Poset of Mesh Patterns

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May 4, 2017

1 The Poset of Mesh Patterns

Mesh patterns were first introduced in [BC11] and a concept mesh pattern containment was introduced in [TU17]. A mesh pattern is of the form $p = (\mathrm{cl}(p), \mathrm{sh}(p))$, where $\mathrm{cl}(p)$ is a permutation and $\mathrm{sh}(p)$ is a set of coordinates corresponding to the boxes that are shaded, where (i, j) denotes the box $[i, i+1] \times [j, j+1]$, see Figure 1.1. We also denote p by $\mathrm{cl}(p)^{\mathrm{sh}(p)}$, for example $132^{\emptyset}$ is the mesh pattern on 132 with no shaded boxes. Let $G(p)$ be the set of coordinates of the points of p.

We let $|\mathrm{cl}(p)|$ represent the length of $\mathrm{cl}(p)$ and $|\mathrm{sh}(p)|$ the size of $\mathrm{sh}(p)$, and define the length of p as $|\mathrm{cl}(p)|$. We define the *rank* of p as $|\pi| + |P|$, that is, the number of points plus the number of shaded boxes.

Given two classical permutations σ and π, an occurrence of σ in π is a map $\alpha : [|\sigma|] \to [|\pi|]$ such that $\alpha(1)\alpha(2)\ldots\alpha(|\sigma|)$ has the same relative order of size as σ. Given any letter $b = \sigma_i$ let $\hat{\alpha}(b) = \pi_{\alpha(i)}$, so $\hat{\alpha}$ maps the value of a letter in σ to its value in π under the map α.

Using the notion of pattern occurrence we can define a partial order on permutations where $\sigma \leq_{\mathcal{P}} \pi$ if there is an occurrence of σ in π. Let $\mathcal{P}$ be the poset of all permutations with the partial order $\leq_{\mathcal{P}}$, we call this the *classical permutation poset*.

We can extend this definition to define an occurrence in a mesh pattern. Consider a pair of mesh patterns m and p and an occurrence α of $\mathrm{cl}(m)$ in $\mathrm{cl}(p)$. Define $R^{\alpha}_{i,j} = [\alpha(i), \alpha(i+1)] \times [\hat{\alpha}(j), \hat{\alpha}(j+1)]$, this is the area in p that corresponds to the box (i, j) in m according to α

An *occurrence* of m in p is an occurrence α of $\mathrm{cl}(m)$ in $\mathrm{cl}(p)$ satisfying $\bigcup_{(i,j)\in \mathrm{sh}(m)} R^{\alpha}_{i,j} \subseteq \mathrm{sh}(p)$ and $G(p) \cap \bigcup_{(i,j)\in \mathrm{sh}(m)} R^{\alpha}_{i,j} = \emptyset$. This is equivalent to saying every shaded point in m corresponds to a fully shaded area in p that contains no points.

2 Möbius Function

We begin with some simple results we can immediately deduce about this poset. First we consider the case that two mesh patterns have the same underlying permutation.

*This research was supported by the EPSRC Grant EP/M027147/1

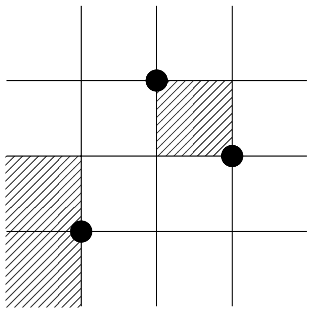

Figure 1.1: The Mesh pattern $(132, \{(0,0),(0,1),(2,2)\})$, or equivalently $132^{(0,0),(0,1),(2,2)}$.

