Math 115AH: Homework set 1

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Problem 1

Show 1, 2, 3, ... , 10 have inverses (mod 11)

$1 \times 1 \equiv 1 \pmod{11} \implies 1^{-1} = 1$
$2 \times 6 \equiv 1 \pmod{11} \implies 2^{-1} = 6$
$3 \times 4 \equiv 1 \pmod{11} \implies 3^{-1} = 4$
$4 \times 3 \equiv 1 \pmod{11} \implies 4^{-1} = 3$
$5 \times 9 \equiv 1 \pmod{11} \implies 5^{-1} = 9$
$6 \times 2 \equiv 1 \pmod{11} \implies 6^{-1} = 2$
$7 \times 8 \equiv 1 \pmod{11} \implies 7^{-1} = 8$
$8 \times 7 \equiv 1 \pmod{11} \implies 8^{-1} = 7$
$9 \times 5 \equiv 1 \pmod{11} \implies 9^{-1} = 5$
$10 \times 10 \equiv 1 \pmod{11} \implies 10^{-1} = 10$

Problem 2

Subproblem 1

Prove: $[m][n] = [mn]$

By definition: $[m][n] = (\sum_{j=1}^{m} 1)(\sum_{j=1}^{n} 1)$
So by the Distributive Property: $[m][n] = \sum_{j=1}^{n}(\sum_{j=1}^{m} 1)(1)$
As follows by the Identity Property: $[m][n] = \sum_{j=1}^{n}(\sum_{j=1}^{m} 1)$
Thus, associatively: $[m][n] = \sum_{j=1}^{mn} 1$
Which, by definition: $\sum_{j=1}^{mn} 1 = [mn]$
Therefore, transitively: $[m][n] = [mn]$

Subproblem 2

Prove: if $\exists\ m > 0$ such that $[m] = 0$, then the smallest positive integer $[p] = 0$ is prime.

It's given that: $m > 0$ such that $[m] = 0$
In order for $1 + 1 + ... + 1 = 0$, the field must be finite.
Since the field is finite, then it must have p^l elements (such that p is a prime and $l \in \mathbb{N}$) and $[p^l] = 0$
Following the prior proof, $[p^l]$ is the same as $[p][p]...[p]$ (l factors).
Since the product within a field can only be zero if one of the factors is also zero, it must be true that $[p] = 0$, and p is a prime number by definition.

Problem 3

Construct a field with four elements: $0, 1, x, x + 1$

+	0	1	x	x+1
0	0	1	x	x+1
1	1	0	x+1	x
x	x	x+1	0	1
x+1	x+1	x	1	0

$\times$	0	1	x	x+1
0	0	0	0	0
1	0	1	x	x+1
x	0	x	x+1	1
x+1	0	x+1	1	x

Problem 4

Prove that if S is a set and V is a vector space over a field F, then the set $f(S, V)$ representing functions from S to V, has a vector space structure defined (for all $s \in S$) by:
$(f_1 + f_2)(s) = f_1(s) + f_2(s)$ and $(\alpha f)(s) = \alpha f(s)$ for all $\alpha \in F$

It's given, in defining the vector space structure, that: $(f_1 + f_2)(s) = f_1(s) + f_2(s)$ and $(\alpha f)(s) = \alpha f(s)$ for all $\alpha \in F$, for all $s \in S$.

Suppose $f_j, f_k \in f$, so by the first given property: $(f_j + f_k)(s) = f_j(s) + f_k(s)$ and $(f_k + f_j)(s) = f_k(s) + f_j(s)$
Since the set f is a function from S to V, $f_j(s), f_k(s) \in V$, which as given, is a vector space. Therefore, they are commutative under vector addition, so: $f_j(s) + f_k(s) = f_k(s) + f_j(s)$
This can be transitively applied to the prior equations to conclude that: $(f_j + f_k)(s) = (f_k + f_j)(s)$ for all $f_j, f_k \in f$, meaning the set is commutative under vector addition.

