Math 115AH: Homework set 2

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Problem 1

Let P_2 = the set of polynomials $p(x)$, with coefficients in $\mathbb{R}$ and degree ≤ 2. Show that P_2 is a vector space over $\mathbb{R}$ of dimension 3.

For P_2 to be a vector space, it must satisfy the following 9 axioms.
The polynomials are in the form of $p_n = a_n x^2 + b_n x + c_n$. Since the RHS is in $\mathbb{R}$, terms of a, b, and c can be added element-wise and follow the existing rules of the established space.
The scalars, for consistency, are represented as $s_n \in \mathbb{R}$.

Commutativity of Addition: $p_1 + p_2 = (a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = (a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2) = (a_2 + a_1)x^2 + (b_2 + b_1)x + (c_2 + c_1) = (a_2x^2 + b_2x + c_2) + (a_1x^2 + b_1x + c_1) = p_2 + p_1$
Associativity of Addition: $p_1 + (p_2 + p_3) = (a_1x^2 + b_1x + c_1) + ((a_2x^2 + b_2x + c_2) + (a_3x^2 + b_3x + c_3)) = (a_1x^2 + b_1x + c_1) + ((a_2 + a_3)x^2 + (b_2 + b_3)x + (c_2 + c_3)) = (a_1 + a_2 + a_3)x^2 + (b_1 + b_2 + b_3)x + (c_1 + c_2 + c_3) = ((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)) + (a_3x^2 + b_3x + c_3) = ((a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) + (a_3x^2 + b_3x + c_3) = (p_1 + p_2) + p_3$
Identity of Addition: $0 = 0x^2 + 0x + 0$; $0 + p_1 = (0x^2 + 0x + 0) + (a_1x^2 + b_1x + c_1) = (0 + a_1)x^2 + (0 + b_1)x + (0 + c_1) = a_1x^2 + b_1x + c_1 = p_1$
Inverse of Addition: $-p_n = -a_nx^2 - b_nx - c_n$; $p_n + -p_n = (a_nx^2 + b_nx + c_n) + (-a_nx^2 - b_nx - c_n) = (a_n - a_n)x^2 + (b_n - b_n)x + (c_n - c_n) = 0x^2 + 0x + 0 = 0$
Identity of Multiplication: $1 * p_n = 1 * (a_nx^2 + b_nx + c_n) = (1 * a_n)x^2 + (1 * b_n)x + (1 * c_n) = a_nx^2 + b_nx + c_n = p_n$
Associativity of Multiplication: $(s_1s_2)p_n = s_1s_2(a_nx^2 + b_nx + c_n) = (s_1s_2a_n)x^2 + (s_1s_2b_n)x + (s_1s_2c_n) = s_1((s_2a_n)x^2 + (s_2b_n)x + (s_2c_n)) = s_1(s_2(a_nx^2 + b_nx + c_n)) = s_1(s_2p_n)$
Distribution over Vectors: $s_n(p_1 + p_2) = s_n((a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) = s_n((a_1 + a_2)x^2 + (b_1 + b_2)x + (c_1 + c_2)) = (s_na_1 + s_na_2)x^2 + (s_nb_1 + s_nb_2)x + (s_nc_1 + s_nc_2) = (s_na_1)x^2 + (s_nb_1)x + (s_nc_1) + (s_na_2)x^2 + (s_nb_2)x + (s_nc_2) = s_n(a_1x^2 + b_1x + c_1) + s_n(a_2x^2 + b_2x + c_2) = s_np_1 + s_np_2$
Distribution over Scalars: $(s_1 + s_2)p_n = (s_1 + s_2)(a_nx^2 + b_nx + c_n) = (s_1a_n + s_2a_n)x^2 + (s_1b_n + s_2b_n)x + (s_1c_n + s_2c_n) = (s_1a_n)x^2 + (s_1b_n)x + (s_1c_n) + (s_2a_n)x^2 + (s_2b_n)x + (s_2c_n) = s_1(a_nx^2 + b_nx + c_n) + s_2(a_nx^2 + b_nx + c_n) = s_1p_n + s_2p_n$

Therefore, P_2 is a vector space over $\mathbb{R}$ of dimension 3.

Problem 2

Is $\{x(x-1), (x-1)(x+1), x(x+1)\}$ a basis for P_2? Justify using a proof.

$\{x(x-1), (x-1)(x+1), x(x+1)\} = \{x^2 - x, x^2 - 1, x^2 + x\}$.
Using the form established in Problem 1 ($x^2, x, 1$), this can then be put into matrix form, and Gaussian Elimination can be used (by setting it equal to zero) to simplify it to Reduced Row-Echelon Form:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since this results in the identity matrix, the only way to achieve zero would be the single linear combination with all values being zero. So, by definition, the vectors are linearly independent.
Since it has 3 linearly independent vectors (dimension of 3), the set fully spans the 3-dimensional P_2, thus, making it a basis for P_2.