Lemma 2.1. *If $cl(m) = cl(p)$, then $[m,p]$ is isomorphic to the boolean lattice $B_{|\,sh(p)|-|\,sh(m)|}$ which implies $\mu(m,p) = (-1)^{|\,sh(p)|\,-\,|\,sh(m)|}$ and $[m,p]$ is shellable.*

Proof. We cannot remove any points from p, we can only unshade boxes, and we can unshaded any boxes from $\mathrm{sh}(p) \setminus \mathrm{sh}(s)$ in any order. $\square$

The simplest mesh patterns are those with no points, that is, the mesh patterns with a single box that is shaded or unshaded, which we denote $\epsilon^{\emptyset}$ and $\epsilon^{(0,0)}$, respectively.

Lemma 2.2. *Consider a mesh pattern p, then:*

$$\mu(\epsilon^A, p) = \begin{cases} 1, & \textit{if } p = \epsilon^A \\ -1, & \textit{if } A = \emptyset \;\&\; rk(p) = 1 \\ 0, & \textit{otherwise} \end{cases}$$

Proof. The cases $p = \epsilon^A$ and $A = \emptyset$ & $\mathrm{rk}(p) = 1$ are trivial. The mesh pattern $\epsilon^{(0,0)}$ is not contained in any large mesh patterns, so the Möbius function is always 0. If $\mathrm{rk}(p) > 1$, then $(\epsilon^{\emptyset}, p)$ contains a unique minimal element $1^{\emptyset}$, so $\mu(\epsilon^{\emptyset}, p)$. $\square$

If two mesh patterns have no shadings then we have an interval from the classical poset.

Lemma 2.3. *If $sh(s) = sh(p) = \emptyset$, then $[s,p]$ is isomorphic to the interval $[cl(s), cl(p)]$ in $\mathcal{P}$, so $\mu_{\mathcal{M}}(s,p) = \mu_{\mathcal{P}}(cl(s), cl(p))$.*

The Möbius function of the classical permutation poset is known to be unbounded [Smi14]. So we get the following corollary:

Corollary 2.4. *The Möbius function is unbounded on $\mathcal{M}$.*

We can also show that the Möbius function is unbounded if we include shadings.

