# The Poset of Mesh Patterns

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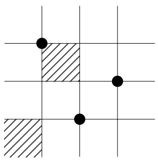
## 1 Introduction

Mesh patterns where first introduced by Brändén and Claesson in [BC11] as a generalisation of permutation patterns, and have been studied extensively in recent years, see e.g., [CTU15, JKR15]. A mesh pattern consist of a pair $(\pi, P)$, where $\pi$ is a permutation and $P$ is a set of coordinates in an $n \times n$ grid. For example, $(312, \{(0,0),(1,2)\})$ is a mesh pattern, which we depict by



A natural definition of when one mesh patterns occurs in another mesh patterns was given in [TU17], which we present in Section 2. This allows us to generalise the classical permutation poset to a poset of mesh patterns, where $(\sigma, S) \leq (\pi, P)$ if there is an occurrence of $(\sigma, S)$ in $(\pi, P)$. The permutation poset has received a lot of attention in recent years, but due to its complicated structure a full understanding of the poset has proven elusive, see [MS15, Smi17]. The poset of mesh patterns, which we define here, contains the poset of permutations as an induced subposet. Therefore, investigating the poset of mesh patterns may lead to a better understanding of the poset of permutations. Moreover, studying this poset may help to answer some of the open questions on mesh patterns.

In Section 2 we introduce the poset of mesh patterns and related definitions, including a brief overview of poset topology. In Section 3 we prove some results on the Möbius function of the poset of mesh patterns. In Section 4 we give a characterisation of the non-pure (or non-graded) intervals of the mesh pattern poset. In Section 5 we give some results on the topology of the mesh pattern poset.

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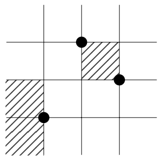
## 2 The Poset of Mesh Patterns

Mesh patterns were first introduced in [BC11] and are a generalisation of permutations. Given a permutation $\pi = \pi_1\pi_2\dots\pi_n$ we can plot $\pi$ on an $n \times n$ grid, where we place a dot at coordinates $(i, \pi_i)$, for all $1 \leq i \leq n$. A mesh pattern is then obtained by shading boxes of this grid, so a mesh pattern takes the from $p = (p_{cl}, p_{sh})$, where $p_{cl}$ is a permutation and $p_{sh}$ is a set of coordinates recording the shaded boxes, which are indexed by their south west corner. For ease of notation we sometimes denote the mesh pattern $p_{cl}^{p_{sh}}$. We let $|p_{cl}|$ represent the length of $p_{cl}$ and $|p_{sh}|$ the size of $p_{sh}$, and define the *length* of $p$ as $|p_{cl}|$. For example, the mesh pattern $(132, \{(0,0),(0,1),(2,2)\})$, or equivalently $132^{(0,0),(0,1),(2,2)}$, has the form:



To define when a mesh pattern occurs within another mesh pattern, first we need to recall two other well-known definitions of occurrence. A permutation $\sigma$ *occurs* in a permutation $\pi$ if there is a subsequence, $\eta$, of $\pi$ whose letters appear in the same relative order of size as the letters of $\sigma$. The subsequence $\eta$ is called an *occurrence*.

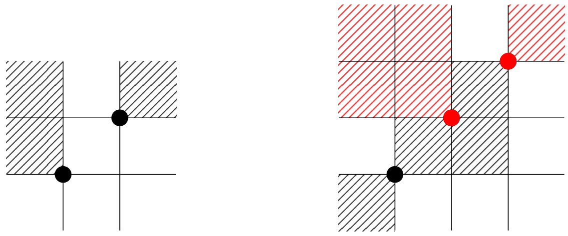
Consider a mesh pattern $(\sigma, S)$ and an occurrence $\eta$ of $\sigma$ in $\pi$, in the classical permutation pattern sense. Each box $(i,j)$ of $(\sigma, S)$ corresponds to an area $R_\eta(i,j)$ in the plot of $\pi$, which is the area bounded by the points in $\pi$ which in $\eta$ correspond to the letters $\sigma_i, \sigma_{i+1}, j, j+1$ of $\sigma$. A point is contained in $R_\eta(i,j)$ if it is in the interior of $R_\eta(i,j)$, that is, not on the boundary. For example, in Figure 2.1 where $\eta$ is the occurence in red, the area of $R_\eta(0,0)$ is the boxes $\{(0,0),(1,0),(0,1),(1,1)\}$, and it contains exactly one point. We say that $\eta$ is an occurrence of the mesh pattern $(\sigma, S)$ in the permutation $\pi$ if there is no point in any of the areas $R_\eta(i,j)$ for any shaded box $(i,j) \in S$.

Using these definitions of occurrences we can recall a concept of mesh pattern containment in another mesh pattern introduced in [TU17]. An example of which is given in Figure 2.1.