Problem 3

Subproblem 1

Use Problem 2 to show that given $a_1, a_2, a_3 \in \mathbb{R}$, $\exists\ P \in P_2 \ni P(-1) = a_1; P(0) = a_2; P(1) = a_3$.

We establish in Problem 2, by the definition of a basis, that all $P \in P_2$ can be represented as a linear combination of the vectors. Therefore, nothing would stop us from plugging in the values of: $P(x) = \frac{1}{2}a_1(x(x-1)) + a_2(x-1)(x+1) + \frac{1}{2}a_3(x(x+1))$

Therefore:
$P(-1) = \frac{1}{2}a_1(-1(-1-1)) + a_2(-1-1)(-1+1) + \frac{1}{2}a_3(-1(-1+1)) = a_1$
$P(0) = \frac{1}{2}a_1(0(0-1)) + a_2(0-1)(0+1) + \frac{1}{2}a_3(0(0+1)) = a_2$
$P(1) = \frac{1}{2}a_1(1(1-1)) + a_2(1-1)(1+1) + \frac{1}{2}a_3(1(1+1)) = a_3$

Subproblem 2

Is P in Subproblem 1 unique?

Yes, each P is unique, by the prior definition, since it only takes 3 points to define a polynomial with degree ≤ 2. Also, since the values of a directly generate a linear combination of independent vectors, multiple sets of points couldn't correspond to the same polynomial.

Problem 4

Find α, β, γ such that for all $P \in P_2$, $\int_{-1}^{+1} P(x)dx = \alpha P(-1) + \beta P(0) + \gamma P(1)$

For convenience, let's choose the same basis and linear combination from Problem 3, so we can use the established fact that $P(-1) = a_1; P(0) = a_2; P(1) = a_3$:
$\int_{-1}^{+1} P(x)dx = \alpha P(-1) + \beta P(0) + \gamma P(1) \implies \int_{-1}^{+1}(\frac{1}{2}(x^2-x)a_1 + (x^2-1)a_2 + \frac{1}{2}(x^2+x)a_3)dx = \alpha a_1 + \beta a_2 + \gamma a_3$

$\alpha a_1 + \beta a_2 + \gamma a_3 = \frac{1}{2}a_1 \int_{-1}^{+1}(x^2-x)dx + a_2 \int_{-1}^{+1}(x^2-1)dx + \frac{1}{2}a_3 \int_{-1}^{+1}(x^2+x)dx$

Separating by elements of each a:
$\alpha a_1 = \frac{1}{2}a_1 \int_{-1}^{+1}(x^2-x)dx$; $\beta a_2 = a_2 \int_{-1}^{+1}(x^2-1)dx$; $\gamma a_3 = \frac{1}{2}a_3 \int_{-1}^{+1}(x^2+x)dx$
$\alpha = \frac{1}{2}\int_{-1}^{+1}(x^2-x)dx$; $\beta = \int_{-1}^{+1}(x^2-1)dx$; $\gamma = \frac{1}{2}\int_{-1}^{+1}(x^2+x)dx$

Problem 5

Generalize Problems 1 and 2 (and 3, if possible) to $P_n =$ the set of polynomials of degree $\leq n$, $n \geq 2$.

For P_n to be a vector space, it must satisfy the following 9 axioms.
The polynomials, $(p_f \in P_n)$, are in the form of $p_f = \{t_f \in P_{n-1}\} + z_f x^n$. Our notation of $\{t_f \in P_{n-1}\}$ will represent the terms of p_f that are shared with P_{n-1}, which we will assume is an established vector space. Since they are part of a vector space, they can already be assumed to have the expected properties. The RHS is in $\mathbb{R}$, so terms can be added element-wise and follow the existing rules of the established space (meaning we can keep z_f and $\{t_f \in P_{n-1}\}$ separate). The scalars, for consistency, are represented as $s_f \in \mathbb{R}$.