We can also prove that for all $f_j, f_k, f_l \in f$, that $(f_j + (f_k + f_l))(s) = ((f_j + f_k) + f_l)(s)$.
We apply the first property twice, as such: $(f_j+(f_k+f_l))(s) = f_j(s)+(f_k+f_l)(s) = f_j(s)+(f_k(s)+f_l(s))$ and $((f_j + f_k) + f_l)(s) = (f_j + f_k)(s) + f_l(s) = (f_j(s) + f_k(s)) + f_l(s)$
Since the set f is a function from S to V, $f_j(s), f_k(s), f_l(s) \in V$, which as given, is a vector space. Therefore, they are associative under addition, so: $f_j(s) + (f_k(s) + f_l(s)) = (f_j(s) + f_k(s)) + f_l(s)$
This can be transitively applied to the prior equations to conclude that: $(f_j + (f_k + f_l))(s) = ((f_j + f_k) + f_l)(s)$ for all $f_j, f_k, f_l \in f$, meaning the set is associative under vector addition.

Since f represents the functions from S to V, there exists a $f_0 \in f$ such that $f_0(s) = 0$. In this case, 0 denotes whatever the additive identity is of the already established vector space V.
Using the first given property, $(f_0 + f_j)(s) = f_0(s) + f_j(s)$, along with the properties of f_0 shown earlier, when added, will result in: $(f_0 + f_j)(s) = f_j(s)$, for all $f_j \in f$, meaning that within the set, there exists an additive identity, f_0 (could also be denoted 0, when necessary), for vector addition.

Since V is already an established vector space, by definition, all elements must have additive inverses. We can then denote the additive inverse of all $f_n(s) \in V$ as $f_{-n}(s)$ for all $s \in S$.
The first given property allows us to assert that $(f_n + f_{-n})(s) = f_n(s) + f_{-n}(s)$, so therefore by the definition stated earlier, $(f_n + f_{-n})(s) = 0)$, (by definition of additive inverses), for all $f_n \in f$, meaning that for all elements of f_n, there exists an additive inverse, f_{-n}.

The established vector space, V, by definition, must have an identity for scalar multiplication that already exists within field F, which we will denote as 1.
Using the second given property, $(1 \times f)(s) = 1 \times f(s)$, along with the known properties of 1 shown earlier, will evaluate to $(1 \times f)(s) = f(s)$, meaning that within the set, there exists a multiplicative identity, 1, for scalar multiplication.

Suppose we have scalars $b, c \in F$ and function $f_n \in f$, and set up the following expression: $(b(cf_n))(s)$
Applying the second given property, we get: $(b(cf_n))(s) = b(cf_n)(s) = b(c(f_n)(s))$
Since we're now fully in V, which is a vector space, it's already been established that multiplicative associativity applies, so $b(c(f_n)(s))$ can be rearranged as: $(bc)(f_n(s))$
By the second given property, $(bc)(f_n(s)) = (bc(f_n))(s)$
Therefore, transitively: $(b(cf_n))(s) = (bc(f_n))(s)$, for all $b, c \in F$ and $f_n \in f$, meaning the set is associative under scalar multiplication.

Suppose we have scalar $b \in F$ and functions $f_j, f_k \in f$, and set up the following expression: $(b(f_j+f_k))(s)$
By applying both the given properties, we can simplify the expression as such: $(b(f_j + f_k))(s) = b((f_j + f_k)(s)) = b(f_j(s) + f_k(s))$

Since V is a known vector space, it already allows us to distribute scalar multiplication over vector addition. Therefore, $b(f_j(s) + f_k(s)) = bf_j(s) + bf_k(s)$
Applying the given properties again: $bf_j(s) + bf_k(s) = (bf_j)(s) + (bf_k)(s) = (bf_j + bf_k)(s)$
Therefore, transitively: $(b(f_j + f_k))(s) = (bf_j + bf_k)(s)$, for all $b \in F$ and $f_j, f_k \in f$, meaning that the set allows distribution of scalar multiplication over vector addition.

