1. (a) Write a derivation to show that $\{Ax\} \vdash_{K_{\mathcal{L}}} \exists x Ax$.
[You may wish to use an instance of the tautology: $((p \to \neg q) \to (q \to \neg p))$.]

(b) Explain how the deduction theorem can be applied to deduce that $\{Ax\} \vdash_{K_{\mathcal{L}}} (Bx \to \exists x Ax)$.

2. Let Σ be a set of closed wffs and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be closed wffs. Recall that for any wff $\mathcal{D}$, $(\mathcal{A} \vee \mathcal{D})$ is an abbreviation for $(\neg\mathcal{A} \to \mathcal{D})$. Suppose that $\Sigma \cup \{(\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C}))\}$ is consistent. Show that either $\Sigma \cup \{\mathcal{A}\}$ is consistent or $\Sigma \cup \{\mathcal{B}\}$ is consistent.

3. Let $\mathcal{L}_G = (\varnothing, \{P^1, L^1, I^2\}, \varnothing)$ be the language of geometry. The intended meaning is that an interpretation of $\mathcal{L}_G$ consists of a set of points and lines. Px means "x is a point", Lx means "x is a line" and Ixy means "line x is incident with point y" (point y is contained in line x). An example of an interpretation of $\mathcal{L}_G$ is illustrated below. We will construct a first-order system with equality, S, with language $\mathcal{L}_G$. [You may use the abbreviations $\wedge, \vee$ and $\exists$.]

(a) Write down axioms to ensure that every element of the domain is either a point or a line, and that no element can be both a point and a line.

(b) Write down axioms to ensure that there is at least one point and that there is at least one line.

(c) Write down an axiom to ensure that two distinct lines intersect in at most one point (in other words, if l_1 and l_2 are lines and $l_1 \neq l_2$ and p_1 and p_2 are points such that l_i is incident with p_j for $i, j = 1, 2$ then $p_1 = p_2$).

(d) Write down an axiom to ensure that for any two distinct points there is at least one line incident with both points.

(e) Show that if $\mathcal{I}$ is a normal model of S with domain D and p_1 and p_2 are points of $\mathcal{I}$ (in other words, $p_1, p_2 \in D$ and $P^{\mathcal{I}}p_1$, $P^{\mathcal{I}}p_2$ both hold and $p_1 \neq p_2$) then there is a unique $l \in D$ such that $L^{\mathcal{I}}l$, $I^{\mathcal{I}}lp_1$ and $I^{\mathcal{I}}lp_2$ all hold. [You need not give a derivation within S of this: it is enough to describe what properties normal models of S must have.]

Example:

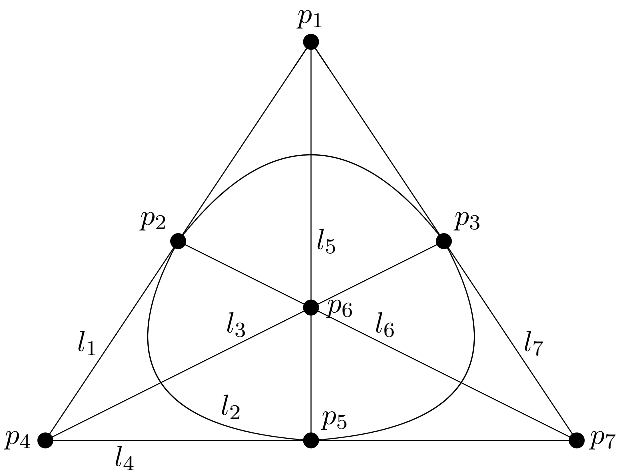

Figure: The finite projective plane of order 7

The domain has 14 elements, $p_1, p_2, \ldots, p_7$ and $l_1, l_2, \ldots, l_7$. For each i, $P^{\mathcal{I}} p_i$ and $L^{\mathcal{I}} l_i$ are true and $P^{\mathcal{I}} l_i$ and $L^{\mathcal{I}} p_i$ are false. $I^{\mathcal{I}}$ is the relation

$$
\begin{gathered}
\{(l_1, p_1), (l_1, p_2), (l_1, p_4), \\
(l_2, p_2), (l_2, p_3), (l_2, p_5), \\
(l_3, p_3), (l_3, p_4), (l_3, p_6), \\
(l_4, p_4), (l_4, p_5), (l_4, p_7), \\
(l_5, p_5), (l_5, p_6), (l_5, p_1), \\
(l_6, p_6), (l_6, p_7), (l_6, p_2), \\
(l_7, p_7), (l_7, p_1), (l_7, p_3)\}
\end{gathered}
$$

In the above diagram, 6 of the lines are represented by straight lines, and line l_2 is represented by a closed curve.