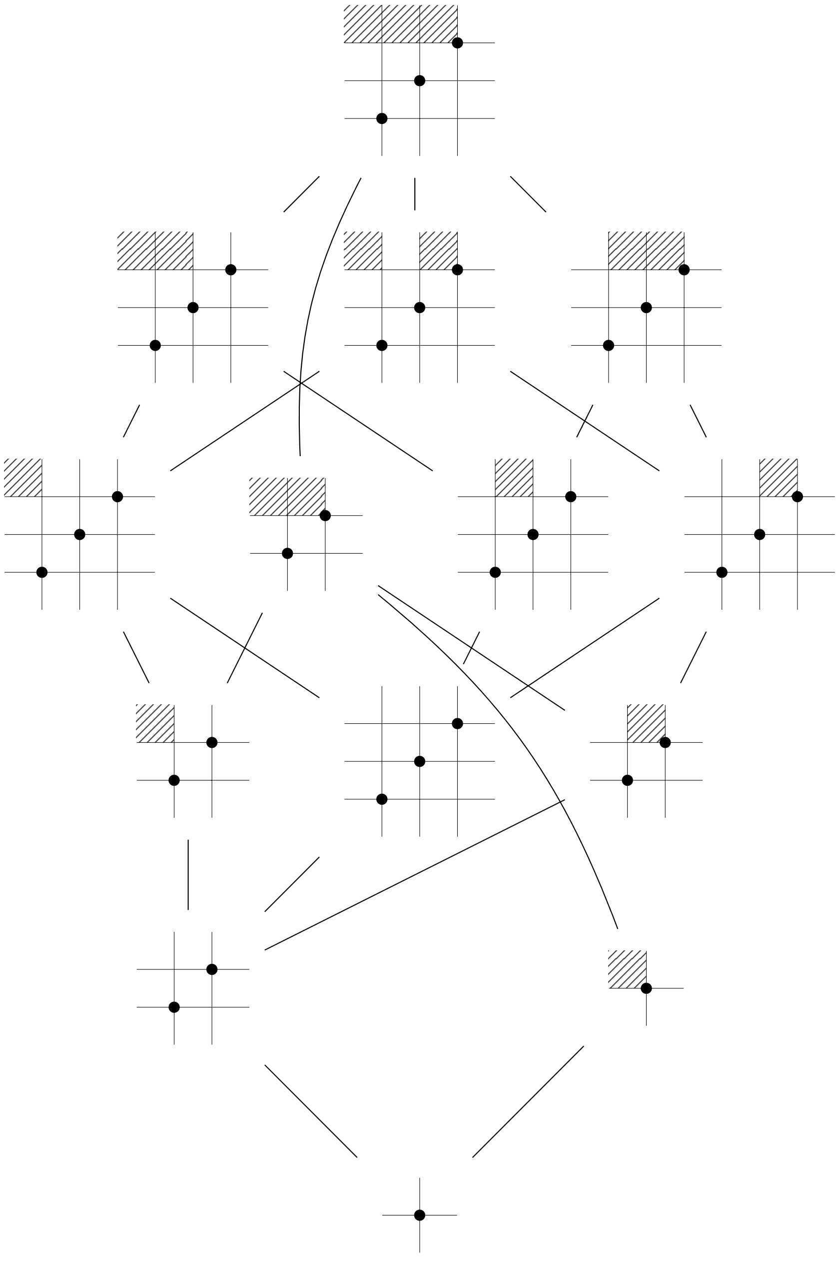

Figure 1.2: The interval $[1^{\emptyset}, 123^{(0,3),(1,3),(2,3)}]$ of $\mathcal{M}$.

Lemma 2.5. *Let m be a mesh pattern with exactly one descent, where the descent bottom is the letter 1, and all boxes south west of the point 1 are shaded, then*

$$\mu(21^{(0,0),(1,0)}, m) = \begin{cases} (-1)^{|m|}\lfloor \frac{n}{2} \rfloor, & \text{if } cl(m) \text{ has no adjacencies} \\ 1, & \text{if } cl(m) \text{ has exactly one letter before the descent in an adjacency tail and none after} \\ 0, & \text{otherwise} \end{cases}$$

Moreover, $[21^{(0,0),(1,0)}, m]$ is shellable.

Proof. Note that every mesh pattern in $I = [21^{(0,0),(1,0)}, m]$ has exactly one descent and everything SW of the point 1 shaded. Define a bijection f from I to the poset of words with subword order where the $i-1$'th letter of $f(p)$ is 0 if i is before the letter 1 in p and 1 if after the letter 1, for all $1 < i \leq n$. So $21^{(0,0),(1,0)}$ maps to the word 0. A mesh pattern in I is uniquely determined by the value of the points before the descent, because these values must appear first followed by the letter 1 and then the remaining letters. Therefore, it is straightforward to see this is a bijection and to check that it is order preserving. So I is isomorphic to an interval of the poset of words with subword order. It was shown in [Bjö90] that these intervals are shellable and the Möbius function equals the number of normal occurrences, where an occurrence is normal if in any run of equal elements every non-initial letter is part of the occurrence, which implies the result. □

We can also see that the Möbius function on $\mathcal{M}$ is not bounded the classical poset, that is, it is not true that $\mu_{\mathcal{M}}(m,p) \leq \mu_{\mathcal{P}}(\mathrm{cl}(m), \mathrm{cl}(p))$. A simple counterexample is the interval $[1^{(0,1)}, 123^{(0,2),(0,3),(1,2),(1,3)}]$, this has Möbius number 1, however $\mu_{\mathcal{P}}(1, 123) = 0$.