Suppose we have scalars $b, c \in F$ and function $f_n \in f$, and set up the following expression: $((b+c)f_n)(s)$
By applying the second given property: $((b + c)f_n)(s) = (b + c)(f_n(s))$
Since V is known vector space, it already allows us to distribute scalar multiplication over scalar addition. Therefore, $(b + c)(f_n(s)) = bf_n(s) + cf_n(s)$
Applying the given properties again: $bf_n(s) + cf_n(s) = (bf_n)(s) + (cf_n)(s) = (bf_n + cf_n)(s)$
Therefore, transitively: $((b + c)f_n)(s) = (bf_n + cf_n)(s)$, for all $b, c \in F$ and $f_n \in f$, meaning that the set allows distribution of scalar multiplication over scalar addition.

As a result of the proof of the previous 9 axioms, the set, $f(S, V)$, is indeed a vector space.

Problem 5

Prove that if V and W are vector spaces over the same field F, then the subset of $f(V, W)$ where f is linear (denoted by $f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2)$) is a vector space.

As given, f_{sub} consists only of the linear functions from V to W. Therefore, all $f_a \in f_{sub}$ fit the form $f_a(v) = \sum_{n=1}^{dim(V)} a_n v_n$ in vector space W where $v \in V$ and $v = \{v_1, v_2, ..., v_{dim(V)}\}$, and is therefore characterized by the set $a = \{a_1, a_2, ..., a_{dim(v)}\}$, where each vector is of $dim(W)$.

Since linear functions are defined by their sets, it reasonably follows that performing addition between functions involves adding corresponding elements of their sets.
Suppose, in addition to f_a, by the same definition, we have another function f_b, hence defined by the set $b = \{b_1, b_2, ..., b_{dim(v)}\}$.
Therefore, addition would be represented as: $(f_a + f_b)(v) = \sum_{n=1}^{dim(V)} (a_n + b_n)v_n$
Since this exists in W, a known vector space, distributive properties apply, so: $\sum_{n=1}^{dim(V)} (a_n + b_n)v_n = \sum_{n=1}^{dim(V)} (a_n v_n + b_n v_n)$
Due to the above logic also allowing associativity within W, we can then rearrange it as: $\sum_{n=1}^{dim(V)} (a_n v_n + b_n v_n) = \sum_{n=1}^{dim(V)} a_n v_n + \sum_{n=1}^{dim(V)} b_n v_n$
However, by the definition of a linear function itself, $\sum_{n=1}^{dim(V)} a_n v_n + \sum_{n=1}^{dim(V)} b_n v_n = f_a(v) + f_b(v)$.
Therefore, addition has the commonly accepted behaviour, such that: $(f_a + f_b)(v) = f_a(v) + f_b(v)$, which we will refer to as our first property.

By the same logic of a linear function being a set of vectors, a scalar should essentially scale each vector by that amount. We'll denote this scalar $\alpha \in F$ within the expression: $(\alpha f_a)(v)$
It logically follows that: $(\alpha f_a)(v) = \sum_{n=1}^{dim(V)} (\alpha f_n)v_n$
However, since this is in vector space W, scalar multiplication follows both associativity and distributivity, so: $\sum_{n=1}^{dim(V)} (\alpha f_n)v_n = \sum_{n=1}^{dim(V)} \alpha(f_n v_n) = \alpha \sum_{n=1}^{dim(V)} f_n v_n$
And by definition: $\alpha \sum_{n=1}^{dim(V)} f_n v_n = \alpha f_a(v)$
So, scalar multiplication also has the commonly accepted behaviour, such that: $(\alpha f_a)(v) = \alpha f_a(v)$, which we will refer to as our second property.