**Definition 2.1** ([TU17])**.** *An occurrence of a mesh pattern $(\sigma, S)$ in another mesh pattern $(\pi, P)$ is an occurrence $\eta$ of $(\sigma, S)$ in $\pi$, where for any $(i,j) \in S$ every box in $R_\eta(i,j)$ is shaded in $(\pi, P)$.*

The classical permutation poset $\mathcal{P}$ is defined as the poset of all permutation, with $\sigma \leq_{\mathcal{P}} \pi$ if and only if $\sigma$ occurs in $\pi$. Using Definition 2.1 we can similarly define the mesh pattern poset $\mathcal{M}$ as the poset of all mesh patterns, with $m \leq_{\mathcal{M}} p$ if $m$ occurs in $p$. Note we drop the subscripts from $\leq$ when it is clear which partial order is being considered. An *interval* $[\alpha, \beta]$ of a poset is defined as subposet induced by the set $\{\kappa \,|\, \alpha \leq \kappa \leq \beta\}$. See Figure 2.2 for an example of an interval of $\mathcal{M}$.

The first result on the mesh pattern poset is that there are infinitely many maximal elements, which shows a significant difference to the permutation poset, where there are no maximal elements.



(a) (b)

Figure 2.1: A pair of mesh patterns, with an occurrence of (a) in (b) depicted in red.

**Lemma 2.2.** *The poset of mesh pattern contains infinitely many maximal elements, which are the mesh patterns in which all boxes are shaded.*

*Proof.* This follows from the easily proven fact that a fully shaded mesh patterns only occurs in itself, and no other mesh patterns. $\square$

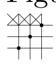
## 2.1 Poset Topology

In this subsection we briefly introduce some poset topology and, refer the reader to [Wac07] for a comprehensive overview of the topic, including any definitions we omit here.

The *Möbius function* on an interval $[\alpha, \beta]$ of a poset is defined by: $\mu(a, a) = 1$, for all $a$, $\mu(a, b) = 0$ if $a \not\leq b$, and

$$\mu(a, b) = -\sum_{c \in [a,b)} \mu(a, c).$$

We refer to $\mu(a, b)$ as the Möbius function of $[a, b]$. See Figure 3.1 for an example.

In a poset we say that $\alpha$ *covers* $\beta$, denoted $\alpha \gtrdot \beta$, if $\alpha > \beta$ and there is no $\kappa$ such that $\alpha > \kappa > \beta$. A *chain* of length $k$ in a poset is a totally ordered subset $c_1 < c_2 < \cdots < c_{k+1}$, and the chain is maximal if $c_i \lessdot c_{i+1}$, for all $1 \leq i \leq k$. A poset is *pure* (also known as *ranked* or *graded*) if all maximal chains have the same length. The *dimension* of a poset $P$, denoted $\dim P$, is the length of the longest maximal chain. For example, the interval in Figure 2.2 is nonpure because there is one maximal chain of length 3 (  $\lessdot$  $\lessdot$  $\lessdot$  ), one maximal chain of length 4 and all other maximal chains have length 5, so the interval has dimension is 5.

The *interior* of an interval $[\alpha, \beta]$ is obtained by removing $\alpha$ and $\beta$, and is denoted $(\alpha, \beta)$. The *order complex* of an interval $[\alpha, \beta]$, denoted $\Delta(\alpha, \beta)$ is the simplicial complex whose faces are the chains of $(\alpha, \beta)$. When we refer to the topology of an interval we mean the topology of the order complex of the interval.

A simplicial complex is *shellable* if we can order the maximal faces $F_1, \dots, F_t$ such that the subcomplex $\left(\cup_{i=1}^{k-1} F_i\right) \cap F_k$ is pure and $(\dim F_k)$-dimensional, for all $k = 2, \dots, t$. Being shellable implies other properties on the topology, such as having the homotopy type of a wedge of spheres.

