STATS 320 Assignment 3 Due: 2pm, 9th May, 2018

Question 1 is about material taught in the first half of the semester

Questions 2-4 are about the second half.

1 In chess, a rook can move either horizontally within its row (left or right) or vertically within its column (up or down), and can move any distance. On an 8×8 chess board, imagine a rook that starts at the top right hand corner at time 0. At each move, a bored child decides to move the rook to a random legal location. Assume a 50/50 chance of moving vertically or horizontally, and then a uniform distribution for where in that row/column it moves to, including the possibility of 'moving back' to the current square.

Let T be the first strictly positive time when the rook is back in the top right hand corner of the board again, that is, $T \in \{1, 2, 3, \dots\}$ is the first time the rook *returns* to the top right hand corner.

Implement a program in R to estimate $\mathbb{E}(T)$, $\mathrm{Var}(T)$, and $\mathrm{sd}(T)$.

Explaining your reasoning, can you describe the stationary distribution for the position of the rook on the chessboard?

2 Suppose the number of arrivals at a queue by time t, N_t, forms a Poisson process of rate λ. Let T_n be the time of the n-th arrival at the queue. Calculate and simplify the following:

(i) $\mathbb{P}(T_2 \leq t)$

(ii) $\mathbb{P}\big(N_3 = 2 \,\big|\, N_5 = 6\big)$

(iii) $\mathbb{E}\big(N_3\,(N_5 - N_3)\,\big)$

(iv) $\mathrm{Cov}\big(N_3, N_5\big)$

(v) $\mathbb{E}\big(T_6 | N_3 = 2\big)$

3 At an industrial complex, chemical alerts occur according to a Poisson process of rate $\mu > 0$ per day. Independently for each alert, it turns out to be a false alarm with probability $p \in (0, 1)$, otherwise it is an alert requiring evacuation with probability $1 - p$.

Explaining your reasoning, answer the following:

(i) What is the probability that no alerts occur on a given day?

(ii) Given that 5 alerts occurred on Monday and Tuesday of one particular week, what is the probability that 3 or more alerts occur on the Wednesday of the same week?

(iii) If there are 6 alerts during a given week, what is the probability less than 2 are evacuations?

(iv) An alert has just occurred, how long on average until the next evacuation alert?

(v) If there are 3 false alarms on Monday and Tuesday, what is the expected total number of alerts during from Monday to Friday?

4 There are 3 super-fast computers in one area of the science building that are accessible 24 hours a day. Suppose students arrive wishing to use one of the computers as a Poisson process of rate $\lambda > 0$ per minute. If all the computers are busy, students will patiently wait there turn in a queue. Once on the computer, each student will stay on the computer an independent exponentially distributed amount of time of rate $\mu > 0$ per minute before leaving.

(i) When exactly one of the computers is busy, what is the probability that the next event is an arrival of a new student?

(ii) When exactly two of the computers are busy, what is the probability that no students either arrive or depart in the next t minutes?

(iii) When all three of the computers are busy, what is the expected time to wait until the next departure of a student?

(iv) Let X_t represent the number of students waiting in the queue to use the computers at time t, including any currently using them. Draw a state transition diagram indicating the rates of moving between the various states of $X = (X_t)_{t \geq 0}$.

(v) Suppose that $\lambda = 5$ and $\mu = 1$. Explain intuitively what you would expect to happen to the queue over the long term.