The Poset of Mesh Patterns

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1 Introduction

Mesh patterns are a generalisation of permutations and have been studied extensively in recent years. A natural definition of when one mesh patterns occurs in another mesh patterns was given in [TU17]. Therefore, we can also generalise the classical permutation poset to a poset of mesh patterns, where $(\sigma, S) \leq (\pi, P)$ if there is an occurrence of (σ, S) in (π, P).

The poset of permutations has recieved a lot of attention in recent years, but due to its complicated structure a full understanding of the poset has proven elusive, see [Smi16, Smi17]. The poset of mesh patterns, which we define here, contains the poset of permutations as an induced subposet. By studying this poset we hope to see if the two posets posses a similar structure, so that a understanding of one may lead to results on the other.

There are many open questions relating to mesh patterns, for example two mesh patterns are *coincident* if their avoidance class is exactly the same. A full classification of such coincidences has proven difficult to obtain, see []. By looking at which mesh patterns are contained within each other, how the structure of the poset this induces, we hope that this may lead to a better understanding of mesh patterns and their properties, such as coincidences.

In Section 2 we introduce the poset of mesh patterns and related definitions, including a brief overview of poset topology. In Section 3 we prove some results on the Möbius function of the poset of mesh patterns. In Section 4 we give a characterisation of the non-pure intervals of the mesh pattern poset. In Section 5 we give some results on the topology of the mesh pattern poset

2 The Poset of Mesh Patterns

Mesh patterns were first introduced in [BC11] and a concept of mesh pattern containment in another mesh pattern was introduced in [TU17]. A mesh pattern is a generalisation of a permutation. Given a permutation $\pi = \pi_1\pi_2 \dots \pi_n$ we can plot π on a $n \times n$ grid, where we place a dot at coordinates (i, π_i), for all $1 \leq i \leq n$. A mesh pattern is then obtained by shading boxes of this grid, so a mesh pattern takes the from $p = (\mathrm{cl}(p), \mathrm{sh}(p))$, where

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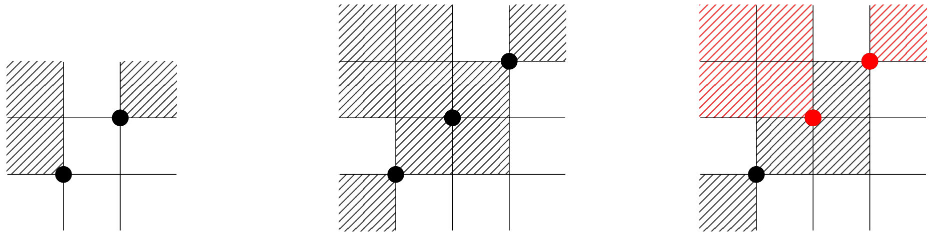
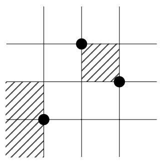


Figure 2.1: A pair of mesh patterns (a) and (b), with (c) showing an occurrence of (a) in (b). Also note (b) does not contain $12^{(0,0),(1,1),(2,2)}$.

$\mathrm{cl}(p)$ is a permutation and $\mathrm{sh}(p)$ is a set of coordinates recording the shaded boxes, which are indexed by their south west corner. For ease of notation we sometimes denote the mesh pattern $\mathrm{cl}(p)^{\mathrm{sh}(p)}$. For example the mesh pattern $(132, \{(0,0),(0,1),(2,2)\})$, or equivalently $132^{(0,0),(0,1),(2,2)}$, has the form:



To define when a mesh pattern occurs within another mesh pattern, first we need to introduce two other well known definitions of occurrence. A permutation σ occurs in a permutation π if there is a subsequence of π whose letters appear in the same relative order of size as the letters of σ

Consider a mesh pattern (σ, S) and an occurrence η of σ in π, in the classical permutation pattern sense. Each box (i,j) of (σ, S) corresponds to an area $R_\eta(i,j)$ in the plot of π, which is the area bounded by the points in π which in η correspond to the letters $\sigma_i, \sigma_{i+1}, j, j+1$ of σ. We say that η is an occurrence of the mesh pattern (σ, S) in the permutation π if there is no point in any of the areas $R_\eta(i,j)$ for any shaded box $(i,j) \in S$.