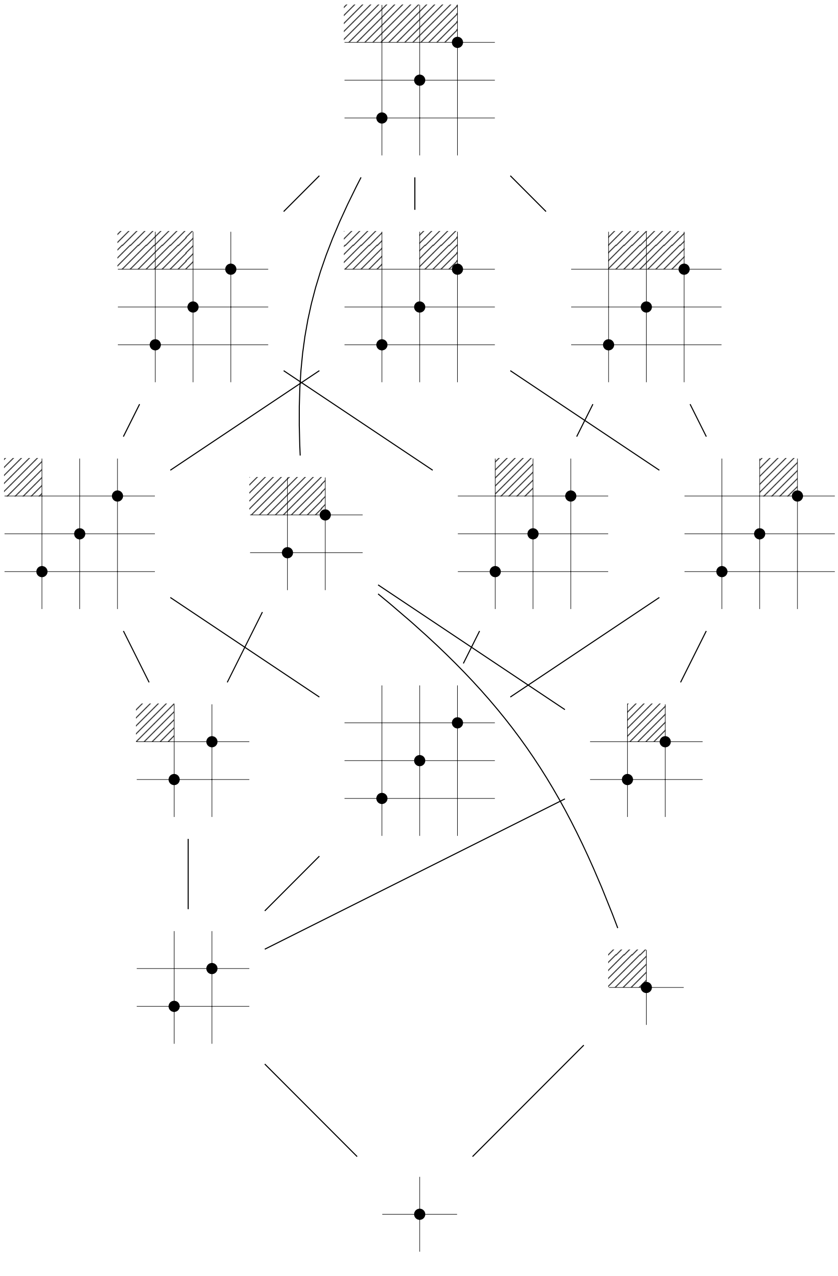


Figure 2.2: The interval $[1^{\emptyset}, 123^{(0,3),(1,3),(2,3)}]$ of $\mathcal{M}$.

An interval $I$ is *disconnected* if the interior can be split into two disjoint pairwise incomparable sets, that is, $I = A \cup B$ with $A \cap B = \emptyset$ and for every $a \in A$ and $b \in B$ we have $a \not\leq b$ and $b \not\leq a$. Each interval $I$ can be decomposed into its smallest connected parts, which we call the *components* of $I$. A component is *nontrivial* if it contains more than one element and we say an interval is *strongly disconnected* if it has at least two nontrivial components. For example, the interval $[1^{\emptyset}, 12^{(0,2),(1,2)}]$ in Figure 2.2 is disconnected but not strongly disconnected. Note that if an interval has dimension less than 3 it can never be strongly disconnected.

We can use disconnectivity as a test for shellability using the following results.

**Lemma 2.3.** *If an interval is strongly disconnected, then it is not shellable.*

*Proof.* Consider any ordering of the maximal chains and let $F_k$, with $k > 1$, be the first chain where every preceding chain belongs to a different component and $F_k$ belongs to a nontrivial component. Note that such an $F_k$ exists in every ordering because the interval is strongly disconnected, and because $F_k$ belongs to a nontrivial component it must have dimension of at least 1. So $\left(\cup_{i=1}^{k-1} F_i\right) \cap F_k = \emptyset$, which has dimension $-1$, so it is not $\dim(F_k - 1)$-dimensional. Therefore, the ordering is not a shelling. □

Since every subinterval of a shellable interval is shellable, [Wac07, Corollary 3.1.9], we obtain the following:

**Corollary 2.4.** *An interval which contains a strongly disconnected interval is not shellable.*

Finally, we present a useful result known as the Quillen fiber lemma [Qui78]. Two simplicial complexes are homotopy equivalent if one can be obtained by deforming the other but not breaking or creating any new "holes", for a formal definition see [Hat02]. A simplicial complex is *contractable* if it is homotopy equivalent to a point and if two posets are homotopy equivalent their Möbius functions are equal. Given a poset $P$, with $p \in P$ define the upper ideal $P_{\geq p} = \{q \in P \,|\, q \geq p\}$.

**Proposition 2.5.** *(Quillen Fiber Lemma) Let $\phi : P \to Q$ be an order-preserving map between posets such that for any $x \in Q$ the complex $\Delta(\phi^{-1}(Q_{\geq x}))$ is contractible. Then $P$ and $Q$ are homotopy equivalent.*

In the subsequent sections we give some results on the Möbius function and topology of the poset of mesh patterns.

# 3 Möbius Function

In this section we present some results on the Möbius function of the mesh pattern poset. We begin with some simple results: on mesh patterns with the same underlying permutations; the mesh patterns with no points $\epsilon^{\emptyset}$ and $\epsilon^{(0,0)}$; and mesh patterns with no shaded boxes. Throughout, the remainder of the paper we assume that $m$ and $p$ are mesh patterns.