Suppose $f_j, f_k \in f_{sub}$, so by the first property: $(f_j + f_k)(v) = f_j(v) + f_k(v)$ and $(f_k + f_j)(v) = f_k(v) + f_j(v)$
Since the subset f_{sub} is a function from V to W, $f_j(v), f_k(v) \in W$, which as given, is a vector space. Therefore, they are commutative under vector addition, so: $f_j(v) + f_k(v) = f_k(v) + f_j(v)$
This can be transitively applied to the prior equations to conclude that: $(f_j + f_k)(v) = (f_k + f_j)(v)$ for all $f_j, f_k \in f_{sub}$, meaning the subset is commutative under vector addition.

We can also prove that for all $f_j, f_k, f_l \in f_{sub}$, that $(f_j + (f_k + f_l))(v) = ((f_j + f_k) + f_l)(v)$.
We apply the first property twice, as such: $(f_j + (f_k + f_l))(v) = f_j(v) + (f_k + f_l)(v) = f_j(v) + (f_k(v) + f_l(v))$ and $((f_j + f_k) + f_l)(v) = (f_j + f_k)(v) + f_l(v) = (f_j(v) + f_k(v)) + f_l(v)$
Since the subset f_{sub} is a function from V to W, $f_j(v), f_k(v), f_l(v) \in W$, which as given, is a vector space. Therefore, they are associative under addition, so: $f_j(v) + (f_k(v) + f_l(v)) = (f_j(v) + f_k(v)) + f_l(v)$

This can be transitively applied to the prior equations to conclude that: $(f_j + (f_k + f_l))(v) = ((f_j + f_k) + f_l)(v)$ for all $f_j, f_k, f_l \in f_{sub}$, meaning the subset is associative under vector addition.

Since f_{sub} represents the functions from V to W, there exists a $f_0 \in f_{sub}$ such that $f_0(v) = 0$. In this case, 0 denotes whatever the additive identity is of the already established vector space W.
Using the first property, $(f_0 + f_j)(v) = f_0(v) + f_j(v)$, along with the properties of f_0 shown earlier, when added, will result in: $(f_0 + f_j)(v) = f_j(v)$, for all $f_j \in f_{sub}$, meaning that within the subset, there exists an additive identity, f_0 (could also be denoted 0, when necessary), for vector addition.

Since W is already an established vector space, by definition, all elements must have additive inverses. We can then denote the additive inverse of all $f_n(v) \in W$ as $f_{-n}(v)$ for all $v \in V$.
The first property allows us to assert that $(f_n + f_{-n})(v) = f_n(v) + f_{-n}(v)$, so therefore by the definition stated earlier, $(f_n + f_{-n})(v) = 0)$, (by definition of additive inverses), for all $f_n \in f_{sub}$, meaning that for all elements of f_n, there exists an additive inverse, f_{-n}.

The established vector space, W, by definition, must have an identity for scalar multiplication that already exists within field F, which we will denote as 1.
Using the second property, $(1 \times f)(v) = 1 \times f(v)$, along with the known properties of 1 shown earlier, will evaluate to $(1 \times f)(v) = f(v)$, meaning that within the subset, there exists a multiplicative identity, 1, for scalar multiplication.

Suppose we have scalars $b, c \in F$ and function $f_n \in f_{sub}$, and set up the following expression: $(b(cf_n))(v)$
Applying the second property, we get: $(b(cf_n))(v) = b(cf_n)(v) = b(c(f_n)(v))$
Since we're now fully in W, which is a vector space, it's already been established that multiplicative associativity applies, so $b(c(f_n)(v))$ can be rearranged as: $(bc)(f_n(v))$
By the second property, $(bc)(f_n(v)) = (bc(f_n))(v)$
Therefore, transitively: $(b(cf_n))(v) = (bc(f_n))(v)$, for all $b, c \in F$ and $f_n \in f_{sub}$, meaning the subset is associative under scalar multiplication.