Using these definitions of occurrences we can define when an occurrence of one mesh patterns appears in another. An example of which is given in Figure 2.1.

Definition 2.1 ([TU17])**.** *An occurrence of a mesh pattern (σ, S) in another mesh pattern (π, P) is an occurrence η of (σ, S) in π, where for any $(i,j) \in S$ every box in $R_\eta(i,j)$ is shaded in (π, P).*

The classical permutation poset $\mathcal{P}$ is defined as the poset of all permutation, with $\sigma \leq_{\mathcal{P}} \pi$ if and only if σ occurs in π. Using Definition 2.1 we can similarly define the poset of mesh patterns $\mathcal{M}$ as the poset of all mesh patterns, with $m \leq_{\mathcal{M}} p$ if m occurs in p. Note we drop the subscripts from $\leq$ when it is clear which partial order is being considered. An *interval* $[\alpha, \beta]$ of a poset is defined as subposet induced by the set $\{\kappa \,|\, \alpha \leq \kappa \leq \beta\}$. See Figure 2.2 for an example of an interval of $\mathcal{M}$.

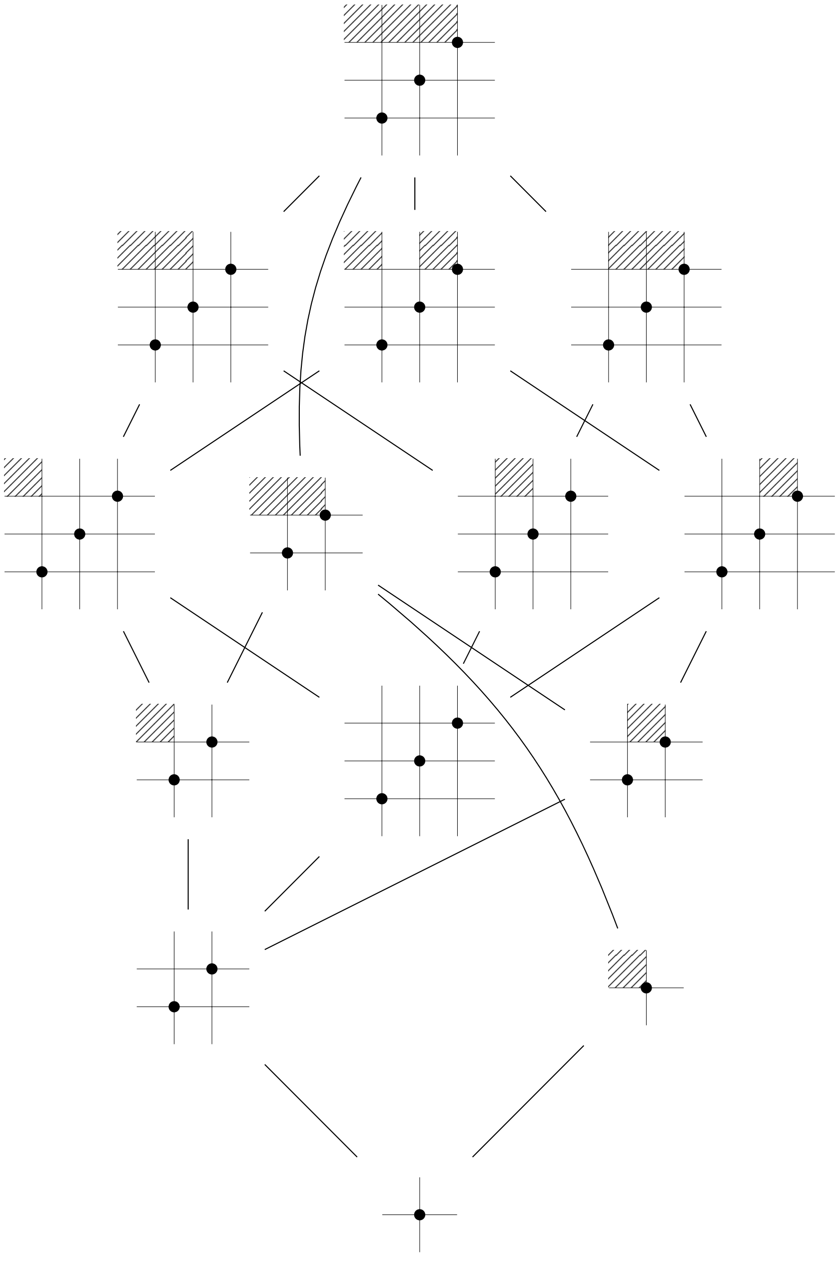


Figure 2.2: The interval $[1^{\emptyset}, 123^{(0,3),(1,3),(2,3)}]$ of $\mathcal{M}$.

We let $|\operatorname{cl}(p)|$ represent the length of $\operatorname{cl}(p)$ and $|\operatorname{sh}(p)|$ the size of $\operatorname{sh}(p)$, and define the *length* of p as $|\operatorname{cl}(p)|$. We define the *rank* of p as $|\operatorname{cl}(p)| + |\operatorname{sh}(p)|$, that is, the number of points plus the number of shaded boxes.

2.1 Poset Topology

Define:

- Möbius function
- Purity
- Shellability
- Quillen Fibre Lemma

The *Möbius function* on an interval $[\alpha, \beta]$ of a poset is defined by: $\mu(a, a) = 1$, for all a, $\mu(a, b) = 0$ if $a \not\leq b$, and

$$\mu(a, b) = -\sum_{c \in [a,b)} \mu(a, c).$$

We refer to $\mu(a, b)$ as the Möbius function of $[a, b]$. See Figure 3.1 for an example.