**Lemma 3.1.** *For any sets $A \subseteq B$ the interval $[s^A, s^B]$ is isomorphic to the boolean lattice $B_{|B|-|A|}$. Therefore, $\mu(s^A, s^B) = (-1)^{|B|-|A|}$ and $[s^A, s^B]$ is shellable.*

*Proof.* We cannot remove any points from $s$, but we can unshade any boxes from $B \setminus A$ in any order. $\square$

**Lemma 3.2.** *Consider $A \in \{\emptyset, (0,0)\}$, then:*

$$\mu(\epsilon^A, p) = \begin{cases} 1, & \textit{if } p = \epsilon^A \\ -1, & \textit{if } A = \emptyset \ \& \ |p_{cl}| + |p_{sh}| = 1 \\ 0, & \textit{otherwise} \end{cases}.$$

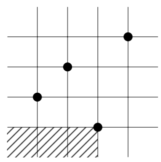
*Proof.* The first two cases are trivial. The mesh pattern $\epsilon^{(0,0)}$ is not contained in any larger mesh patterns, so the Möbius function is always 0. If $|p_{cl}| + |p_{sh}| > 1$, then $(\epsilon^\emptyset, p)$ contains a unique minimal element $1^\emptyset$, so $\mu(\epsilon^\emptyset, p) = 0$. $\square$

**Lemma 3.3.** *The interval $[s^\emptyset, t^\emptyset]$ is isomorphic to $[s, t]$ in $\mathcal{P}$, so $\mu_{\mathcal{M}}(s^\emptyset, t^\emptyset) = \mu_{\mathcal{P}}(s, t)$.*

The Möbius function of the classical permutation poset is known to be unbounded [Smi14]. So we get the following corollary:

**Corollary 3.4.** *The Möbius function is unbounded on $\mathcal{M}$.*

We can also show that the Möbius function is unbounded if we include shadings. We do this by mapping to the poset $\mathcal{W}$ of words with subword order, which is analogous to that used in [Smi16, Section 2]. This is the poset made up of all words and $u \leq w$ if there is a subword of $w$ that equals $u$. A *descent* in a permutation $\pi = \pi_1\pi_2 \dots \pi_n$ is a pair of letters $\pi_i, \pi_{i+1}$ with $\pi_i > \pi_{i+1}$. We call $\pi_{i+1}$ the *descent bottom*. An *adjacency tail* is a letter $\pi_i$ with $\pi_i = \pi_{i-1} \pm 1$. Let $adj(\pi)$ be the number of adjacency tails in $\pi$. Consider the set of mesh patterns $M^1$ where the permutation has exactly one descent, the descent bottom is 1 and we shade everything south west of 1. For example, the mesh pattern $2314^{(0,0),(1,0),(2,0)}$:

 .

**Lemma 3.5.** *Consider a mesh pattern $m \in M^1$, then $[21^{(0,0),(1,0)}, m]$ is shellable and*

$$\mu(21^{(0,0),(1,0)}, m) = \begin{cases} (-1)^{|m|} \lfloor \frac{n}{2} \rfloor, & \textit{if } adj(m_{cl}) = 0 \\ (-1)^{|m|}, & \textit{if } adj(m_{cl}) = 1 \textit{ and the tail is before the descent} \\ 0, & \textit{otherwise} \end{cases}.$$

*Proof.* First note that every mesh pattern in $I = [21^{(0,0),(1,0)}, m]$ is also in $M^1$. We define a map $f$ from $M^1$ to binary words in the following way. Let $b(m)$ be the set of letters that appear before 1 in $m \in M^1$. Set $f(m)$ as the word where the $i$th letter is 0 if it is in $b(m)$ and 1 otherwise. The inverse of this map is obtained by taking a binary word $w$ put the positions that are 0's in increasing order then the positions that are 1 in increasing order, and shade everything southwest of 1. So $f$ is a bijection.

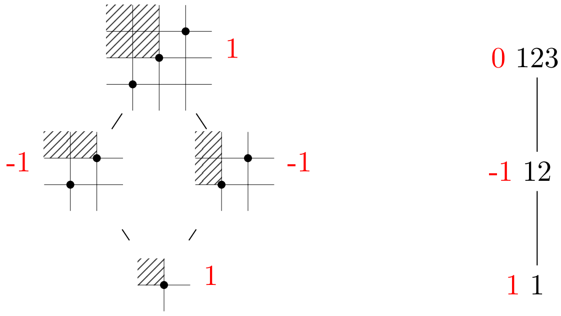



Figure 3.1: The interval $[1^{(0,1)}, 123^{(0,2),(0,3),(1,2),(1,3)}]$ (left) in $\mathcal{M}$ and $[1, 123]$ (right) in $\mathcal{P}$, with the Möbius function in red.

The map $f$ is a bijection from $\mathcal{M}$ to $\mathcal{W}$ and it is straightforward to check that $f$ is order preserving. Therefore, the interval $[21^{(0,0),(1,0)}, m]$ is isomorphic to $[10, f(m)]$ in $\mathcal{W}$. It was shown in [Bjö90] that intervals of $\mathcal{W}$ are shellable, which proves the shellability part, and the Möbius function equals the number of normal occurrences with the sign given by the dimension, where an occurrence is *normal* if in any consecutive sequence of equal elements every non-initial letter is part of the occurrence. In our bijection a non-initial letter of such a sequence maps to an adjacency tail. Let an occurrence of $21^{(0,0),(1,0)}$ in $m$ be normal if the corresponding occurrence of 10 in $f(m)$ is normal.

