

The Poset of Mesh Patterns

Jason P Smith*

June 8, 2017

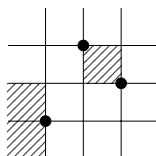
1 Introduction

Mesh patterns are a generalisation of permutations and have been studied extensively in recent years. A natural definition of when one mesh pattern occurs in another mesh pattern was given in [TU17]. Therefore, we can also generalise the classical permutation poset to a poset of mesh patterns, where $(\sigma, S) \leq (\pi, P)$ if there is an occurrence of (σ, S) in (π, P) .

To define when a mesh pattern occurs within another mesh pattern, first we need to recall when a mesh pattern (σ, S) occurs in a permutation π . Given any occurrence η of σ in π , in the classical permutation pattern sense, then each box (i, j) of (σ, S) corresponds to an area $R_\eta(i, j)$ in the plot of π . We say that η is an occurrence of the mesh pattern (σ, S) in the permutation π if there is no point in any of the areas $R_\eta(i, j)$ for any shaded box $(i, j) \in S$. We define an occurrence of a mesh pattern (σ, S) in another mesh pattern (π, P) as an occurrence η of (σ, S) in π where if (i, j) is shaded in (σ, S) , then every box in $R_\eta(i, j)$ is shaded in (π, P) .

2 The Poset of Mesh Patterns

Mesh patterns were first introduced in [BC11] and a concept of mesh pattern containment in another mesh pattern was introduced in [TU17]. A mesh pattern is of the form $p = (\text{cl}(p), \text{sh}(p))$, where $\text{cl}(p)$ is a permutation and $\text{sh}(p)$ is a set of coordinates corresponding to the boxes that are shaded, where (i, j) denotes the box $[i, i+1] \times [j, j+1]$. We also denote p by $\text{cl}(p)^{\text{sh}(p)}$, for example 132^\emptyset is the mesh pattern on 132 with no shaded boxes. For example the mesh pattern $(132, \{(0, 0), (0, 1), (2, 2)\})$, or equivalently $132^{(0,0),(0,1),(2,2)}$, has the form:



*This research was supported by the EPSRC Grant EP/M027147/1

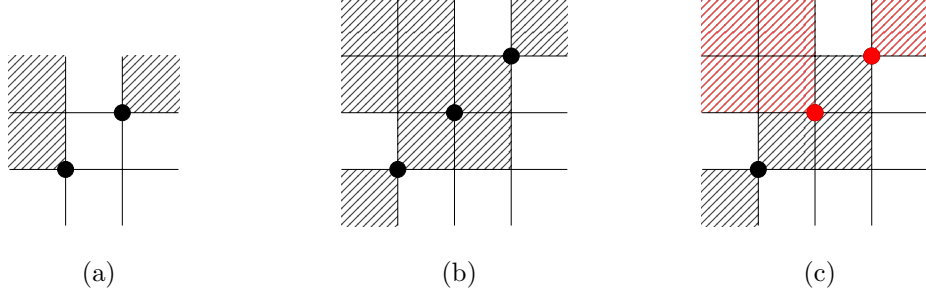


Figure 2.1: A pair of mesh patterns (a) and (b), with (c) showing an occurrence of (a) in (b). Also note (b) does not contain $12^{(0,0),(1,1),(2,2)}$.

We let $|\text{cl}(p)|$ represent the length of $\text{cl}(p)$ and $|\text{sh}(p)|$ the size of $\text{sh}(p)$, and define the length of p as $|\text{cl}(p)|$. We define the *rank* of p as $|\pi| + |P|$, that is, the number of points plus the number of shaded boxes. Let $G(p)$ be the set of coordinates of the points of p .

Given two classical permutations σ and π , an occurrence of σ in π is a map $\alpha : [|\sigma|] \rightarrow [|\pi|]$ such that $\alpha(1)\alpha(2)\dots\alpha(|\sigma|)$ has the same relative order of size as σ . Given any letter $b = \sigma_i$ let $\hat{\alpha}(b) = \pi_{\alpha(i)}$, so $\hat{\alpha}$ maps the value of a letter in σ to its value in π under the map α .