If we consider intervals where the bottom mesh pattern has no shadings, then we get the following result:

Lemma 2.6. *Consider the interval $[m,p]$ in $\mathcal{M}$. If $sh(m) = emptyset \neq sh(p)$ and there is no $s \in (m,p)$ with $cl(s) = cl(m)$, then $\mu(m,p) = 0$.*

Proof. Define a map $f(x) = \mathrm{cl}(x)^{\emptyset}$, for any $x \in (m,p)$. Let A order-preserving image $f((m,p))$. Note that $\mu(A)$ is contractible because it has a unique maximal element $\mathrm{cl}(p)^{\emptyset}$. Moreover, for any $y \in A$ $f^{-1}(A_{\geq y})$ equals $[y,p)$ which is contractible. Therefore, (m,p) is homotopically equivalent to A using the Quillen Fiber Lemma, which implies $\mu(m,p) = 0$. This is also called a retraction. □

We can combine Lemma 2.6 with he following result to see that the Möbius function is almost always zero on the interval $[1^{\emptyset}, p]$.

Lemma 2.7. *As n tends to infinity the proportion of permutations that contain one of $\{1^{(0,0)}, 1^{(1,0)}, 1^{(0,1)}, 1^{(1,1)}\}$ approaches 0.*

Proof. Let $P(n,i)$ be the probability that the letter i is an occurrence of $1^{(0,0)}$ in a length n mesh pattern. And let $P(n)$ be the probability that a length n mesh pattern contains $1^{(0,0)}$.

The probability that i is an occurrence of $1^{(0,0)}$ is given by selecting the location k of i, each has probability $\frac{1}{n}$, and then we require that all boxes south west of i are filled, of which there are 2^{ik}. Note that this over estimates the probability, because it is possible that there is a point south west of i, which would imply i is not an occurrence of $1^{(0,0)}$, however this argument still counts them. We can formulate this as:

$$P(n,i) \leq \sum_{k=1}^{n+1-i} \frac{1}{n}\left(\frac{1}{2^i}\right)^k = \frac{1}{n}\left(\frac{2^{-i(n+2-i)}}{2^{-i}-1} - 1\right) = \frac{1}{n2^i}\left(\frac{1-2^{-i(n+1-i)}}{1-2^{-i}}\right) \leq \frac{2}{n2^i}$$

To compute the probability that a length n permutation contains $1^{(0,0)}$ we can sum over all letters i and test if i is an occurrence of $1^{(0,0)}$. Note again this is an over estimate because if a permutation contains multiple occurrences of $1^{(0,0)}$ it counts that permutation more than once.

$$P(n) \leq \sum_{i=1}^{n} P(n,i) \leq \sum_{i=1}^{n} \frac{2}{n2^i} = \frac{2}{n}\left(\frac{\left(\frac{1}{2}\right)^{n+1}-1}{\frac{1}{2}-1} - 1\right) \leq \frac{2}{n}$$

We can repeat this calculation for the other three one shadings of 1 so we get that $P(n) \leq \frac{4}{n} \to 0$. □

Corollary 2.8. *As n tends to infinity the proportion of mesh patterns p of length n such that $\mu(1^{\emptyset}, p) = 0$ approaches 1.*

In the classical case it is true that given a permutation σ the probability a permutation of length n contains σ tends to 1 as n tends to infinity, this follows from the Marcus-Tardos Theorem [MT04]. By the above result we can see the same is not true in the mesh pattern case. In fact we conjecture the opposite is true:

Conjecture 2.9. *Given a mesh pattern m, the probability that a random mesh pattern of length n contains m tends to 0 as n tends to infinity.*

3 Purity

Interestingly the mesh pattern poset is not a pure poset, that is, not every maximal chain has the same length. We say an edge $x \lessdot y$ is *impure* if $\mathrm{rk}(y) - \mathrm{rk}(x) > 1$. Let $m - p$ be the mesh pattern obtained by deleting the point p and let $m \setminus p$ be the occurrence of $m - p$ in m. We say that deleting a point p *merges shadings* if there is a shaded box in $m - p$ that corresponds to more than 1 shaded box in $m \setminus p$.

