mathcal{B} \rightarrow p)) = 1$ and $\nu((p \rightarrow \mathcal{B})) = 1$.
Thus by induction we have that $\nu(\mathcal{A}) = 1$, meaning that negation is not capable of being defined under the set $\{\rightarrow\}$ only. Therefore, $\neg p \not\Leftrightarrow \mathcal{A}$ under this set, and $\{\rightarrow\}$ is unable to fit the definition of a complete set, making it incomplete.

## 2

### (a)

$x$ is a theorem of $\mathcal{F}_1$, iff $x$ is of the form $AA^{2i}$, for some $i \in \mathbb{N}$

### (b)

$x$ is a theorem of $\mathcal{F}_2$, iff $x$ is of the form $A^{2^i}$, for some $i \in \mathbb{N}$
To see this, first consider the derivation;

$$A = A^{2^0},\ AA = A^{2^1},\ AAAA = A^{2^2},\ \dots,\ \underbrace{AA\dots AA}_{2^i} = A^{2^i}$$

Now suppose $\mathcal{B}_1, \dots, B_n = x$ is a derivation. We show by induction that each $\mathcal{B}_k$, $1 \leq k \leq n$ is of the required form.
For $k = 1$, $\mathcal{B}$ is $A$. So it is of the required form where $i = 0$.
For $k > 1$, if $\mathcal{B}_k = A$ we are done. Otherwise, $\mathcal{B}_k$ was obtained from some $\mathcal{B}_l$, $l < k$, from the single rule $x \implies xx$. By our inductive hypothesis, $\mathcal{B}_l = A^{2^i}$. Then $\mathcal{B}_k = A^{2^i}A^{2^i} = A^{2^{i+1}}$.
Thus $\mathcal{B}_k$ has the required form, ending our inductive proof.

## 3

We first show that $(\mathcal{A} \to \neg\neg\mathcal{A}$ is a tautology with the following truth table:

| $\mathcal{A}$ | $(\mathcal{A}$ | $\to$ | $\neg\neg\mathcal{A})$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 0 |
| | | $*$ | |

so $\nu((\mathcal{A} \to \neg\neg\mathcal{A})) = 1$
From **Theorem 2.36** (The Adequacy Theorem for $L$), we have **Corollary 2.37**; Every tautology is a theorem of $L$. Thus we have $\vdash_L (\mathcal{A} \to \neg\neg\mathcal{A})$
Then by **Theorem 2.7** (The Deduction Theorem for $L$) we have that $\{\mathcal{A}\} \vdash_L \neg\neg\mathcal{A}$ iff $\vdash_L (\mathcal{A} \to \neg\neg\mathcal{A})$
Thus $\{\mathcal{A}\} \vdash_L \neg\neg\mathcal{A}$, without requiring derivation in $L$

## 4

We first show that $\mathbb{Z}$ is countable. We do this by demonstrating that we can enumerate $\mathbb{Z}$ as $\{z_n | n \in \mathbb{N}\}$, i.e. there is an onto map $g : \mathbb{N} \mapsto \mathbb{Z}$. Consider the following map:

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

We can show this is onto: first let $z \in \mathbb{Z}$. We can now show that there exists $n \in \mathbb{N}$ such that $g(n) = z$. Choose $n = g^{-1}(z)$, where

$$g^{-1}(z) = \begin{cases} 2z & \text{if } z > 0 \\ -2z + 1 & \text{if } z \leq 0 \end{cases}$$

and so $g(g^{-1}(z)) = z$. So for all $z \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $g(n) = z$. Hence, $\mathbb{Z}$ is countable.
Now consider some bijective encoding for each $z \in \mathbb{Z}$ as a string of symbols, represented by the set $Z$. Decimal representation of a string of digits is suitable. Then by **Proposition 2.30**, the set $Z^*$ of all finite non-empty strings of symbols from $Z$ is countable.
The encoding is bijective, so upon being applied inversely, we will have the same property of countability for the set of all finite subsets of $\mathbb{Z}$

## 5

### (a)

We have from **Theorem 2.36** (The Adequacy Theorem for $L$), **Corollary 2.3.7**, that every tautology is a theorem of $L$.

Hence we have that $\Sigma \vdash_L A$.
Therefore there does exist a wff $A$ such that $\Sigma \vdash_L A$ and $\Sigma \vdash_L \neg A$, going against the definition of consistency, **Definition 2.16**.
Thus $\Sigma$ is inconsistent.

### (b)

From **Lemma 2.20**, we have that if $\Sigma \nvdash_L \mathcal{A}$, then $\Sigma \cup \{\neg\mathcal{A}\}$ is consistent.
From **Theorem 2.35**, $\Sigma \cup \{\neg\mathcal{A}\}$ is consistent iff $\Sigma \cup \{\neg\mathcal{A}\}$ is satisfiable.
Hence we have that ($\Sigma \nvdash_L \mathcal{A} \rightarrow \Sigma \cup \{\neg\mathcal{A}\}$ is satisfiable)

Using **Theorem 2.35**, $\Sigma \cup \{\neg\mathcal{A}\}$ is satisfiable iff $\Sigma \cup \{\neg\mathcal{A}\}$ is consistent. If it is consistent, then under **Definition 2.16**, we do no have both $\Sigma \cup \{\neg\mathcal{A}\} \vdash_L \mathcal{A}$ and $\Sigma \cup \{\neg\mathcal{A}\} \vdash_L \neg A$. Because we certainly have $\Sigma \cup \{\neg\mathcal{A}\} \vdash_L \neg\mathcal{A}$, as $\neg\mathcal{A} \in \Sigma \cup \{\neg\mathcal{A}\}$, then we must not have $\Sigma \cup \{\neg\mathcal{A}\} \vdash_L \mathcal{A}$.
Therefore $A \notin \Sigma \cup \{\neg\mathcal{A}\}$, else $\Sigma \cup \{\neg\mathcal{A}\}$ would entail $\mathcal{A}$ under $L$. So $\mathcal{A} \notin \Sigma$, as $\Sigma \subseteq \Sigma \cup \{\neg\mathcal{A}\}$.
As $\mathcal{A} \notin \Sigma$, then $\Sigma \nvdash_L \mathcal{A}$, because $\mathcal{A}$ wouldn't be a wff, due to not belonging to $\Sigma$, meaning not belonging to $form(L)$.
Hence we have that ($\Sigma \cup \{\neg\mathcal{A}\}$ is satisfiable$\rightarrow \Sigma \nvdash_L \mathcal{A}$)

Combining these, we have that $\Sigma$ does not entail $A$ iff $\Sigma \cup \{\neg A\}$ is satisfiable.

### (c)

By **Theorem 2.14** (The Soundness Theorem for $L$), we have that if $\{\mathcal{A}\} \vdash_L \mathcal{B}$, then $\mathcal{A} \vDash \mathcal{B}$. From **Definition 2.12**, as $\mathcal{A} \vDash \mathcal{B}$, then for the truth assignment $\nu$ such that $\nu(\mathcal{A}) = 1$, then $\nu(\mathcal{B}) = 1$.
Then by **Definition 2.10**, we have that $\nu((\mathcal{A} \rightarrow \mathcal{B})) = 1$.
Finally, from **Definition 2.11**, we can call this a tautology, as $\nu((\mathcal{A} \rightarrow \mathcal{B})) = 1$ under every truth assignment $\nu$.