Using the notion of pattern occurrence we can define a partial order on permutations where $\sigma \leq_{\mathcal{P}} \pi$ if there is an occurrence of σ in π . Let \mathcal{P} be the poset of all permutations with the partial order $\leq_{\mathcal{P}}$, we call this the *classical permutation poset*.

We can extend this definition to define an occurrence in a mesh pattern. Consider a pair of mesh patterns m and p and an occurrence α of $\text{cl}(m)$ in $\text{cl}(p)$. Define $R_{i,j}^{\alpha} = [\alpha(i), \alpha(i+1)] \times [\hat{\alpha}(j), \hat{\alpha}(j+1)]$, this is the area in p that corresponds to the box (i, j) in m according to α .

An *occurrence* of m in p is an occurrence α of $\text{cl}(m)$ in $\text{cl}(p)$ satisfying $\bigcup_{(i,j) \in \text{sh}(m)} R_{i,j}^{\alpha} \subseteq \text{sh}(p)$ and $G(p) \cap \bigcup_{(i,j) \in \text{sh}(m)} R_{i,j}^{\alpha} = \emptyset$. This is equivalent to saying every shaded box in m corresponds to a fully shaded area in p that contains no points.

2.1 Poset Topology

Define:

- Möbius function
- Purity
- Shellability
- Quillen Fibre Lemma

Given any poset a natural question to ask is how does its Möbius function behave? The Möbius function on any interval $[\alpha, \beta]$ is defined in the following way: $\mu(a, a) = 1$, for all a , $\mu(a, b) = 0$ if $a \not\leq b$, and