Lemma 3.1. *An edge between $x \lessdot y$ is impure if and only if all occurrences of x in y use all shaded boxes of y and are obtained by deleting a point that merges shading.*

Proof. First we show the backwards direction. Because x is obtained by deleting a point that merges shadings x must have one less point and at least one less shading so $\mathrm{rk}(y) - \mathrm{rk}(x) \geq 2$. So it suffices to show that there is no z such that $x < z < y$. Suppose such a z exists, then if z is obtained by deshading a box in y it can no longer contain x because all occurrences of x in y use all shaded areas of y. If z is obtained by deleting a point, then that cannot remove shadings, only merge shadings, otherwise it wouldn't contain x, and it implies $cl(x) = cl(z)$. Moreover, if $x < z$ then we can deshade some boxes of z to get x which implies there is an occurrence of x in y that doesn't use all the shaded boxes of y.

Now consider the forwards direction, so suppose $x \lessdot y$ is impure. So $\mathrm{rk}(y) - \mathrm{rk}(x) \geq 2$, which implies x is obtained by deleting a single point which merges shadings but does not delete shadings, because any other combination of deleting points and deshading can be done in successive steps. Furthermore, this must be true for any point that can be deleted to get x, that is, for all occurrences of x in y. Moreover, if there is an occurrence that doesn't use all the shaded boxes of y, we can deshade the box it doesn't use and get an element that lies between x and y. $\square$

Lemma 3.2. *If there is an impure edge in $[1^{\emptyset}, m]$, then there is an impure edge $a \lessdot b$ where $cl(m) = cl(b)$.*

Proof. If $x \lessdot y$ is an impure edge in $[1^{\emptyset}, m]$, then let b be a mesh pattern obtained by adding points to y to get the classical pattern b. Pick an occurrence of x in y and add the points to x in the order induced by how they are added to y and the occurrence, call this a. The points added will not have any shadings in the four boxes touching it, therefore no point touching a shading in a can embed in a new point of b. Moreover, the set of embeddings of a in b is a subset of x in y, after adding the new points to each. These two conditions imply that if every embedding of x in y uses all the shadings of y, this is also true for every embedding of a in b. Therefore, the result follows by Lemma 3.1. $\square$

Proposition 3.3. *Consider a mesh pattern m. The interval $[1^{\emptyset}, m]$ is non-pure if and only if there exists a point p in m such that $m - p$ merges shadings and there is no other occurrence of $m - p$ in m with a subset of shadings of $m \setminus p$.*

Proof. First we show the backwards direction. Let x be the mesh pattern obtained by inserting p back into m^p, and η the corresponding embedding of m^p in x. Note that it is not always true that $x = m$ because some shaded shadings of m are lost when deleting p. We claim that $m^p \lessdot x$ is an impure edge. This follows by Lemma 3.1 because by η uses all the shaded boxes in x and there is no subshading occurrence.

To see the other direction suppose there is an impure edge in $[\omega, m]$. By Lemma 3.2 there is an impure edge $a \lessdot b$ where $cl(b) = cl(m)$. If m is impure then it must remove both a point and a shading, so it must merge shadings by deleting some point p and there is no element between them so there can be no subshading of b that contains a. $\square$

Corollary 3.4. *There is an impure edge in the interval $[m, p]$ if and only if there exists a point x in p such that $p - x$ merges shadings and there is no other occurrence of $p - x$ in p with a subset of shadings of $p \setminus x$, and $p - x \geq m$.*

Note that containing an impure edge in $[m, p]$ does not necessarily imply there that $[m, p]$ is non-pure. For example, if $[m, p]$ contains only one edge and that edge is impure, then $[m, p]$ is still pure. So it is necessary that there is an impure edge and a pure edge in the interval. Although it is also possible to have an non-pure poset that only contains impure edges, if the difference between the ranks of the elements at ends of the edges varies.

