STATS 320 **Assignment 4** **Due: 2pm, 28th May, 2018**

1 **[25 marks]** Consider an $M/E_4/1$ queue that has Poisson arrivals at rate $\lambda = 2$ customers per minute and the Erlang distribution for for service times is $\Gamma(4, 10)$ distributed. Assume the system is in steady state.

(i) Use the Pollaczek-Khinchin formula to find the expected queue length of the system.[3 marks]

(ii) What is the expected total time a customer spends in the system? [3 marks]

(iii) What is the expected length of a busy period for this queue? [3 marks]

(iv) How long should an arriving customer expect to wait in the queue before reaching the server? [3 marks]

(v) What is the probability that an arriving customer finds the server busy? [3 marks]

(vi) Write an R program to simulate this queue, and plot a realization of $N(t)$ for the first 20-minutes run of the process, starting with $N(0) = 0$. [5 marks]

(vii) Use your code to estimate the average queue length L in steady state. Make sure you run your simulation for a long time but discard the first 20% of the simulation period so that it has time to settle down into steady state. Compare with the exact formula from part (a). [5 marks]

2 **[25 marks]** Suppose that a new ferry route has opened up to a previously pristine island. Unfortunately, the ferry introduces rats to the island and wherever female rats go, male rats always manage to follow.

Suppose that the size of the population of female rats as a function of time, $N(t)$, can be modelled by the classical birth death process with arrival rates

$$\lambda_n := n\lambda + \theta, \qquad \text{for} \quad n = 0, 1, 2, \dots,$$

and departure rates

$$\mu_n := n\mu, \qquad \text{for} \quad n = 0, 1, 2, \dots,$$

for some constants $\theta, \lambda, \mu > 0$.

(a) Suppose that $\theta > 0$ and $\lambda < \mu$, find all the stationary distributions that exist (if any). [5 marks]

(b) Write an R function that simulates the stochastic process $N(t)$. Your function should be called `ratpopulation` and should take five arguments: `T`, the maximum duration of the simulation, `N0`, the initial number of female rats, `lambda`, `theta` and `mu`. You can start by copying the `mm1()` function from lectures and modifying it appropriately. [5 marks]

For the following questions, suppose that it is known that $\lambda = 0.2$ per week, $\theta = 0.4$ per week and $\mu = 0.25$ per week.

(c) With initially 10 female rats, that is $N(0) = 10$, use your function to to generate 3 random realizations of $N(t)$ for 250 weeks, and plot all 3 realizations in different colours on the same graph. Use the `xlim=c(0, 250)` and `ylim=c(0, 50)` to set the axis limits. Label the axes appropriately by using `xlab` and `ylab`. [5 marks]

(d) If initially there are no female rats on the island, use your simulation to estimate the distribution of the population 1 year later. Provide a plot of this distribution. [5 marks]

(e) Use simulation to estimate the long term average number of female rats on the island. [5 marks]

3 [15 marks] Customers arrive at a fast food outlet as a Poisson process at rate 2 per minute. There are two assistants on duty and the time it takes an assistant to serve a customer is exponentially distributed with mean 30 seconds. Once served the customers leave. As the outlet is small and customers are are always in a rush, any customers arriving when there are already 4 customers in the outlet (including 2 being served) will leave immediately without waiting to be served.

(i) If a new customer arrives when both assistants are busy but nobody else is waiting, what is the expected time to be served? [3 marks]

(ii) Suppose that both assistants are busy and one other customer is waiting. What is the probability that the outlet empties of customers before any other new customers arrive? [3 marks]

(iii) Find the stationary distribution for the number of customers in the outlet. [3 marks]

(iv) Calculate the long term average queue length. [3 marks]

(v) Over a one hour period in steady state, how many customers are expected to leave immediately without being served? [3 marks]

4 [15 marks] Consider the continuous-time Markov chain with state space $S = \{A, B, C\}$ and transition intensity matrix

$$\mathbf{Q} = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 0 & 3 & -3 \end{pmatrix}.$$

- **a** Starting at state A, what is the probability that the next three transitions are $A \to B \to C \to B$? [3 marks]
- **b** Starting at state A, what is the probability that the chain returns to state A after three transitions? [3 marks]
- **c** Find the stationary distribution of the Markov chain. [3 marks]
- **d** When the chain is in steady state, what is the expected number of state transitions made during 100 units of time? [3 marks]
- **e** Starting in state A, what is the average time until leaving state A? If the chain has just left state A, what is the average time until it first returns to state A? [3 marks]