Suppose we have scalar $b \in F$ and functions $f_j, f_k \in f_{sub}$, and set up the following expression: $(b(f_j + f_k))(v)$
By applying both the properties, we can simplify the expression as such: $(b(f_j+f_k))(v) = b((f_j+f_k)(v)) = b(f_j(v) + f_k(v))$
Since W is a known vector space, it already allows us to distribute scalar multiplication over vector addition. Therefore, $b(f_j(v) + f_k(v)) = bf_j(v) + bf_k(v)$
Applying the properties again: $bf_j(v) + bf_k(v) = (bf_j)(v) + (bf_k)(v) = (bf_j + bf_k)(v)$
Therefore, transitively: $(b(f_j + f_k))(v) = (bf_j + bf_k)(v)$, for all $b \in F$ and $f_j, f_k \in f_{sub}$, meaning that the subset allows distribution of scalar multiplication over vector addition.

Suppose we have scalars $b, c \in F$ and function $f_n \in f_{sub}$, and set up the following expression: $((b+c)f_n)(v)$
By applying the second property: $((b + c)f_n)(v) = (b + c)(f_n(v))$
Since W is known vector space, it already allows us to distribute scalar multiplication over scalar addition. Therefore, $(b + c)(f_n(v)) = bf_n(v) + cf_n(v)$
Applying the properties again: $bf_n(v) + cf_n(v) = (bf_n)(v) + (cf_n)(v) = (bf_n + cf_n)(v)$
Therefore, transitively: $((b + c)f_n)(v) = (bf_n + cf_n)(v)$, for all $b, c \in F$ and $f_n \in f_{sub}$, meaning that the subset allows distribution of scalar multiplication over scalar addition.

As a result of the proof of the previous 9 axioms, the subset of $f(V, W)$ is a vector space.

Problem 6

Show that the subset of $f([0, 1], \mathbb{R})$ consisting of continuous functions from $[0, 1]$ to $\mathbb{R}$ is a vector space.

Because the functions within the subset are continuous within the range $[0, 1]$, by definition, each function has a continuous set of values within $\mathbb{R}$, corresponding to all possible inputs within the domain.

Therefore, we can directly add these sets of values to, in turn, add functions.
By matching them up for all $s \in [0, 1]$, because it's in $\mathbb{R}$, the vector space properties lead it to result in the same value as would be obtained by adding the functions after they are evaluated, so: $(f_1 + f_2)(s) =$

$f_1(s) + f_2(s)$, which we'll call our first property.

By similar logic, we can also directly multiply functions by a scalar.
By multiplying each element of the subset within $\mathbb{R}$ by the scalar, the vector space properties again result in the same exact value as would be obtained by multiplying with the scalar after evaluating the function, so: $(\alpha f_n)(s) = \alpha f_n(s)$, which we'll call our second property.

Consider the following with $s \in [0, 1]$.

Suppose $f_j, f_k \in f$, so by the first property: $(f_j + f_k)(s) = f_j(s) + f_k(s)$ and $(f_k + f_j)(s) = f_k(s) + f_j(s)$
Since the subset f is a function from $[0, 1]$ to $\mathbb{R}$, $f_j(s), f_k(s) \in \mathbb{R}$, which as given, is a vector space. Therefore, they are commutative under vector addition, so: $f_j(s) + f_k(s) = f_k(s) + f_j(s)$
This can be transitively applied to the prior equations to conclude that: $(f_j + f_k)(s) = (f_k + f_j)(s)$ for all $f_j, f_k \in f$, meaning the subset is commutative under vector addition.

We can also prove that for all $f_j, f_k, f_l \in f$, that $(f_j + (f_k + f_l))(s) = ((f_j + f_k) + f_l)(s)$.
We apply the first property twice, as such: $(f_j + (f_k + f_l))(s) = f_j(s) + (f_k + f_l)(s) = f_j(s) + (f_k(s) + f_l(s))$ and $((f_j + f_k) + f_l)(s) = (f_j + f_k)(s) + f_l(s) = (f_j(s) + f_k(s)) + f_l(s)$
Since the subset f is a function from $[0, 1]$ to $\mathbb{R}$, $f_j(s), f_k(s), f_l(s) \in \mathbb{R}$, which as given, is a vector space. Therefore, they are associative under addition, so: $f_j(s) + (f_k(s) + f_l(s)) = (f_j(s) + f_k(s)) + f_l(s)$
This can be transitively applied to the prior equations to conclude that: $(f_j + (f_k + f_l))(s) = ((f_j + f_k) + f_l)(s)$ for all $f_j, f_k, f_l \in f$, meaning the subset is associative under vector addition.