$$\mu(a, b) = - \sum_{c \in [a, b)} \mu(a, c).$$

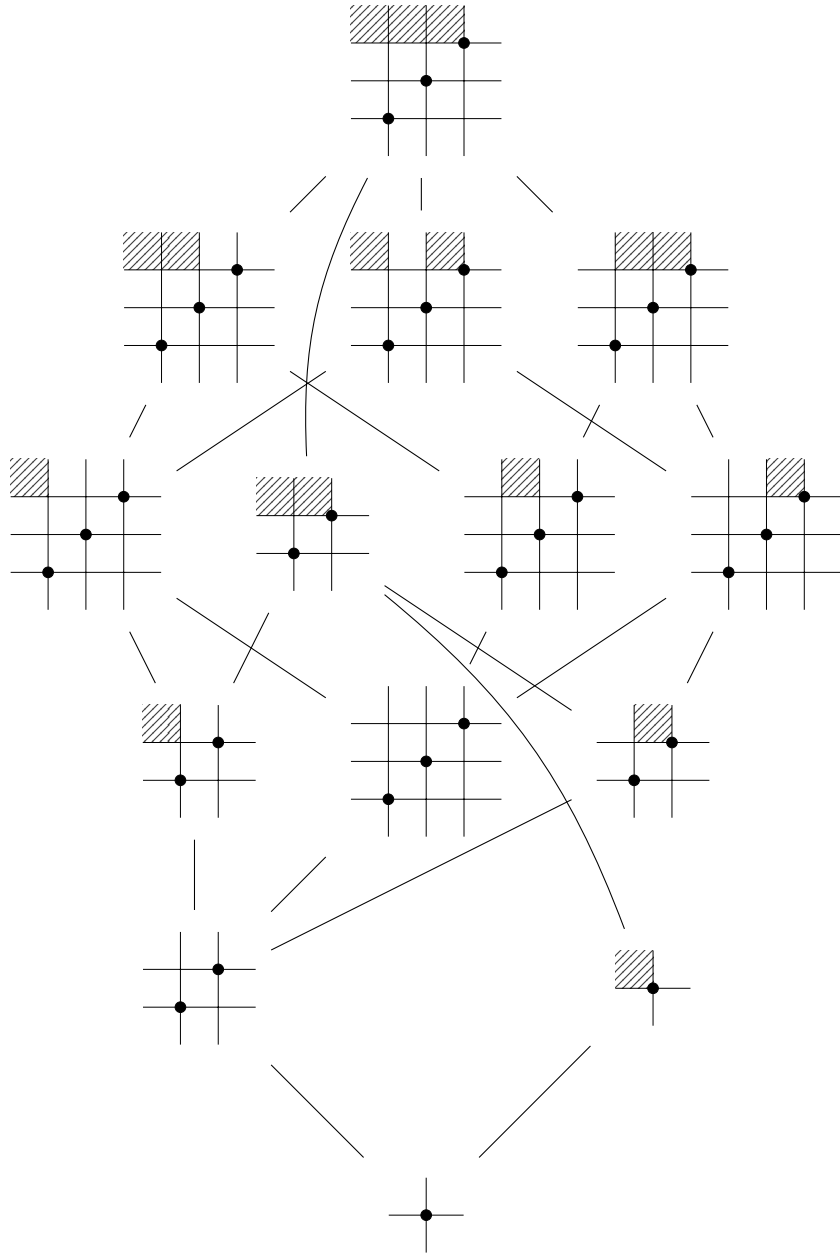


Figure 2.2: The interval $[1^{\emptyset}, 123^{(0,3),(1,3),(2,3)}]$ of \mathcal{M} .

We refer to $\mu(a, b)$ as the Möbius function of $[a, b]$. See Figure 3.1 for an example.

3 Möbius Function

In this section we present some preliminary results on the Möbius function of the mesh pattern poset. First we consider the case that two mesh patterns have the same underlying permutation.

Lemma 3.1. *If $cl(m) = cl(p)$, then $[m, p]$ is isomorphic to the boolean lattice $B_{|sh(p)| - |sh(m)|}$, which implies $\mu(m, p) = (-1)^{|sh(p)| - |sh(m)|}$ and $[m, p]$ is shellable.*

Proof. We cannot remove any points from p , we can only unshade boxes, and we can unshade any boxes from $sh(p) \setminus sh(s)$ in any order. \square

The simplest mesh patterns are those with no points, that is, the mesh patterns with a single box that is shaded or unshaded, which we denote ϵ^\emptyset and $\epsilon^{(0,0)}$, respectively.

Lemma 3.2. *Consider a mesh pattern p , then:*

$$\mu(\epsilon^A, p) = \begin{cases} 1, & \text{if } p = \epsilon^A \\ -1, & \text{if } A = \emptyset \text{ \& } cl(p) + sh(p) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. The first two cases are trivial. The mesh pattern $\epsilon^{(0,0)}$ is not contained in any larger mesh patterns, so the Möbius function is always 0. If $rk(p) > 1$, then (ϵ^\emptyset, p) contains a unique minimal element 1^\emptyset , so $\mu(\epsilon^\emptyset, p) = 0$. \square

If two mesh patterns have no shadings then we have an interval from the classical poset.

Lemma 3.3. *If $sh(s) = sh(p) = \emptyset$, then $[s, p]$ is isomorphic to the interval $[cl(s), cl(p)]$ in \mathcal{P} , so $\mu_{\mathcal{M}}(s, p) = \mu_{\mathcal{P}}(cl(s), cl(p))$.*

The Möbius function of the classical permutation poset is known to be unbounded [Smi14]. So we get the following corollary:

Corollary 3.4. *The Möbius function is unbounded on \mathcal{M} .*

We can also show that the Möbius function is unbounded if we include shadings.

Lemma 3.5. *Let m be a mesh pattern with exactly one descent, where the descent bottom is the letter 1, and all boxes south west of the point 1 are shaded, then*

$$\mu(21^{(0,0),(1,0)}, m) = \begin{cases} (-1)^{|m| \lfloor \frac{n}{2} \rfloor}, & \text{if } cl(m) \text{ has no adjacencies} \\ 1, & \text{if } cl(m) \text{ has exactly one letter before the descent} \\ & \text{in an adjacency tail and none after} \\ 0, & \text{otherwise} \end{cases}$$

Moreover, $[21^{(0,0),(1,0)}, m]$ is shellable.

Proof. Let $P(n, i)$ be the probability that the letter i is an occurrence of $1^{(0,0)}$ in a length n mesh pattern. And let $P(n)$ be the probability that a length n mesh pattern contains $1^{(0,0)}$.

