Mathematics 4MB3 Assignment 2

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Question One

Part a

Let the total infectious period be the time between when an individual enters I_1 and leaves I_n. Assume that at time $t = 0$, we have I_{1_0} infectives in the first serially linked infectious compartment, I_1. Then, if we prevent contact with any susceptible individuals, the differential equation for I_1 becomes

$$\frac{dI}{dt} = n\gamma I_1$$

If we solve this differential equation using separation of variables, we obtain

$$I_1(t) = I_{1_0}e^{-n\gamma t}$$

If at time t, $I_1(t) = I_{1_0}e^{-n\gamma t}$ individuals are in I_1, then after time t, the proportion of individuals in I_1 is reduced by a factor of $e^{-n\gamma t}$, so that the proportion of individuals who have an infectious period shorter than t is $1 - e^{-n\gamma t}$. Since this is the cumulative density function of the time an individual spends in I_1, the probability density function is the derivative of this function, or $n\gamma e^{-n\gamma t}$. The mean of this distribution is.

$$\int_0^\infty tn\gamma e^{-n\gamma t}dt = \frac{1}{\gamma n}$$

So the mean time spent in I_1 is $\frac{1}{\gamma n}$ Since the removal rate for every infective compartment is the same, the average person should spend the same amount of time in each compartment. Thus, the total infectious period is $n\frac{1}{\gamma n} = \frac{1}{\gamma}$, which is unchanged from the standard SIR model with one infectious compartment.

Part b

$\mathcal{R}_0$ captures the speed of infectious disease spread, which the linear chain trick does not change, when applied to a model. By expanding I into n sub compartments, but forcing people to move between the compartments at n times the normal rate, the extra compartments do not change the speed of disease spread in the model. In addition, since the sum of the infectious compartments $\sum_1^n I_i$ is the same as I in the standard SIR model, the rate of new infections is identical in both models, which means $\mathcal{R}_0$ should be too.