Since f represents the functions from $[0, 1]$ to $\mathbb{R}$, there exists a $f_0 \in f$ such that $f_0(s) = 0$. In this case, 0 denotes whatever the additive identity is of the already established vector space $\mathbb{R}$.
Using the first property, $(f_0 + f_j)(s) = f_0(s) + f_j(s)$, along with the properties of f_0 shown earlier, when added, will result in: $(f_0 + f_j)(s) = f_j(s)$, for all $f_j \in f$, meaning that within the subset, there exists an additive identity, f_0 (could also be denoted 0, when necessary), for vector addition.

Since $\mathbb{R}$ is already an established vector space, by definition, all elements must have additive inverses. We can then denote the additive inverse of all $f_n(s) \in \mathbb{R}$ as $f_{-n}(s)$ for all $s \in [0, 1]$.
The first property allows us to assert that $(f_n + f_{-n})(s) = f_n(s) + f_{-n}(s)$, so therefore by the definition stated earlier, $(f_n + f_{-n})(s) = 0)$, (by definition of additive inverses), for all $f_n \in f$, meaning that for all elements of f_n, there exists an additive inverse, f_{-n}.

The established vector space, $\mathbb{R}$, by definition, must have an identity for scalar multiplication that already exists within field F, which we will denote as 1.
Using the second property, $(1 \times f)(s) = 1 \times f(s)$, along with the known properties of 1 shown earlier, will evaluate to $(1 \times f)(s) = f(s)$, meaning that within the subset, there exists a multiplicative identity, 1, for scalar multiplication.

Suppose we have scalars $b, c \in F$ and function $f_n \in f$, and set up the following expression: $(b(cf_n))(s)$
Applying the second property, we get: $(b(cf_n))(s) = b(cf_n)(s) = b(c(f_n)(s))$
Since we're now fully in $\mathbb{R}$, which is a vector space, it's already been established that multiplicative associativity applies, so $b(c(f_n)(s))$ can be rearranged as: $(bc)(f_n(s))$
By the second property, $(bc)(f_n(s)) = (bc(f_n))(s)$
Therefore, transitively: $(b(cf_n))(s) = (bc(f_n))(s)$, for all $b, c \in F$ and $f_n \in f$, meaning the subset is associative under scalar multiplication.

Suppose we have scalar $b \in F$ and functions $f_j, f_k \in f$, and set up the following expression: $(b(f_j + f_k))(s)$
By applying both the properties, we can simplify the expression as such: $(b(f_j + f_k))(s) = b((f_j + f_k)(s)) = b(f_j(s) + f_k(s))$
Since $\mathbb{R}$ is a known vector space, it already allows us to distribute scalar multiplication over vector addition. Therefore, $b(f_j(s) + f_k(s)) = bf_j(s) + bf_k(s)$
Applying the properties again: $bf_j(s) + bf_k(s) = (bf_j)(s) + (bf_k)(s) = (bf_j + bf_k)(s)$
Therefore, transitively: $(b(f_j + f_k))(s) = (bf_j + bf_k)(s)$, for all $b \in F$ and $f_j, f_k \in f$, meaning that the subset allows distribution of scalar multiplication over vector addition.

Suppose we have scalars $b, c \in F$ and function $f_n \in f$, and set up the following expression: $((b + c)f_n)(s)$
By applying the second property: $((b + c)f_n)(s) = (b + c)(f_n(s))$

Since $\mathbb{R}$ is known vector space, it already allows us to distribute scalar multiplication over scalar addition.
Therefore, $(b+c)(f_n(s)) = bf_n(s) + cf_n(s)$
Applying the properties again: $bf_n(s) + cf_n(s) = (bf_n)(s) + (cf_n)(s) = (bf_n + cf_n)(s)$
Therefore, transitively: $((b+c)f_n)(s) = (bf_n + cf_n)(s)$, for all $b, c \in F$ and $f_n \in f$, meaning that the subset allows distribution of scalar multiplication over scalar addition.