In a poset we say that α *covers* β, denoted $\alpha \gtrdot \beta$, if $\alpha > \beta$ and there is no κ such that $\alpha > \kappa > \beta$. A *chain* of length k in a poset is a totally ordered subset $c_1 < c_2 < \cdots < c_{k+1}$, and the chain is maximal if $c_i \lessdot c_{i+1}$, for all $1 \leq i \leq k$. A poset is *pure* (also known as *ranked* or *graded*) if all maximal chains have the same length. For example, the interval in Figure 2.2 is nonpure because there is one maximal chain of length 3 ( $\lessdot$  $\lessdot$  $\lessdot$ ) and all other maximal chains have length 5.

In the subsequent sections we give some results on the Möbius function and topology of the poset of mesh patterns.

3 Möbius Function

In this section we present some preliminary results on the Möbius function of the mesh pattern poset. First we consider the case that two mesh patterns have the same underlying permutation.

Lemma 3.1. *If $cl(m) = cl(p)$, then $[m, p]$ is isomorphic to the boolean lattice $B_{|sh(p)| - |sh(m)|}$, which implies $\mu(m, p) = (-1)^{|sh(p)| - |sh(m)|}$ and $[m, p]$ is shellable.*

Proof. We cannot remove any points from p, we can only unshade boxes, and we can unshaded any boxes from $\operatorname{sh}(p) \setminus \operatorname{sh}(s)$ in any order. $\square$

The simplest mesh patterns are those with no points, that is, the mesh patterns with a single box that is shaded or unshaded, which we denote $\epsilon^{\emptyset}$ and $\epsilon^{(0,0)}$, respectively.

Lemma 3.2. *Consider a mesh pattern p, then:*

$$\mu(\epsilon^A, p) = \begin{cases} 1, & \text{if } p = \epsilon^A \\ -1, & \text{if } A = \emptyset \ \& \ cl(p) + sh(p) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. The first two cases are trivial. The mesh pattern $\epsilon^{(0,0)}$ is not contained in any larger mesh patterns, so the Möbius function is always 0. If $\mathrm{rk}(p) > 1$, then $(\epsilon^{\emptyset}, p)$ contains a unique minimal element $1^{\emptyset}$, so $\mu(\epsilon^{\emptyset}, p) = 0$. □

If two mesh patterns have no shadings then we have an interval from the classical poset.