4 Topology

If $[\mathrm{cl}(s), \mathrm{cl}(p)]$ is (dis)connected in $\mathcal{P}$ it does no imply $[s, p]$ has the same property in $\mathcal{M}$. For example, the interval $[123, 456123]$ is disconnected in the classical permutation poset but in $\mathcal{M}$ the interval $[123^{(3,0),(3,1),(3,2)}, 456123^{(6,0),(6,1),(6,2)}]$, see Figure 4.1, is chain, so is connected. Furthermore, the interval $[321, 521643]$ is connected in $\mathcal{P}$ but $[321^{(1,3)}, 521643^{(1,5),(1,6),(4,6)}]$, see Figure 4.2, is disconnected in $\mathcal{M}$.

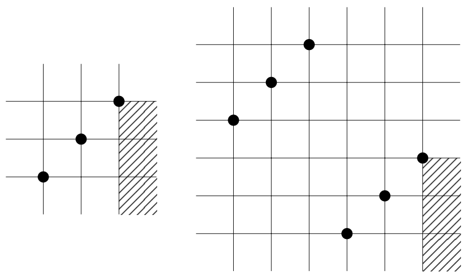

Figure 4.1: The mesh patterns $123^{(3,0),(3,1),(3,2)}$ and $456123^{(6,0),(6,1),(6,2)}$

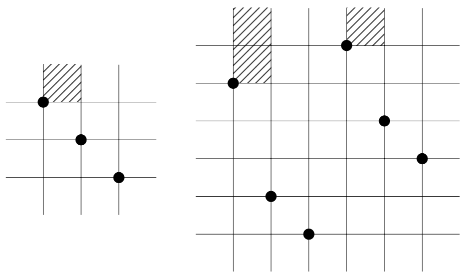

Figure 4.2: The mesh patterns $321^{(1,3)}$ and $521643^{(1,5),(1,6),(4,6)}$

This also implies that if $[\mathrm{cl}(s), \mathrm{cl}(p)]$ is (non-)shellable in $\mathcal{P}$ then it is not true that $[s, p]$ has the same property in $\mathcal{M}$. This follows by the previous examples, $[123, 456123]$ is not shellable but $[123^{(3,0),(3,1),(3,2)}, 456123^{(6,0),(6,1),(6,2)}]$ is shellable, and $[321, 521643]$ is shellable but $[321^{(1,3)}, 521643^{(1,5),(1,6),(4,6)}]$ is not shellable.

We can define a direct sum operation on mesh patterns, where given two mesh patterns s and t, where the top right corner of s and bottom left corner of t are not shaded, the direct sum $s \oplus t$ has classical pattern $\mathrm{cl}(s) \oplus \mathrm{cl}(t)$ and the shadings are given by placing t north east of s and shading any borders so they extend to the edge. We can similarly define the skew-sum.

Lemma 4.1. *If m is indecomposable and $(0, 0), (|m|, |m|) \notin sh(m)$, then $[m, m \oplus m]$ is disconnected.*

Proof. There are exactly two occurrences of m in $m \oplus m$, the first $|m|$ letters η_1 or the last $|m|$ letters η_2. If you delete a point or deshade any box that is not in η_1 then that point or box must be part of η_2, so the resulting mesh pattern only contains one occurrence of m which corresponds to η_1 so then you can only delete points or shadings that are not part of η_1 so must be part of η_2. A similar argument applies if you initial delete a point or shading not part of η_2. Therefore, any two points/shading removals must both be part of η_1 or η_2, thus the poset can be split into components where on one side we remove elements not in η_1 and the other elements not in η_2. □

Corollary 4.2. *If m is indecomposable and $(|m|, 0), (0, |m|) \notin sh(m)$, then $[m, m \ominus m]$ is disconnected.*

A analogous result is used in the classical case in [MS15] to show that almost all intervals of the classical permutation poset are not shellable. The proof of this follows form the Marcus-Tardos theorem, we have seen this result does not apply in the mesh pattern case so we cannot prove a similar result using this technique. A similar problem was studied for boxed mesh patterns in permutations in [AKV13], which is equivalent to boxed mesh patterns in fully shaded mesh patterns. So we leave it as an open question

Question 4.3. *What proportion of intervals of $\mathcal{M}$ are shellable?*

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