As a result of the proof of the previous 9 axioms, the subset of $f([0,1], \mathbb{R})$ is a vector space.

Problem 7

Subproblem 1

Define $I : C([0,1]) \to \mathbb{R}$ by $I(f) = \int_0^1 f(x)\,dx$. Show that $I \in Hom(C([0,1]), \mathbb{R})$.

By definition, $Hom(C([0,1]), \mathbb{R})$ represents the set of all possible morphisms mapping $C([0,1])$ to $\mathbb{R}$. Therefore, we can prove $I \in Hom(C([0,1]), \mathbb{R})$ by first proving that I itself is a morphism.
Since $I = \int_0^1 f(x)\,dx$, it's a definite integral, so it holds the established properties surrounding both addition and scalar multiples for definite integrals, which can be written as follows: for all possible $f_1, f_2 \in C([0,1])$ and $c \in \mathbb{R}$, $\int_0^1 (f_1(x) + f_2(x))\,dx = \int_0^1 f_1(x)\,dx + \int_0^1 f_2(x)\,dx$ and $\int_0^1 cf_1(x)\,dx = c\int_0^1 f_1(x)\,dx$
By using the definition of I, we get: $I(f_1 + f_2) = I(f_1) + I(f_2)$ and $I(cf_1) = cI(f_1)$, preserving both vector addition and scalar multiplication.
Since both vector addition and scalar multiplication are preserved, I is inherently a morphism from $C([0,1])$ to $\mathbb{R}$. This consequently results in I being an element of $Hom(C([0,1]), \mathbb{R})$.

Subproblem 2

Define $E_\alpha : C([0,1]) \to \mathbb{R}$ by $E_\alpha(f) = f(\alpha)$, $\alpha \in [0,1]$. Show $E_\alpha \in Hom(C([0,1]), \mathbb{R})$.

By definition, $Hom(C([0,1]), \mathbb{R})$ represents the set of all possible morphisms mapping $C([0,1])$ to $\mathbb{R}$. Therefore, we can prove $I \in Hom(C([0,1]), \mathbb{R})$ by first proving that E_α itself is a morphism.
Any addition we perform of E_α would fit the form $E_\alpha(f_1 + f_2)$, and any scalar multiplication we perform would fit the form $E_\alpha(cf_1)$, for all possible $f_1, f_2 \in C([0,1])$ and $c \in \mathbb{R}$.
By definition of E_α, it follows that: $E_\alpha(f_1 + f_2) = (f_1 + f_2)(\alpha)$ and $E_\alpha(cf_1) = (cf_1)(\alpha)$
Due to the established behavior of functions within $\mathbb{R}$: $(f_1 + f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$ and $(cf_1)(\alpha) = cf_1(\alpha)$
By using the definition of E_α again, we get: $f_1(\alpha) + f_2(\alpha) = E_\alpha(f_1) + E_\alpha(f_2)$ and $cf_1(\alpha) = cE_\alpha(f_1)$
Thus, transitively: $E_\alpha(f_1 + f_2) = E_\alpha(f_1) + E_\alpha(f_2)$ and $E_\alpha(cf_1) = cE_\alpha(f_1)$, preserving both vector addition and scalar multiplication.
Since both vector addition and scalar multiplication are preserved, E_α is inherently a morphism from $C([0,1])$ to $\mathbb{R}$. This consequently results in E_α being an element of $Hom(C([0,1]), \mathbb{R})$.

Subproblem 3

Show that there is no collection $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$ such that $I = \sum_{j=1}^n \beta_j E_{\alpha_j}$.