Lemma 3.3. *If $sh(s) = sh(p) = \emptyset$, then $[s, p]$ is isomorphic to the interval $[cl(s), cl(p)]$ in $\mathcal{P}$, so $\mu_{\mathcal{M}}(s, p) = \mu_{\mathcal{P}}(cl(s), cl(p))$.*

The Möbius function of the classical permutation poset is known to be unbounded [Smi14]. So we get the following corollary:

Corollary 3.4. *The Möbius function is unbounded on $\mathcal{M}$.*

We can also show that the Möbius function is unbounded if we include shadings.

Lemma 3.5. *Let m be a mesh pattern with exactly one descent, where the descent bottom is the letter 1, and all boxes south west of the point 1 are shaded, then*

$$\mu(21^{(0,0),(1,0)}, m) = \begin{cases} (-1)^{|m|} \lfloor \frac{n}{2} \rfloor, & \text{if } cl(m) \text{ has no adjacencies} \\ 1, & \text{if } cl(m) \text{ has exactly one letter before the descent in an adjacency tail and none after} \\ 0, & \text{otherwise} \end{cases}$$

Moreover, $[21^{(0,0),(1,0)}, m]$ is shellable.

Proof. Note that every mesh pattern in $I = [21^{(0,0),(1,0)}, m]$ has exactly one descent and everything SW of the point 1 shaded. Define a bijection f from I to the poset of words with subword order where the $i-1$'th letter of $f(p)$ is 0 if i is before the letter 1 in p and 1 if after the letter 1, for all $1 < i \leq n$. So $21^{(0,0),(1,0)}$ maps to the word 0. A mesh pattern in I is uniquely determined by the value of the points before the descent, because these values must appear first followed by the letter 1 and then the remaining letters. Therefore, it is straightforward to see this is a bijection and to check that it is order preserving. So I is isomorphic to an interval of the poset of words with subword order. It was shown in [Bjö90] that these intervals are shellable and the Möbius function equals the number of normal occurrences, where an occurrence is normal if in any run of equal elements every non-initial letter is part of the occurrence, which implies the result. □

We can also see that the Möbius function on $\mathcal{M}$ is not bounded by the classical poset, that is, it is not true that $\mu_{\mathcal{M}}(m, p) \leq \mu_{\mathcal{P}}(\mathrm{cl}(m), \mathrm{cl}(p))$. A simple counterexample is the interval $[1^{(0,1)}, 123^{(0,2),(0,3),(1,2),(1,3)}]$, this has Möbius function 1, however $\mu_{\mathcal{P}}(1, 123) = 0$, see Figure 3.1.

If we consider intervals where the bottom mesh pattern has no shadings, then we get the following result:

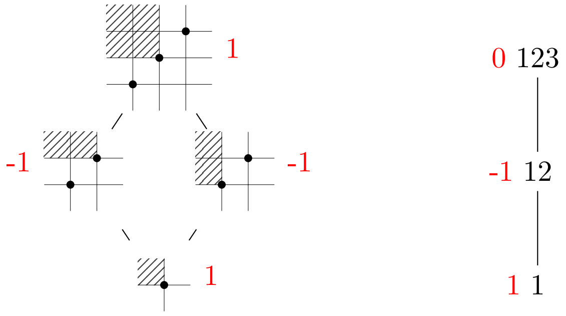


Figure 3.1: The interval $[1^{(0,1)}, 123^{(0,2),(0,3),(1,2),(1,3)}]$ (left) in $\mathcal{M}$ and $[1, 123]$ (right) in $\mathcal{P}$, with the Möbius function in red.

Lemma 3.6. *Consider the interval $[m, p]$ in $\mathcal{M}$. If $sh(m) = \emptyset \neq sh(p)$ and there is no $s \in (m, p)$ with $cl(s) = cl(m)$, then $\mu(m, p) = 0$.*

Proof. Define a map $f(x) = \mathrm{cl}(x)^{\emptyset}$, for any $x \in (m, p)$, and let $A := f((m, p))$, then $A = (\mathrm{cl}(m), \mathrm{cl}(p)]$. So A is contractible, because it has the unique maximal element $\mathrm{cl}(p)^{\emptyset}$. Moreover, for any $y \in A$ $f^{-1}(A_{\geq y})$ equals $[y, p)$ which is contractible. Therefore, (m, p) is homotopically equivalent to A using the Quillen Fiber Lemma, which implies $\mu(m, p) = 0$. $\square$

We can combine Lemma 3.6 with the following result to see that the Möbius function is almost always zero on the interval $[1^{\emptyset}, p]$.

