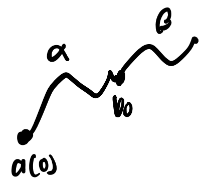
Section 54, #3,5,6



$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

③ Let $p: E \to B$ be a covering map with E, B top. spaces.

Let $\alpha, \beta : I \to B$ be paths in B with $\alpha(1) = \beta(0)$.

Let $\tilde{\alpha}, \hat{\beta} : I \to E$ be liftings of α, β resp. s.t. $\tilde{\alpha}(1) = \hat{\beta}(0)$

s.t. $\alpha = p \circ \tilde{\alpha}$ and $\beta = p \circ \tilde{\beta}$.

(1) Since $\tilde{\alpha}(1) = \tilde{\beta}(1)$, $\tilde{\alpha} * \tilde{\beta}$ is well-defined.

(2) Define $f = \alpha * \beta = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \beta(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$ and $g = \tilde{\alpha} * \tilde{\beta} = \begin{cases} \tilde{\alpha}(2s) & s \in [0, \frac{1}{2}] \\ \hat{\beta}(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$.

Show $f = p \circ g$

For $s \in [0, \frac{1}{2}]$, $(p \circ g)(s) = p(g(s)) = p(\tilde{\alpha}(s)) = \alpha(s)$.

For $s \in [\frac{1}{2}, 1]$, $(p \circ g)(s) = p(g(s)) = p(\tilde{\beta}(s)) = \beta(s)$.

$\therefore$ g is a lifting of f.

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MTH 532 Topology II Homework 1.

Problem 1.

Recall that $\alpha = p \circ \tilde{\alpha}$ and $\beta = p \circ \tilde{\beta}$ by definition of lifting.

Since $\alpha(1) = \beta(0)$, $\alpha * \beta$ is well-defined. Similarly, $\tilde{\alpha} \circ \tilde{\beta}$ is well-defined.

Define $f = \alpha * \beta = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$ and $g = \tilde{\alpha} \circ \tilde{\beta} = \begin{cases} \tilde{\alpha}(2t) & t \in [0, \frac{1}{2}] \\ \tilde{\beta}(2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$

To show that $g: I \to E$ is a lifting of $f: I \to B$, it suffices to show that $f = p \circ g$.

For $t \in [0, \frac{1}{2}]$, $(p \circ g)(t) = p(g(t)) = p(\tilde{\alpha}(2t)) = \alpha(2t) = f(t)$.

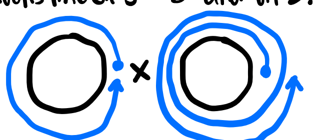
For $t \in [\frac{1}{2}, 1]$, $(p \circ g)(t) = p(g(t)) = p(\tilde{\beta}(2t-1)) = \beta(2t-1) = f(t)$.

$\therefore$ g is a lifting of f.

Problem 2.

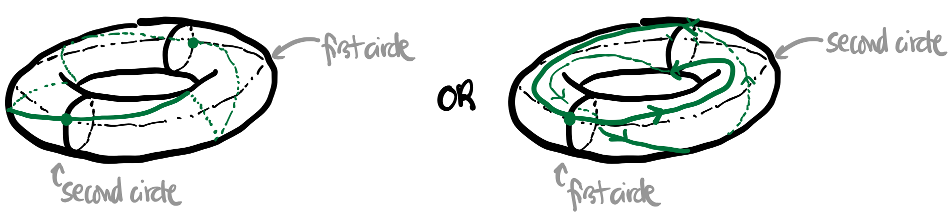
Part (1). Sketch what f looks like on $S^1 \times S^1$ and on D.

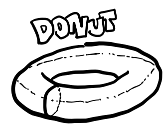
On $S^1 \times S^1$:



Here, the path loops around the first circle once and around the second circle twice.

On D:





Part (2). Find a lifting $\tilde{f}$ of f from $\mathbb{R} \times \mathbb{R}$.

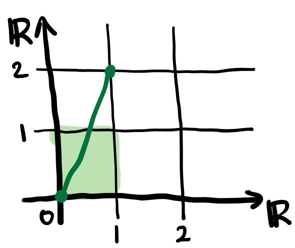
From Theorem 53.1, $p: \mathbb{R} \to S^1$ is given by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$.

We claim that $\tilde{f}: I \to \mathbb{R} \times \mathbb{R}$ given by $t \mapsto (t, 2t)$ is a lifting of f.

And so, $((p \times p) \circ \tilde{f})(t) = (p \times p)(\tilde{f}(t)) = (p \times p)(t, 2t) = (\cos(2\pi t), \sin(2\pi t)) \times (\cos(4\pi t), \sin(4\pi t)) = f(t)$.

Although, $\tilde{f}': I \to \mathbb{R} \times \mathbb{R}$ given by $t \mapsto (t+a, 2t+b)$ with $a, b \in \mathbb{Z}$ also works in general.

Part (3). Sketch of $\tilde{f}$.



Problem 3.

Fix $x_0 = e^0 = 1 \in S^1$ to be the base point.

Part (1). Consider the map $g(z) = z^n$.

Then, $g_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is given by $[f] \mapsto [g \circ f]$.

Recall that $\pi_1(S^1, 1) \cong \mathbb{Z}$ given by the covering map $p : \mathbb{R} \to S^1$ given by $x \mapsto e^{i2\pi x}$ and its lifting correspondence $\phi : \pi_1(S^1, 1) \to p^{-1}(1) \cong \mathbb{Z}$ given by $\phi([f]) = \tilde{f}(1)$ where $\tilde{f}$ is the lifting of f.

To compute g_*, we'll map $\mathbb{Z} \cong p^{-1}(1) \xrightarrow{\phi^{-1}} \pi_1(S^1, 1) \xrightarrow{g_*} \pi_1(S^1, 1) \xrightarrow{\phi} p^{-1}(1) \cong \mathbb{Z}$.

Also, recall that $\pi_1(S^1, 1) \cong \mathbb{Z}$ is an infinite cyclic group. So, $\mathbb{Z} = \langle 1 \rangle$ and $\pi_1(S^1, 1) = \langle f \rangle$ with $f : I \to S^1$ given by $t \mapsto e^{2\pi i t}$.

Since g_* is a homomorphism, it suffices to show what $g_*([f])$ is to compute g_*.

So, $g_*([f]) = [g \circ f]$ and $(g \circ f)(t) = g(e^{i2\pi t}) = e^{i2\pi tn}$ with $t \in I$ and $g \circ f : I \to S^1$.

Observe that $\widetilde{g \circ f} : I \to \mathbb{R}$ given by $t \mapsto nt$ is a lifting of $g \circ f$.

Then, $\phi([g \circ f]) = \widetilde{g \circ f}(1) = n$.

$\therefore g_*([1]) = g_*([f]) = n$.

$\therefore \forall a \in \mathbb{Z}$, $g_*(a) = an$. That is, if f is a loop based at 1 on S^1, $g \circ f$ is the loop repeated n times.

Part (2). Consider the map $h(z) = \frac{1}{z^n} = z^{-n}$.

By a similar argument, $\forall a \in \mathbb{Z}$, $h_*(a) = -an$.

That is, if f is a loop based at 1 on S^1, $h \circ f$ is the reversed loop repeated n times.