Define f_1, for any possible $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$, as a function that satisfies the given condition, meaning that: $I(f_1) = \sum_{j=1}^n \beta_j E_{\alpha_j}|f_1$.
Since there are less elements within $\alpha_1, ..., \alpha_n$ than in the domain, $[0,1]$, there must exist another function, f_2, such that for all $\alpha \in \{\alpha_1, ..., \alpha_n\}$, $E_\alpha(f_1) = E_\alpha(f_2)$, but it differs from f_1 on values not included in α, leading to: $I(f_1) \neq I(f_2)$
Since f_1 and f_2 share all relevant values: $\sum_{j=1}^n \beta_j E_{\alpha_j}|f_1 = \sum_{j=1}^n \beta_j E_{\alpha_j}|f_2$
So, transitively, it follows that: $\sum_{j=1}^n \beta_j E_{\alpha_j}|f_2 = \sum_{j=1}^n \beta_j E_{\alpha_j}|f_1 = I(f_1) \neq I(f_2) \implies I(f_2) \neq \sum_{j=1}^n \beta_j E_{\alpha_j}|f_2$
Therefore, there is no collection $\alpha_1, ..., \alpha_n$ and $\beta_1, ..., \beta_n$ such that $I = \sum_{j=1}^n \beta_j E_{\alpha_j}$.

Subproblem 4

Show that if $\alpha_1, ..., \alpha_n \in [0,1]$ and $\beta_1, ..., \beta_n \in \mathbb{R}$ then $\sum_{j=1}^{n} \beta_j E_{\alpha_j} = 0 \in Hom(C([0,1]), \mathbb{R})$ implies that $\beta_1 = ... = \beta_n = 0$.

Assume that, while $\sum_{j=1}^{n} \beta_j E_{\alpha_j} = 0 \in Hom(C([0,1]), \mathbb{R})$, given that p is the amount of nonzero factors of the set $\{\beta_1, ..., \beta_n \in \mathbb{R}\}$, any case where $p \geq 1$, can't exist because it would be unable to evaluate all possible functions in $C([0,1])$ as 0.

First, attempt the case of $p = 1$. The order of elements β_n and the corresponding α_n doesn't matter, but for convenience, assign β_1 to be the nonzero value. Since the function that always evaluates to 1 is an element of $C([0,1])$, the summation, by its given definition, must ultimately evaluate it to 0. This however, is impossible because in the summation step $j = 1$, it adds the resulting product of β_1 and E_{α_1}, both of which are, by definition, non-zero, leading to the sum also being established as non-zero. All other $\beta_2, ..., \beta_n$ are defined as zero, so the proceeding summation steps will only add zero values, which can never make the sum non-zero again.

Under the assumption that value p, therefore, fails to exist, attempt the case of $p+1$. Now that there is more than 1 nonzero value (as this is $p+1$), the previous issue is avoided, so whenever $\sum_{j=1}^{p+1} \beta_j = 0$, the values, $\beta_1, ..., \beta_{p+1}$, correctly evaluate the previous function (due to the sum condition used to set these values being just the desired property itself, simplified as all $E_\alpha = 1$). However, $C([0,1])$ must also include another function, where all values evaluate to 1 besides α_1, which goes to 0 (and the values infinitesimally close to α_1 within the domain take on values necessary to maintain continuity but ultimately don't affect any other α value). Using the same sets of β and α, the summation fails to evaluate this new function to zero because, while otherwise identical for the following summation steps, in the summation step $j = 1$, the new value of E_{α_1} being zero leads the resulting product of β_1 and E_{α_1} to also be zero. This results in the summation ultimately having a value that is β_1 less than the original function, hence: $\sum_{j=1}^{n} \beta_j E_{\alpha_j} = \sum_{j=1}^{p+1} \beta_j - \beta_1 = 0 - \beta_1 = -\beta_1 \neq 0$.

This exhaustively proves, by induction, that for all values of $p \geq 1$, the proposed summation fails to evaluate all $f \in C([0,1])$ to 0. Therefore, the only possible values for $\beta_1, ..., \beta_n$ is the case where $p = 0$, so $\beta_1 = ... = \beta_n = 0$.