Lemma 3.7. *As n tends to infinity the proportion of permutations of length n that contain one of $\{1^{(0,0)}, 1^{(1,0)}, 1^{(0,1)}, 1^{(1,1)}\}$ approaches* 0.

Proof. Let $P(n, i)$ be the probability that the letter i is an occurrence of $1^{(0,0)}$ in a length n mesh pattern. And let $P(n)$ be the probability that a length n mesh pattern contains $1^{(0,0)}$.

The probability that i is an occurrence of $1^{(0,0)}$ is given by selecting the location k of i, each has probability $\frac{1}{n}$, and then we require that all boxes south west of i are filled, of which there are 2^{ik}. Note that this over estimates the probability, because it is possible that there is a point south west of i, which would imply i is not an occurrence of $1^{(0,0)}$, however this argument still counts them. We can formulate this as:

$$P(n, i) \leq \sum_{k=1}^{n+1-i} \frac{1}{n} \left(\frac{1}{2^i}\right)^k = \frac{1}{n} \left(\frac{2^{-i(n+2-i)}}{2^{-i} - 1} - 1\right) = \frac{1}{n2^i} \left(\frac{1 - 2^{-i(n+1-i)}}{1 - 2^{-i}}\right) \leq \frac{2}{n2^i}$$

To compute the probability that a length n permutation contains $1^{(0,0)}$ we can sum over all letters i and test if i is an occurrence of $1^{(0,0)}$. Note again this is an over estimate because if a permutation contains multiple occurrences of $1^{(0,0)}$ it counts that permutation more than once.

$$P(n) \leq \sum_{i=1}^{n} P(n, i) \leq \sum_{i=1}^{n} \frac{2}{n2^i} = \frac{2}{n} \left(\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} - 1\right) \leq \frac{2}{n}$$

We can repeat this calculation for the other three one shadings of 1 so we get that $P(n) \leq \frac{8}{n} \to 0$. $\square$

Corollary 3.8. *As n tends to infinity the proportion of mesh patterns p of length n such that $\mu(1^{\emptyset}, p) = 0$ approaches 1.*

In the classical case it is true that given a permutation σ the probability a permutation of length n contains σ tends to 1 as n tends to infinity, this follows from the Marcus-Tardos Theorem [MT04]. By the above result we can see the same is not true in the mesh pattern case. In fact we conjecture the opposite is true:

Conjecture 3.9. *Given a mesh pattern m, the probability that a random mesh pattern of length n contains m tends to 0 as n tends to infinity.*

4 Purity

Interestingly the mesh pattern poset is not a pure poset, that is, not every maximal chain has the same length. First we consider the length of the maximal chain in any interval $[1^{\emptyset}, m]$, that is, the rank of $[1^{\emptyset}, m]$.

Lemma 4.1. *For any mesh pattern m, we have $rk(1^{\emptyset}, m) = |\, cl(m)| + |\, sh(m)|$.*

Proof. We can create a chain from m to $1^{\emptyset}$ by deshading all boxes, in any order, and then deleting all but one point, in any order, this is a chain of length $|\operatorname{cl}(m)| + |\operatorname{sh}(m)|$. Moreover, to create a smaller element at least one shading or point must be removed, so we cannot create a chain of length greater than $|\operatorname{cl}(m)| + |\operatorname{sh}(m)|$. $\square$

So we define the rank of a mesh pattern as $\operatorname{rk}(m) = |\operatorname{cl}(m)| + |\operatorname{sh}(m)|$ and we say an edge $m \lessdot p$ is *impure* if $\operatorname{rk}(p) - \operatorname{rk}(m) > 1$, this is equivalent to the edge being between different ranked elements in any interval $[1^{\emptyset}, x]$, with $x \geq p$. We begin with a classification of impure edges.