The probability that i is an occurrence of $1^{(0,0)}$ is given by selecting the location k of i , each has probability $\frac{1}{n}$, and then we require that all boxes south west of i are filled, of which there are 2^{ik} . Note that this over estimates the probability, because it is possible that there is a point south west of i , which would imply i is not an occurrence of $1^{(0,0)}$, however this argument still counts them. We can formulate this as:

$$P(n, i) \leq \sum_{k=1}^{n+1-i} \frac{1}{n} \left(\frac{1}{2^i} \right)^k = \frac{1}{n} \left(\frac{2^{-i(n+2-i)}}{2^{-i} - 1} - 1 \right) = \frac{1}{n2^i} \left(\frac{1 - 2^{-i(n+1-i)}}{1 - 2^{-i}} \right) \leq \frac{2}{n2^i}$$

To compute the probability that a length n permutation contains $1^{(0,0)}$ we can sum over all letters i and test if i is an occurrence of $1^{(0,0)}$. Note again this is an over estimate because if a permutation contains multiple occurrences of $1^{(0,0)}$ it counts that permutation more than once.

$$P(n) \leq \sum_{i=1}^n P(n, i) \leq \sum_{i=1}^n \frac{2}{n2^i} = \frac{2}{n} \left(\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} - 1 \right) \leq \frac{2}{n}$$

We can repeat this calculation for the other three one shadings of 1 so we get that $P(n) \leq \frac{8}{n} \rightarrow 0$. \square

Corollary 3.8. *As n tends to infinity the proportion of mesh patterns p of length n such that $\mu(1^\emptyset, p) = 0$ approaches 1.*

In the classical case it is true that given a permutation σ the probability a permutation of length n contains σ tends to 1 as n tends to infinity, this follows from the Marcus-Tardos Theorem [MT04]. By the above result we can see the same is not true in the mesh pattern case. In fact we conjecture the opposite is true:

Conjecture 3.9. *Given a mesh pattern m , the probability that a random mesh pattern of length n contains m tends to 0 as n tends to infinity.*

4 Purity

Interestingly the mesh pattern poset is not a pure poset, that is, not every maximal chain has the same length. First we consider the length of the maximal chain in any interval $[1^\emptyset, m]$, that is, the rank of $[1^\emptyset, m]$.

Lemma 4.1. *For any mesh pattern m , we have $rk(1^\emptyset, m) = |cl(m)| + |sh(m)|$.*

Proof. We can create a chain from m to 1^\emptyset by deshading all boxes, in any order, and then deleting all but one point, in any order, this is a chain of length $|cl(m)| + |sh(m)|$. Moreover, to create a smaller element at least one shading or point must be removed, so we cannot create a chain of length greater than $|cl(m)| + |sh(m)|$. \square

So we define the rank of a mesh pattern as $\text{rk}(m) = |\text{cl}(m)| + |\text{sh}(m)|$ and we say an edge $m \leq p$ is *impure* if $\text{rk}(p) - \text{rk}(m) > 1$, this is equivalent to the edge being between different ranked elements in any interval $[1^\emptyset, x]$, with $x \geq p$. We begin with a classification of impure edges.

Let $m - x$ be the mesh pattern obtained by deleting the point x and let $m \setminus x$ be the occurrence of $m - x$ in m that does not use the point x . We say that deleting a point x *merges shadings* if there is a shaded box in $m - x$ that corresponds to more than one shaded box in $m \setminus x$.

Lemma 4.2. *An edge between $m \leq p$ is impure if and only if all occurrences of m in p use all shaded boxes of p and are obtained by deleting a point that merges shading.*

Proof. First we show the backwards direction. Because m is obtained by deleting a point that merges shadings, m must have one less point and at least one less shading so $\text{rk}(p) - \text{rk}(m) \geq 2$. So it suffices to show that there is no z such that $m < z < p$. Suppose such a z exists, then if z is obtained by deshading a box in p it can no longer contain m because all occurrences of m in p use all shaded areas of p . If z is obtained by deleting a point, then that cannot remove shadings, only merge shadings, otherwise it wouldn't contain m , and it implies $\text{cl}(m) = \text{cl}(z)$. Moreover, if $m < z$ then we can deshade some boxes of z to get m which implies there is an occurrence of m in p that doesn't use all the shaded boxes of p .