Commutativity of Addition: $p_1 + p_2 = (\{t_1 \in P_{n-1}\} + z_1x^n) + (\{t_2 \in P_{n-1}\}x + z_2x^n) = \{t_1 + t_2 \in P_n\} + (z_1+z_2)x^n = \{t_2+t_1 \in P_{n-1}\} + (z_2+z_1)x^n = (\{t_2 \in P_{n-1}\}x + z_2x^n) + (\{t_1 \in P_{n-1}\} + z_1x^n) = p_2 + p_1$

Associativity of Addition: $p_1 + (p_2 + p_3) = (\{t_1 \in P_{n-1}\} + z_1x^n) + ((\{t_2 \in P_{n-1}\}x + z_2x^n) + (\{t_3 \in P_{n-1}\}x + z_3x^n)) = (\{t_1 \in P_{n-1}\} + z_1x^n) + (\{t_2 + t_3 \in P_{n-1}\} + (z_2 + z_3)x^n) = \{t_1 + t_2 + t_3 \in P_{n-1}\} + (z_1 + z_2 + z_3)x^n = (\{t_1 + t_2 \in P_n\} + (z_1 + z_2)x^n) + (\{t_3 \in P_{n-1}\}x + z_3x^n) = ((\{t_1 \in P_{n-1}\} + z_1x^n) + (\{t_2 \in P_{n-1}\}x + z_2x^n)) + (\{t_3 \in P_{n-1}\}x + z_3x^n) = (p_1 + p_2) + p_3$
Identity of Addition: $0 = \{0 \in P_{n-1}\} + 0x^n$; $0 + p_1 = (\{0 \in P_{n-1}\} + 0x^n) + (\{t_1 \in P_{n-1}\} + z_1x^n) = \{t_1 \in P_{n-1}\} + (0 + z_1)x^n = \{t_1 \in P_{n-1}\} + z_1x^n = p_1$
Inverse of Addition: $-p_f = \{-t_f \in P_{n-1}\} + -z_fx^n$; $p_f + -p_f = (\{t_f \in P_{n-1}\} + z_fx^n) + (\{-t_f \in P_{n-1}\} + -z_fx^n) = (\{t_f - t_f \in P_{n-1}\}) + (z_f - z_f)x^n = \{0 \in P_{n-1}\} + 0x^n = 0$
Identity of Multiplication: $1 * p_f = 1 * (\{t_f \in P_{n-1}\} + z_fx^n) = \{1 * t_f \in P_{n-1}\} + (1 * z_f)x^n = \{t_f \in P_{n-1}\} + z_fx^n = p_f$
Associativity of Multiplication: $(s_1s_2)p_f = s_1s_2(\{t_f \in P_{n-1}\} + z_fx^n) = \{s_1s_2t_f \in P_{n-1}\} + (s_1s_2z_f)x^n = s_1\{s_2t_f \in P_{n-1}\} + s_1(s_2z_f)x^n = s_1(s_2(\{t_f \in P_{n-1}\} + z_fx^n)) = s_1(s_2p_f)$
Distribution over Vectors: $s_f(p_1+p_2) = s_f((\{t_1 \in P_{n-1}\}+z_1x^n)+(\{t_2 \in P_{n-1}\}x+z_2x^n)) = s_f(\{t_1+t_2 \in P_n\} + (z_1 + z_2)x^n) = \{s_ft_1 + s_ft_2 \in P_n\} + (s_fz_1 + s_fz_2)x^n = \{s_ft_1 \in P_n\} + (s_fz_1)x^n + \{s_ft_2 \in P_n\} + (s_fz_2)x^n = s_f(\{t_1 \in P_{n-1}\} + z_1x^n) + s_f(\{t_2 \in P_{n-1}\}x + z_2x^n) = s_fp_1 + s_fp_2$
Distribution over Scalars: $(s_1 + s_2)p_f = (s_1 + s_2)(\{t_f \in P_{n-1}\} + z_fx^n) = \{s_1t_f + s_2t_f \in P_{n-1}\} + (s_1z_f + s_2z_f)x^n = \{s_1t_f \in P_{n-1}\} + \{s_2t_f \in P_{n-1}\} + s_1z_fx^n + s_2z_fx^n = s_1(\{t_f \in P_{n-1}\} + z_fx^n) + s_2(\{t_f \in P_{n-1}\} + z_fx^n) = s_1p_f + s_2p_f$

Therefore, P_n is a vector subspace over $\mathbb{R}$ of dimension $n + 1$.

It is also possible to create a basis for P_n by using the same form as seen in Problem 2.
All we'd have to do is use the basis of P_{n-1} (which we assume exists in the form) and take all unique factors that exist throughout the n amount of existing terms, multiplying them together to append as a new term. Then, multiply each previously existing term by $(x - (n - 1))$.
Effectively, this transforms the already linearly independent vectors (by definition) of P_{n-1}, so they are still linearly independent. The added term has to be linearly independent from the others since all others include a factor of $(x - (n - 1))$, so linear combinations of a nontrivial last term are the only way to define a nontrivial value for when $x = n-1$, proving that it's not dependent on the others. By extension, the same logic applies to all of the previous terms since they each can also be defined by being the only term that doesn't have a factor of $(x - k)$.
Since we have a set of $n + 1$ linearly independent vectors, matching the $n + 1$ dimensional P_n, the set must fully span P_n, making it a basis.