Let $m - x$ be the mesh pattern obtained by deleting the point x and let $m \setminus x$ be the occurrence of $m - x$ in m that does not use the point x. We say that deleting a point x *merges shadings* if there is a shaded box in $m - x$ that corresponds to more than one shaded box in $m \setminus x$.

Lemma 4.2. *An edge between $m \lessdot p$ is impure if and only if all occurrences of m in p use all shaded boxes of p and are obtained by deleting a point that merges shading.*

Proof. First we show the backwards direction. Because m is obtained by deleting a point that merges shadings, m must have one less point and at least one less shading so $\operatorname{rk}(p) - \operatorname{rk}(m) \geq 2$. So it suffices to show that there is no z such that $m < z < p$. Suppose such a z exists, then if z is obtained by deshading a box in p it can no longer contain m because all occurrences of m in p use all shaded areas of p. If z is obtained by deleting a point, then that cannot remove shadings, only merge shadings, otherwise it wouldn't contain m, and it implies $cl(m) = cl(z)$.

Moreover, if $m < z$ then we can deshade some boxes of z to get m which implies there is an occurrence of m in p that doesn't use all the shaded boxes of p.

Now consider the forwards direction, so suppose $m \lessdot p$ is impure. So $\mathrm{rk}(p) - \mathrm{rk}(m) \geq 2$, which implies m is obtained by deleting a single point which merges shadings but does not delete shadings, because any other combination of deleting points and deshading can be done in successive steps. Furthermore, this must be true for any point that can be deleted to get m, that is, for all occurrences of m in p. Moreover, if there is an occurrence that doesn't use all the shaded boxes of p, we can deshade the box it doesn't use and get an element that lies between m and p. □

Lemma 4.3. *If there is an impure edge in* $[1^{\emptyset}, m]$*, then there is an impure edge* $a \lessdot b$ *where* $cl(m) = cl(b)$*.*

Proof. If $x \lessdot y$ is an impure edge in $[1^{\emptyset}, m]$, then let b be a mesh pattern obtained by adding points to y so $cl(b) = cl(m)$. Pick an occurrence of x in y and add the points to x in the positions induced by how they are added to y and the occurrence, call this a. The points added will not have any shadings in the four boxes touching it, therefore no point touching a shading in a can embed in a new point of b. Moreover, the set of embeddings of a in b is a subset of x in y, after adding the new points to each. These two conditions imply that every embedding of x in y uses all the shadings of y, this is also true for every embedding of a in b. Therefore, the result follows by Lemma 4.2. □

Proposition 4.4. *Consider a mesh pattern m. The interval $[1^{\emptyset}, m]$ is non-pure if and only if there exists a point p in m such that $m - p$ merges shadings and there is no other occurrence of $m - p$ in m with a subset of shadings of $m \setminus p$.*

Proof. First we show the backwards direction. Let x be the mesh pattern obtained by inserting p back into $m - p$, and η the corresponding embedding of $m - p$ in x. Note that it is not always true that $x = m$ because some shaded shadings of m are lost when deleting p. We claim that $m - p \lessdot x$ is an impure edge. This follows by Lemma 4.2 because η uses all the shaded boxes in x and there is no subshading occurrence.

To see the other direction suppose there is an impure edge in $[\omega, m]$. By Lemma 4.3 there is an impure edge $a \lessdot b$ where $cl(b) = cl(m)$. If m is impure then it must remove both a point and a shading, so it must merge shadings by deleting some point p and there is no element between them so there can be no subshading of b that contains a. □

Corollary 4.5. *There is an impure edge in the interval $[m, p]$ if and only if there exists a point x in p such that $p - x$ merges shadings and there is no other occurrence of $p - x$ in p with a subset of shadings of $p \setminus x$, and $p - x \geq m$.*

Note that containing an impure edge in $[m, p]$ does not necessarily imply that $[m, p]$ is non-pure. For example, if $[m, p]$ contains only one edge and that edge is impure, then $[m, p]$ is still pure. Although it is also possible to have a pure poset that contains impure and pure edges, see Figure 4.1.