Now consider the forwards direction, so suppose $m \leq p$ is impure. So $\text{rk}(p) - \text{rk}(m) \geq 2$, which implies m is obtained by deleting a single point which merges shadings but does not delete shadings, because any other combination of deleting points and deshading can be done in successive steps. Furthermore, this must be true for any point that can be deleted to get m , that is, for all occurrences of m in p . Moreover, if there is an occurrence that doesn't use all the shaded boxes of p , we can deshade the box it doesn't use and get an element that lies between m and p . \square

Lemma 4.3. *If there is an impure edge in $[1^\emptyset, m]$, then there is an impure edge $a \leq b$ where $\text{cl}(m) = \text{cl}(b)$.*

Proof. If $x \leq y$ is an impure edge in $[1^\emptyset, m]$, then let b be a mesh pattern obtained by adding points to y so $\text{cl}(b) = \text{cl}(m)$. Pick an occurrence of x in y and add the points to x in the positions induced by how they are added to y and the occurrence, call this a . The points added will not have any shadings in the four boxes touching it, therefore no point touching a shading in a can embed in a new point of b . Moreover, the set of embeddings of a in b is a subset of x in y , after adding the new points to each. These two conditions imply that every embedding of x in y uses all the shadings of y , this is also true for every embedding of a in b . Therefore, the result follows by Lemma 4.2. \square

Proposition 4.4. *Consider a mesh pattern m . The interval $[1^\emptyset, m]$ is non-pure if and only if there exists a point p in m such that $m - p$ merges shadings and there is no other occurrence of $m - p$ in m with a subset of shadings of $m \setminus p$.*

Proof. First we show the backwards direction. Let x be the mesh pattern obtained by inserting p back into $m - p$, and η the corresponding embedding of $m - p$ in x . Note that it

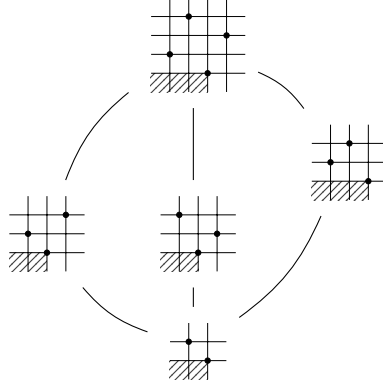


Figure 4.1: The interval $[21^{(0,0),(1,0)}, 2413^{(0,0),(1,0),(2,0)}]$, which is pure but contains both pure and impure edges.

is not always true that $x = m$ because some shaded shadings of m are lost when deleting p . We claim that $m - p \leq x$ is an impure edge. This follows by Lemma 4.2 because η uses all the shaded boxes in x and there is no subshading occurrence.

To see the other direction suppose there is an impure edge in $[\omega, m]$. By Lemma 4.3 there is an impure edge $a \leq b$ where $cl(b) = cl(m)$. If m is impure then it must remove both a point and a shading, so it must merge shadings by deleting some point p and there is no element between them so there can be no subshading of b that contains a . \square

Corollary 4.5. *There is an impure edge in the interval $[m, p]$ if and only if there exists a point x in p such that $p - x$ merges shadings and there is no other occurrence of $p - x$ in p with a subset of shadings of $p \setminus x$, and $p - x \geq m$.*

Note that containing an impure edge in $[m, p]$ does not necessarily imply that $[m, p]$ is non-pure. For example, if $[m, p]$ contains only one edge and that edge is impure, then $[m, p]$ is still pure. Although it is also possible to have a pure poset that contains impure and pure edges, see Figure 4.1.