In every occurrence of $21^{(0,0),(1,0)}$ in $m$ the 1 of 21 must occur as the 1 in $m$. So all that remains is to place the 2 somewhere. For the occurrence to be normal every adjacency tail must be used in the occurrence. If there are no adjacency tails, we can place 2 anywhere before the descent, which must consist of all the even letters so there are $\lfloor \frac{n}{2} \rfloor$ options. If there is one adjacency tail and it is before the descent we place the 2 in it and we have one normal occurrence, otherwise there are no normal occurrences. $\square$

The Möbius function on $\mathcal{P}$ often takes larger values than on $\mathcal{M}$, but it is not always true that $\mu_{\mathcal{M}}(m, p) \leq \mu_{\mathcal{P}}(m_{cl}, p_{cl})$. A simple counterexample is the interval

$$[1^{(0,1)}, 123^{(0,2),(0,3),(1,2),(1,3)}],$$

this has Möbius function 1, however $\mu_{\mathcal{P}}(1, 123) = 0$, see Figure 3.1.

If we consider intervals where the bottom mesh pattern has no shadings, then we get the following result:

**Lemma 3.6.** *Consider an interval $[s^{\emptyset}, p]$ in $\mathcal{M}$ with $p_{sh} \neq \emptyset$. If there is no $s^A \in (m, p)$ with $A \neq \emptyset$, then $\mu(s^{\emptyset}, p) = 0$.*

*Proof.* Consider the map $f : (s^{\emptyset}, p) \to A : x \mapsto x^{\emptyset}_{cl}$, that is, $f$ removes all shadings from $x$. This implies that $A = (s^{\emptyset}, p^{\emptyset}_{cl}]$, so $A$ is contractible, because it has the unique maximal element $p^{\emptyset}_{cl}$, hence $\mu(A) = 0$. Moreover, $f^{-1}(A_{\geq y}) = [y, p)$, for all $y \in A$, which is contractible. Therefore, $(s^{\emptyset}, p)$ is homotopy equivalent to $A$ by the Quillen Fiber Lemma (Proposition 2.5), which implies $\mu(s^{\emptyset}, p) = 0$. $\square$

**Example 3.7.** *Consider the subinterval* $[1^{\emptyset}, 12^{(0,2)}]$ *in Figure 2.2, applying Lemma 3.6 implies* $\mu(1^{\emptyset}, 12^{(0,2)}) = 0$. *However, we cannot apply Lemma 3.6 to* $[1^{\emptyset}, 12^{(0,2),(1,2)}]$ *because it contains the element* $1^{(0,1)}$.

We can combine Lemma 3.6 with the following result to see that the Möbius function is almost always zero on the interval $[1^{\emptyset}, p]$.

**Lemma 3.8.** *As* $n$ *tends to infinity the proportion of permutations of length* $n$ *that contain any of* $\{1^{(0,0)}, 1^{(1,0)}, 1^{(0,1)}, 1^{(1,1)}\}$ *approaches* 0.

*Proof.* Let $P(n, i)$ be the probability that the letter $i$ is an occurrence of $1^{(0,0)}$ in a length $n$ mesh pattern. And let $P(n)$ be the probability that a length $n$ mesh pattern contains $1^{(0,0)}$.

The probability that $i$ is an occurrence of $1^{(0,0)}$ is given by selecting the location $k$ of $i$, each has probability $\frac{1}{n}$, and then we require that all boxes south west of $i$ are filled, of which there are $2^{ik}$. Note that this over estimates the probability, because it is possible that there is a point south west of $i$, which would imply $i$ is not an occurrence of $1^{(0,0)}$, however this argument still counts them. We can formulate this as:

$$P(n, i) \leq \sum_{k=1}^{n+1-i} \frac{1}{n} \left(\frac{1}{2^i}\right)^k = \frac{1}{n} \left(\frac{2^{-i(n+2-i)}}{2^{-i} - 1} - 1\right) = \frac{1}{n2^i} \left(\frac{1 - 2^{-i(n+1-i)}}{1 - 2^{-i}}\right) \leq \frac{2}{n2^i}$$

To compute the probability that a length $n$ mesh pattern contains $1^{(0,0)}$ we can sum the over all the letters $i$. Note again this is an over estimate because if a mesh pattern contains multiple occurrences of $1^{(0,0)}$ it counts that mesh pattern more than once.

$$P(n) \leq \sum_{i=1}^{n} P(n, i) \leq \sum_{i=1}^{n} \frac{2}{n2^i} = \frac{2}{n} \left(\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} - 1\right) \leq \frac{2}{n}$$

Repeating this calculation for the other three shadings of 1 implies that the probability of containing any of the forbidden mesh pattern is bounded by $\frac{8}{n} \to 0$. $\square$

**Corollary 3.9.** *As* $n$ *tends to infinity the proportion of mesh patterns* $p$ *of length* $n$ *such that* $\mu(1^{\emptyset}, p) = 0$ *approaches* 1.

In the classical case it is true that given a permutation $\sigma$ the probability a permutation of length $n$ contains $\sigma$ tends to 1 as $n$ tends to infinity, this follows from the Marcus-Tardos Theorem [MT04]. By the above result we can see the same is not true in the mesh pattern case. In fact we conjecture the opposite is true:

**Conjecture 3.10.** *Given a mesh pattern* $m$*, the probability that a random mesh pattern of length* $n$ *contains* $m$ *tends to* 0 *as* $n$ *tends to infinity.*

# 4 Purity

Recall that a poset is pure (also known as graded or ranked) if all the maximal chains have the same length, and as we can see from Figure 2.2 the mesh pattern poset is non-pure. In this section we classify which intervals $[1^\emptyset, m]$ are non-pure. First we consider the length of the longest maximal chain in any interval $[1^\emptyset, m]$, that is, the dimension of $[1^\emptyset, m]$.