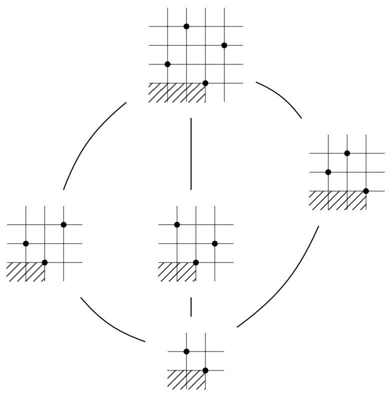


Figure 4.1: The interval $[21^{(0,0),(1,0)}, 2413^{(0,0),(1,0),(2,0)}]$, which is pure but contains both pure and impure edges.

5 Topology

The connectivity of the interval $[\mathrm{cl}(m), \mathrm{cl}(p)]$ in $\mathcal{P}$ does not necessarily imply the same property for $[m, p]$ in $\mathcal{M}$. For example, the interval $[123, 456123]$ is disconnected in $\mathcal{P}$ but the interval

$$[a, b] = [123^{(3,0),(3,1),(3,2)}, 456123^{(6,0),(6,1),(6,2)}],$$

see Figure 5.1a, is a chain in $\mathcal{M}$, so is connected. Furthermore, the interval $[321, 521643]$ is connected in $\mathcal{P}$ but the interval

$$[x, y] = [321^{(1,3)}, 521643^{(1,5),(1,6),(4,6)}],$$

see Figure 5.1b, is disconnected in $\mathcal{M}$. This also implies that if $[\mathrm{cl}(m), \mathrm{cl}(p)]$ is (non-)shellable in $\mathcal{P}$ then it is not true that $[m, p]$ has the same property in $\mathcal{M}$. For example, $[123, 456123]$ is not shellable but $[a, b]$ is shellable, and $[321, 521643]$ is shellable but $[x, y]$ is not shellable.

(a) $123^{(3,0),(3,1),(3,2)}$ and $456123^{(6,0),(6,1),(6,2)}$ (b) $321^{(1,3)}$ and $521643^{(1,5),(1,6),(4,6)}$

We can define a direct sum operation on mesh patterns, where given two mesh patterns s and t, the top right corner of s and bottom left corner of t are not shaded. The direct sum $s \oplus t$ has classical pattern $\mathrm{cl}(s) \oplus \mathrm{cl}(t)$ and the shadings are given by placing t north east of s and shading any borders so they extend to the edge. We can similarly define the skew-sum.

Lemma 5.1. *If m is indecomposable and $(0, 0), (|m|, |m|) \notin sh(m)$, then $[m, m \oplus m]$ is disconnected.*

Proof. There are exactly two occurrences of m in $m \oplus m$, the first $|m|$ letters η_1 or the last $|m|$ letters η_2. If you delete a point or deshade any box that is not in η_1 then that point or box must be part of η_2, so the resulting mesh pattern only contains one occurrence of m which corresponds to η_1 so then you can only delete points or shadings that are not part of η_1 so must be part of η_2. A similar argument applies if you initial delete a point or shading not part of η_2. Therefore, any two points/shading removals must both be part of η_1 or η_2, thus the poset can be split into components where on one side we remove elements not in η_1 and the other elements not in η_2. □

Corollary 5.2. *If m is skew-indecomposable and $(|m|, 0), (0, |m|) \notin sh(m)$, then $[m, m \ominus m]$ is disconnected.*

An analogous result is used in the classical case in [MS15] to show that almost all intervals of the classical permutation poset are not shellable. The proof of this follows from the Marcus-Tardos theorem, we have seen this result does not apply in the mesh pattern case so we cannot prove a similar result using this technique. A similar problem was studied for boxed mesh patterns in permutations in [AKV13], which is equivalent to boxed mesh patterns in fully shaded mesh patterns. So we leave it as an open question:

Question 5.3. *What proportion of intervals of $\mathcal{M}$ are shellable?*

6 Open Questions

Conjecture 6.1. *Every interval of $\mathcal{M}$ is unimodal.*

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