Because of the unique form of the basis, we can also generalize Problem 3 into defining $n + 1$ values of a such that $P(k) = a_{k+2}$.
Since we've maintained the property that all terms can be defined by being the only term that doesn't have a factor of $(x - k)$, $P(k)$ results in all other terms being zeroed out, and the term itself evaluating to a singular value multiplied by whatever the combination coefficient will be. We can then set the coefficient within our linear combination as the reciprocal of the evaluated value times a_{k+2}, ensuring that $P(k) = a_{k+2}$.
This can be repeated for all terms to correctly set the linear combination to fulfill the stated conditions.
The P, defined by $n + 1$ a values is also unique because all polynomials of degree n can be defined by $n + 1$ points.

Since the initial solutions of Problems 1, 2, and 3 established the properties for $n = 2$, and we have just proved them again for any n under the assumption of $n - 1$ fulfilling them, then by induction, the properties must hold for all $n \geq 2$.

Problem 6

Do problem 3 with $-1, 0, 1$ replaced by an arbitrary set of three numbers x_1, x_2, x_3, all different.

We can instead choose the basis of $\{(x-x_2)(x-x_3), (x-x_1)(x-x_3), (x-x_1)(x-x_2)\}$. Define the kernel as any linear combination that will result in polynomial $0x^2 + 0x + 0$, which has the unique property that all values of x lead to zero.

For each term of position $k \in \{1, 2, 3\}$, it's the only one that doesn't have $(x - x_k)$ as a factor. The zero polynomial would lead any value of x to zero, including x_k. However, $x = x_k$ would lead to all elements other than the 'k'th being evaluated to zero, so the only linear combination that would achieve the zero polynomial would be with the coefficient for k being zero. Since this logic applies for all terms, the kernel must only include the single trivial linear combination, indicating that the terms are linearly independent.

Since it's now established as a basis, all $P \in P_2$ can be represented as a linear combination of the vectors. Therefore, nothing would stop us from plugging in the values of:
$P(x) = \frac{a_1}{(x_1-x_2)(x_1-x_3)}(x-x_2)(x-x_3) + \frac{a_2}{(x_2-x_1)(x_2-x_3)}(x-x_1)(x-x_3) + \frac{a_3}{(x_3-x_1)(x_3-x_2)}(x-x_1)(x-x_2)$

So:
$P(x_1) = \frac{a_1}{(x_1-x_2)(x_1-x_3)}(x_1-x_2)(x_1-x_3) + \frac{a_2}{(x_2-x_1)(x_2-x_3)}(x_1-x_1)(x_1-x_3) + \frac{a_3}{(x_3-x_1)(x_3-x_2)}(x_1-x_1)(x_1-x_2) = a_1$
$P(x_2) = \frac{a_1}{(x_1-x_2)(x_1-x_3)}(x_2-x_2)(x_2-x_3) + \frac{a_2}{(x_2-x_1)(x_2-x_3)}(x_2-x_1)(x_2-x_3) + \frac{a_3}{(x_3-x_1)(x_3-x_2)}(x_2-x_1)(x_2-x_2) = a_2$
$P(x_3) = \frac{a_1}{(x_1-x_2)(x_1-x_3)}(x_3-x_2)(x_3-x_3) + \frac{a_2}{(x_2-x_1)(x_2-x_3)}(x_3-x_1)(x_3-x_3) + \frac{a_3}{(x_3-x_1)(x_3-x_2)}(x_3-x_1)(x_3-x_2) = a_3$

Problem 7

Show that if x_1, x_2, x_3 are all different, $det \begin{pmatrix} 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_3 & {x_3}^2 \end{pmatrix} \neq 0$

$det \begin{pmatrix} 1 & x_1 & {x_1}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_3 & {x_3}^2 \end{pmatrix} = 1(x_2{x_3}^2 - {x_2}^2x_3) - x_1({x_3}^2 - {x_2}^2) + {x_1}^2(x_3 - x_2) = x_2{x_3}^2 - {x_2}^2x_3 - x_1{x_3}^2 + x_1{x_2}^2 + {x_1}^2x_3 - {x_2}^2x_3 = (x_1 - x_2)(-{x_3}^2 - x_1x_2 + x_3(x_1 + x_2)) = -(x_1 - x_2)(x_3 - x_1)(x_3 - x_2)$

Since we're given that all x_1, x_2, x_3 are different, we therefore know that the difference between them is a nonzero value. The determinant can be simplified into only factors of these differences, so there is no way for the determinant to equal zero.