**Lemma 4.1.** *For any mesh pattern $m$, we have* $\dim(1^\emptyset, m) = |m_{cl}| + |m_{sh}|$.

*Proof.* We can create a chain from $m$ to $1^\emptyset$ by deshading all boxes, in any order, and then deleting all but one point, in any order. The length of this chain is $|m_{cl}| + |m_{sh}|$. Moreover, we cannot create a longer chain because at every step of a chain we must deshade a box or delete a point. □

So we define the *dimension* of a mesh pattern as $\dim(m) = |m_{cl}| + |m_{sh}|$ and we say an edge $m \lessdot p$ is *impure* if $\dim(p) - \dim(m) > 1$. Next we give a classification of impure edges.

Let $m^{-x}$ be the mesh pattern obtained by deleting the point $x$ and let $\eta_m^{-x}$ be the occurrence of $m^{-x}$ in $m$ that does not use the point $x$. An occurrence $\eta$ of $m$ in $p$ *uses the shaded box* $(a, b) \in p_{sh}$ if $(a, b) \in R_\eta(i, j)$ for some shaded box $(i, j) \in m_{sh}$. We say that deleting a point $x$ *merges shadings* if there is a shaded box in $m^{-x}$ that corresponds to more than one shaded box in $\eta_m^{-x}$, see Figure 4.1.

**Lemma 4.2.** *Two mesh patterns $m < p$ form an impure edge if and only if all occurrences of $m$ in $p$ use all shaded boxes of $p$ and are obtained by deleting a point that merges shadings.*

*Proof.* First we show the backwards direction. Because $m$ is obtained by deleting a point that merges shadings, $m$ must have one less point and at least one less shading so $\dim(p) - \dim(m) \geq 2$. So it suffices to show that there is no $z$ such that $m < z < p$. Suppose such a $z$ exists, then if $z$ is obtained by deshading a box in $p$ it can no longer contain $m$ because all occurrences of $m$ in $p$ use all shaded areas of $p$. If $z$ is obtained by deleting a point and $m < z$, then $m_{cl} = z_{cl}$. Therefore, we can deshade some boxes of $z$ to get $m$, which implies there is an occurrence of $m$ in $p$ that does not use all the shaded boxes of $p$.

Now consider the forwards direction. Suppose $m \lessdot p$ is impure, so $\dim(p) - \dim(m) \geq 2$. Therefore, $m$ is obtained by deleting a single point which merges shadings, but does not delete shadings because any other combination of deleting points and deshading can be done in successive steps. Furthermore, this must be true for any point that can be deleted to get $m$, that is, for all occurrences of $m$ in $p$. Moreover, if there is an occurrence that does not use all the shaded boxes of $p$, we can deshade the box it doesn't use and get an element that lies between $m$ and $p$. □

**Lemma 4.3.** *If $[m, p]$ contains an impure edge, then it contains an impure edge $a \lessdot b$ where $p_{cl} = b_{cl}$.*

*Proof.* Let $x \lessdot y$ be an impure edge in $[m, p]$. So $x$ is obtained from $y$ by deleting a point $i$. Consider an occurrence $\eta$ of $y$ in $p$ and let $b$ be the mesh pattern where $b_{cl} = p_{cl}$ and $b_{sh}$ are

$a = 12^{(0,2),(1,2)}$ $\qquad$ $b = 123^{(1,3),(2,3)}$

Figure 4.1: Two mesh patterns with a point $x$ in black which merges shadings and the occurrences $a^{-x}$ and $b^{-x}$ in red. By Lemma 4.2 $a^{-x} \lessdot a$ is impure, but $b^{-x} < b$ is not an impure edge because there is a second occurrence of $b^{-x}$ in $b$, using points 23, that does not use all shaded boxes in $b$.

the boxes used by $\eta$. Let $a$ be the mesh pattern obtained from $b$ by deleting the point which corresponds to $i$ in $\eta$.

The mesh pattern $b$ is constructed from $y$ by adding a collection of points. None of these added points can be touching a shaded box in $b$, as they must be added to empty boxes of $y$. Moreover, the set of occurrences of $a$ in $b$ correspond to the set of occurrences of $x$ in $y$, after adding the new points. This implies that if the as the occurrences of $x$ in $y$ satisfy the conditions of Lemma 4.2, the occurrences of $a$ in $b$ also satisfy these conditions. So $a \lessdot b$ is an impure edge by Lemma 4.2. $\square$

**Proposition 4.4.** *The interval $[1^{\emptyset}, m]$ is non-pure if and only if there exists a point $x$ in $m$ such that $m^{-x}$ merges shadings and there is no other occurrence of $m^{-x}$ in $m$ which uses a subset of the shadings used by $\eta_m^{-x}$.*

*Proof.* First we show the backwards direction. Let $t$ be the mesh pattern obtained by inserting $x$ back into $m^{-x}$, and $\phi$ the corresponding occurrence of $m^{-p}$ in $t$. Note that there are no other occurrences of $m^{-x}$ in $t$ because there is no occurrence of $m^{-x}$ in $m$ which uses a subset of the shadings used by $\eta_m^{-x}$. Therefore, by Lemma 4.2 we get that $m^{-x} \lessdot t$ is an impure edge.