5 Topology

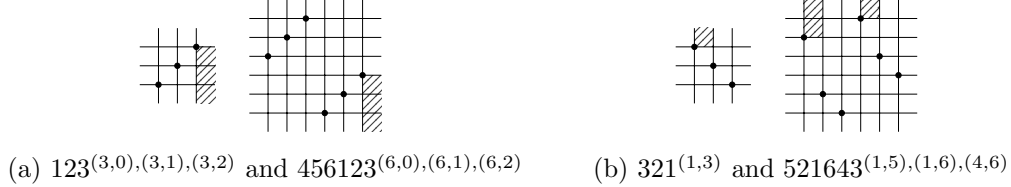
The connectivity of the interval $[cl(m), cl(p)]$ in \mathcal{P} does not necessarily imply the same property for $[m, p]$ in \mathcal{M} . For example, the interval $[123, 456123]$ is disconnected in \mathcal{P} but the interval

$$[a, b] = [123^{(3,0),(3,1),(3,2)}, 456123^{(6,0),(6,1),(6,2)}],$$

see Figure 5.1a, is a chain in \mathcal{M} , so is connected. Furthermore, the interval $[321, 521643]$ is connected in \mathcal{P} but the interval

$$[x, y] = [321^{(1,3)}, 521643^{(1,5),(1,6),(4,6)}],$$

see Figure 5.1b, is disconnected in \mathcal{M} . This also implies that if $[\text{cl}(m), \text{cl}(p)]$ is (non-)shellable in \mathcal{P} then it is not true that $[m, p]$ has the same property in \mathcal{M} . For example, $[123, 456123]$ is not shellable but $[a, b]$ is shellable, and $[321, 521643]$ is shellable but $[x, y]$ is not shellable.



We can define a direct sum operation on mesh patterns, where given two mesh patterns s and t , the top right corner of s and bottom left corner of t are not shaded. The direct sum $s \oplus t$ has classical pattern $\text{cl}(s) \oplus \text{cl}(t)$ and the shadings are given by placing t north east of s and shading any borders so they extend to the edge. We can similarly define the skew-sum.

Lemma 5.1. *If m is indecomposable and $(0, 0), (|m|, |m|) \notin \text{sh}(m)$, then $[m, m \oplus m]$ is disconnected.*

Proof. There are exactly two occurrences of m in $m \oplus m$, the first $|m|$ letters η_1 or the last $|m|$ letters η_2 . If you delete a point or deshade any box that is not in η_1 then that point or box must be part of η_2 , so the resulting mesh pattern only contains one occurrence of m which corresponds to η_1 so then you can only delete points or shadings that are not part of η_1 so must be part of η_2 . A similar argument applies if you initial delete a point or shading not part of η_2 . Therefore, any two points/shading removals must both be part of η_1 or η_2 , thus the poset can be split into components where on one side we remove elements not in η_1 and the other elements not in η_2 . \square

Corollary 5.2. *If m is skew-indecomposable and $(|m|, 0), (0, |m|) \notin \text{sh}(m)$, then $[m, m \ominus m]$ is disconnected.*

An analogous result is used in the classical case in [MS15] to show that almost all intervals of the classical permutation poset are not shellable. The proof of this follows from the Marcus-Tardos theorem, we have seen this result does not apply in the mesh pattern case so we cannot prove a similar result using this technique. A similar problem was studied for boxed mesh patterns in permutations in [AKV13], which is equivalent to boxed mesh patterns in fully shaded mesh patterns. So we leave it as an open question:

Question 5.3. *What proportion of intervals of \mathcal{M} are shellable?*

References

- [AKV13] Sergey Avgustinovich, Sergey Kitaev, and Alexandr Valyuzhenich. Avoidance of boxed mesh patterns on permutations. *Discrete Applied Mathematics*, 161(12):43 – 51, 2013.

- [BC11] Petter Brändén and Anders Claesson. Mesh patterns and the expansion of permutation statistics as sums of permutation patterns. *Electron. J. Combin.*, 18(2):P5, 2011.
- [Bjö90] Anders Björner. The Möbius function of subword order. *Institute for Mathematics and its Applications*, 19:118, 1990.
- [MS15] Peter R. W. McNamara and Einar Steingrímsson. On the topology of the permutation pattern poset. *Journal of Combinatorial Theory, Series A*, 134:1–35, 2015.
- [MT04] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the stanley–wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.
- [Smi14] Jason P. Smith. On the Möbius function of permutations with one descent. *The Electronic Journal of Combinatorics*, 21:2.11, 2014.
- [TU17] Murray Tannock and Henning Ulfarsson. Equivalence classes of mesh patterns with a dominating pattern. *arXiv preprint arXiv:1704.07104*, 2017.