MATHS 315 Assignment 3

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September 23, 2018

1

(a)

Let $\mathcal{I}$ be an interpretation with domain D, and let v be an $\mathcal{I}$-assignment.

Case 1 If $\mathcal{I} \not\models_v \forall x \forall y \forall z Axyz$, then it is vacuously true that

$$\mathcal{I} \models_v (\forall x \forall y \forall z Axyz \to \forall y \exists x \exists z Axyz)$$

Case 2 Suppose then that $\mathcal{I} \models_v \forall x \forall y \forall z Axyz$. Let $d \in D$.
Since $\mathcal{I} \models_v \forall x \forall y \forall z Axyz$, by definition 3.10, 7, for any $d_1, d_2, d_3 \in D$,

$$\mathcal{I} \models_{v \frac{d_1}{x} \frac{d_2}{y} \frac{d_3}{z}} Axyz$$

But $v \frac{d_1}{x} \frac{d_2}{y} \frac{d_3}{z} = v \frac{d_2}{y} \frac{d_1}{x} \frac{d_3}{z}$

Since we have D as a non-empty set, then there exists some d_1 and d_3, such that

$$\mathcal{I} \models_{v \frac{d_2}{y} \frac{d_1}{x} \frac{d_3}{z}}$$

So by definition 3.10,8,

$$\mathcal{I} \models_{v \frac{d_2}{y}} \exists x \exists z Axzy$$

And since d_2 was chosen arbitrarily, we have that

$$\mathcal{I} \models_v \forall y \exists x \exists z Axyz$$

Therefore, by definition 3.10, 5,

$$\mathcal{I} \models_v (\forall x \forall y \forall z Axyz \to \forall y \exists x \exists z Axyz)$$

Which, as v and $\mathcal{I}$ were chosen arbitrarily, by definitions 3.12, 3.14, gives that

$$(\forall x \forall y \forall z Axyz \to \forall y \exists x \exists z Axyz)$$

is logically valid.

(b)

The predicate form

$$(\forall x \exists y Bxy \to \forall y Byy)$$

is not logically valid.

Let $\mathcal{I}$ be the interpretation with domain $\mathbb{N}$, with $B^{\mathcal{I}}(m,n)$ holding iff $m < n$. Let v be an $\mathcal{I}$-assignment. For every $m \in \mathbb{N}$, $B^{\mathcal{I}}(m, m+1)$ holds, so for any v we have $\mathcal{I} \vDash_{v\frac{m}{x}\frac{m+1}{y}} Bxy$. Thus by definition 3.10,8, $\mathcal{I} \vDash_v \forall x \exists y Bxy$.
There is no $n \in \mathbb{N}$ for which $B^{\mathcal{I}}(n,n)$ holds, so here is no n for which $\mathcal{I} \vDash_{v\frac{n}{y}} Byy$, so by definition 3.10, 7, $\mathcal{I} \nvDash \forall Byy$. Therefore, by definition 3.10,5, $\mathcal{I} \nvDash_v (\forall x \exists y Bxy \to \forall y Byy)$. Thus by definitions 3.12, 3.14, $(\forall x \exists y Bxy \to \forall y Byy)$ is not logically valid.

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(a)

Case 1 If $\mathcal{I} \nvDash A$, then by definition 3.10,5, $\mathcal{I} \vDash_v (A \to \exists A)$, and so $A \Rightarrow \exists A$.
Case 2 Suppose then that $\mathcal{I} \vDash A$. Then for every $\mathcal{I}$-assignment v , $\mathcal{I} \vDash_v A$ in every interpretation $\mathcal{I}$ of $\mathcal{L}$, by definitions 3.12, 3.14. Consider $v(x) = d$ for some $d \in D$. Then we have $\mathcal{I} \vDash_{v\frac{d}{x}} A$ for some $d \in D$. As $\mathcal{I} \vDash A$, this is alid. Be definition 3.10, 8, and as D is non-empty, this is equivalent to $\mathcal{I} \vDash_v \exists x A$. Thus, $A \Rightarrow \exists A$.

(b)

Let $A^{\mathcal{I}}(x)$ be the unary supremum predicate with domain D, that is, $\forall d \in D$, $d \leq x$.
Set D to some finite partially ordered set, with cardinality > 1.
We have for any $\mathcal{I}$-assignment v that, based on the properties of D, $\mathcal{I} \vDash_{v\frac{e}{x}} Ax$ for some $e \in D$, as every finite, non-empty partially ordered set has some element that exists as the supremum.
Now, consider the $\mathcal{I}$-assignment v such that, for all $d \in D$,

$$v(x) = \begin{cases} \text{Sup}(D) & \text{if } x \neq \text{Sup}(D) \\ \text{Inf}(D) & \text{if } x = \text{ Sup}(D) \end{cases}$$

Therefore $\mathcal{I} \nvDash_v Ax$
Thus we have that $\exists x Ax \not\Rightarrow Ax$.

3

(a)

i

$$(\exists z \forall x(Axy \to Bz) \vee \exists z Axz)\left(\frac{fzy}{x}\right) = (\exists z \forall x(Axy \to Bz) \vee \exists z Afzyz)$$

ii

$$((\exists x \forall y Axy \to Ayy) \to \forall y(Afxyz \wedge Bz))\left(\frac{fzy}{x}\right)$$
$$= (\exists x \forall y Axy \to Ayy) \to \forall y(Affzyyz \wedge Bz))$$

(b)

z for x	$\times$	✓
x for y	$\times$	$\times$
y for z	✓	$\times$
z for y	$\times$	✓
fxz for x	$\times$	✓
fygz for x	$\times$	$\times$

4

(a)

1.	$(B \to A)$	Hypothesis
2.	$(A \to (\neg A \to \neg B)$	Tautology Instance
3.	$(\neg A \to \neg B)$	1, 2, Modus Ponens
4.	$((\neg A \to \neg B) \to ((\neg A \to B) \to A))$	K3
5.	$((\neg A \to B) \to A)$	3, 4, Modus Ponens
6.	$\forall x((\neg A \to B) \to A)$	5, Generalisation

(b)

No - by Theorem 4.12 (The Deduction Theorem for $K_{\mathcal{L}}$), we can deduce

$$\vdash_{K_{\mathcal{L}}} ((B \to A) \to \forall x((\neg A \to B) \to A))$$

from

$$\{(B \to A)\} \vdash_{K_{\mathcal{L}}} \forall x((\neg A \to B) \to A)$$

iff when generalisation is used, it is not used on a free variable. Here, generalisation was applied on x, which may be a free variable in $(B \to A)$.