To see the other direction suppose there is an impure edge in $[1^{\emptyset}, m]$. By Lemma 4.3 there is an impure edge $a \lessdot b$ where $b_{cl} = m_{cl}$. By Lemma 4.2 all occurrences of $a$ in $b$ use all shaded boxes of $b$ and are obtained by deleting a point that merges shadings. Moreover, if deleting a point merges shadings in $b$, then its deletion merges shadings in $m$, which implies the result. $\square$

**Corollary 4.5.** *There is an impure edge in the interval $[m, p]$ if and only if there exists a point $x$ in $p$ such that $p^{-x}$ merges shadings and there is no other occurrence of $p^{-x}$ in $p$ with a subset of shadings of $\eta_p^{-x}$, and $p^{-x} \geq m$.*

Note that containing an impure edge in $[m, p]$ does not necessarily imply that $[m, p]$ is non-pure. For example, if $[m, p]$ contains only one edge and that edge is impure, then $[m, p]$ is still pure. Although it is also possible to have a pure poset that contains impure and pure edges, see Figure 4.2.

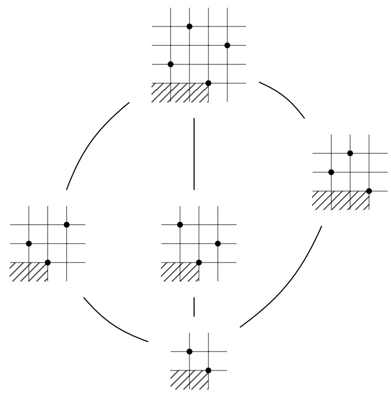
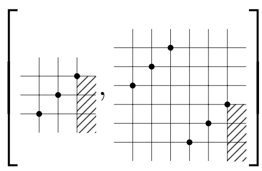


Figure 4.2: The interval $[21^{(0,0),(1,0)}, 2413^{(0,0),(1,0),(2,0)}]$, which is pure but contains both pure and impure edges.

# 5 Topology

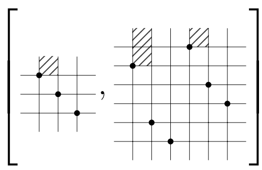
A full classification of shellable intervals has not been obtained for the classical permutation poset, so finding such a classification for the mesh pattern poset would be equally difficult, if not more so. However, in [MS15] all disconnected intervals are described, and containing a disconnected subinterval implies a pure interval is not shellable. So this gives a large class of non-shellable intervals, in fact it is shown that almost all intervals are not shellable. By Lemma 2.3 containing a strongly disconnected interval implies a non-pure interval is not shellable. So in this section we consider when an interval is strongly disconnected. Firstly we look at the relationship between connectivity in $\mathcal{P}$ and $\mathcal{M}$.

The connectivity of the interval $[m_{cl}, p_{cl}]$ in $\mathcal{P}$ does not necessarily imply the same property for $[m, p]$ in $\mathcal{M}$. For example, the interval $[123, 456123]$ is disconnected in $\mathcal{P}$ but the interval



(5.1)

is a chain in $\mathcal{M}$, so is connected. Furthermore, the interval $[321, 521643]$ is connected in $\mathcal{P}$ but the interval



(5.2)

is strongly disconnected in $\mathcal{M}$. Therefore, if $[m_{cl}, p_{cl}]$ is (non-)shellable in $\mathcal{P}$, then it is not true that $[m, p]$ has the same property in $\mathcal{M}$. For example, $[123, 456123]$ is not shellable but (5.1) is shellable, and $[321, 521643]$ is shellable but (5.2) is not shellable.

In [MS15] the direct sum operation is used to show that almost all intervals are not

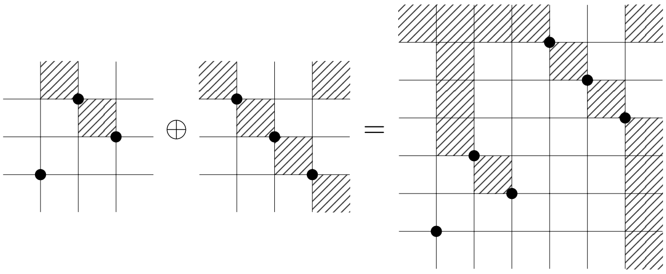


Figure 5.1: The direct sum of two mesh patterns.

shellable in $\mathcal{P}$. We can generalise the direct sum operation to mesh patterns. Given two permutations $\alpha = \alpha_1 \dots \alpha_a$ and $\beta = \beta_1 \dots \beta_b$ the direct sum of the two is defined as $\alpha \oplus \beta = \alpha_1 \dots \alpha_a(\beta_1 + a)(\beta_2 + a) \dots (\beta_b + a)$, that is, we append $\beta$ to $\alpha$ and increase the value of each letter by the length of $\alpha$. This can also be thought of in terms of the plots of $\alpha$ and $\beta$ by placing a copy of $\beta$ to the north east of $\alpha$. Similarly we can define the skew-sum $\alpha \ominus \beta$ by prepending $\alpha$ to $\beta$ and increasing the value of each letter of $\alpha$ by the length of $\beta$. We extend these definition to mesh patterns in the following way:

**Definition 5.1.** *Consider two mesh patterns $s$ and $t$, where the top right corner of $s$ and bottom left corner of $t$ are not shaded. The direct sum $s \oplus t$ has the classical pattern $s_{cl} \oplus t_{cl}$ and shaded boxes $s_{sh} \cup \{(i, j + |s_{cl}|) \mid (i, j) \in t_{sh}\}$, and also for any shaded boxes $(i, |s_{cl}|)$, $(|s_{cl}|, i)$, $(j, |s_{cl}|)$ or $(|s_{cl}|, j)$, shaded all the boxes north, east, south or west of the box, respectively, for all $0 \leq i < |s_{cl}|$ and $|s_{cl}| < j \leq |s_{cl}| + |t_{cl}|$. We similarly define the skew-sum for when the bottom right corner of $s$ and top left corner of $t$ are not shaded.*

The direct product $s \oplus t$ can be consider as placing a copy of $t$ north east of $s$ and any shaded box that was on a boundary we extend to the new boundary, see Figure 5.1. We define the direct sum in this way because it maintains one of the most important properties in the permutation sense, that the first $|s_{cl}|$ letters are an occurrence of $s$ and the final $|t_{cl}|$ letters are an occurrence of $t$.

A permutation is said to be indecomposable if it cannot be written as the direct sum of smaller permutation. We can generalise this to mesh patterns.

**Definition 5.2.** *A mesh pattern $m$ is* indecomposable *(resp.* skew-indecomposable*) if it cannot be written $m = a \oplus b$ (resp. $m = a \ominus b$), where neither $a$ or $b$ is $m$.*

**Remark 5.3.** *It is well known that a permutation has a unique decomposition into indecomposable permutations. Which implies that a mesh pattern also has a unique decomposition.*

Using these definitions we can give a large class of strongly disconnected intervals, which is a mesh pattern generalisation of Lemma 4.2 in [MS15].

**Lemma 5.4.** *If $m$ is indecomposable,* $\dim m > 1$ *and* $(0, 0), (|m|, |m|) \notin m_{sh}$*, then $[m, m \oplus m]$ is strongly disconnected.*

*Proof.* By Lemma 4.2 in [MS15] the interval $[m_{cl}, m_{cl} \oplus m_{cl}]$ is strongly disconnected, with components $P_1 = \{m_{cl} \oplus x \mid x \in [1, m_{cl})\}$ and $P_2 = \{x \oplus m_{cl} \mid x \in [1, m_{cl})\}$. Consider any pair $\alpha, \beta \in [m, m \oplus m]$, if $\alpha_{cl}$ and $\beta_{cl}$ are not in the same component of $[m_{cl}, m_{cl} \oplus m_{cl}]$, then $\alpha$ and $\beta$ are incomparable. Let $\hat{P}_1 = \{\alpha \mid \alpha_{cl} \in P_1\}$ and $\hat{P}_2 = \{\alpha \mid \alpha_{cl} \in P_2\}$. However, $\hat{P}_1 \cup \hat{P}_2 \neq (m_{cl}, m \oplus m_{cl})$ because it does not include the mesh patterns with $\alpha_{cl} = m_{cl} \oplus m_{cl}$.

There are exactly two occurrences of $m$ in $m \oplus m$. These are $\eta_1$ the first $|m|$ letters and $\eta_2$ the last $|m|$ letters. Note that each shaded box of $m \oplus m$ is used by at least one of $\eta_1$ and $\eta_2$, so if we deshade a box the resulting pattern $x$ contains at most one occurrence of $m$, either the first or last $|m|$ letters. Let $Q_1$ and $Q_2$ be sets of patterns with underlying permutation $m_{cl} \oplus m_{cl}$ where the first and last $|m|$ letters are the only occurrence of $m$, respectively. So any element $Q_1$ cannot contain any element in $P_2 \cup Q_2$ and similarly any element of $Q_2$ cannot contain an element of $P_1 \cup Q_1$. Therefore, $P_1 \cup Q_1$ and $P_2 \cup Q_2$ are disconnected nontrivial components of $[m, m \oplus m]$. $\square$

**Corollary 5.5.** *If $m$ is skew-indecomposable,* $\dim m > 1$ *and* $(|m|, 0), (0, |m|) \notin m_{sh}$*, then* $[m, m \ominus m]$ *is strongly disconnected.*

Using Lemma 4.2 in [MS15] it is shown that almost all intervals of the classical permutation poset are not shellable. The proof of this follows from the Marcus-Tardos theorem, we have seen this result does not apply in the mesh pattern case so we cannot prove a similar result using this technique. A similar problem was studied for boxed mesh patterns in permutations in [AKV13], which is equivalent to boxed mesh patterns in fully shaded mesh patterns. So we leave it as an open question:

**Question 5.6.** *What proportion of intervals of $\mathcal{M}$ are shellable?*

The Möbius function in the permutation poset can be computed more easily by decomposing the permutations into smaller parts using the direct sum, or skew-sum, see [BJJS11, MS15]. Which leads to the following question:

**Question 5.7.** *Can a formula for the Möbius function of $\mathcal{M}$ be obtained by decomposing mesh patterns using direct sums and skew sums?*

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