PERFECT AND SEMI-PERFECT SHUFFLES

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ABSTRACT. We discuss a new class of mathematically precise shuffles and discuss how they interact when considered with perfect shuffles, even seeming to form a group. New card control methods are examined, allowing for more precise manipulation of a deck of cards than possible through perfect shuffles alone. The unshuffling of a camp scene drawn on the side of a deck is used to illustrate.

1. Introduction

Perfect shuffles (also known as *faro shuffles*) are well known tools in the world of card manipulation, for both card cheats and magicians alike. They can be used to shuffle a deck into seeming disarray, and yet can return the shuffled deck to its original order thereafter. In this paper we will see how greater control is possible when using *semi-perfect* shuffles, named for their similarity to the faro.

In Section 2 we look at the physical handling of the shuffles, and how they can be done by an eager card manipulator. In Section 3 we give mathematical descriptions of the various shuffles and briefly touch on some of their properties. In Section 4 we see the kind of card manipulations these shuffles are capable of. In Section 5 we show how the drawing of a campsite on the side of a deck can be shuffled and mixed into an unrecognizable state, and yet through further shuffling the drawn scene can be brought back. The final drawn scene can be different from the initial one though, with parts having been moved or even inverted as desired.

2. The Handling of the Shuffles

2.1. **The Faro.** For a deck with an even amount of cards the faro shuffle entails splitting the deck into two equally sized halves and interweaving them into one another. The interweave must be perfect, having a perfect alternation of individual cards from the two halves. The first version of this shuffle is the *out-faro*, where the card that was initially on the top of the deck stays on the top (and thus stays *out*-side). The *in-faro* is the second version, and has the initial top card move *into* the deck by becoming the second card from the top.

There are a few ways to handle a faro. For an out-faro we assume the card manipulator takes the top half of the deck in their right hand, holds the bottom half in their left, and weaves the cards such that the top card stays on the top as required. For the in-faro we assume the card manipulator moves the bottom half to their right hand, holds the top half in their left, and weaves the cards as they would for an out-faro. For such an in-faro, the top card of the right hand's packet goes to the top of the deck. Figure 1 shows how this is done, and how it acts as an in-faro. If your own personal handling of the faro differs, you will need to figure out small adjustments for the semi-perfect shuffles later.

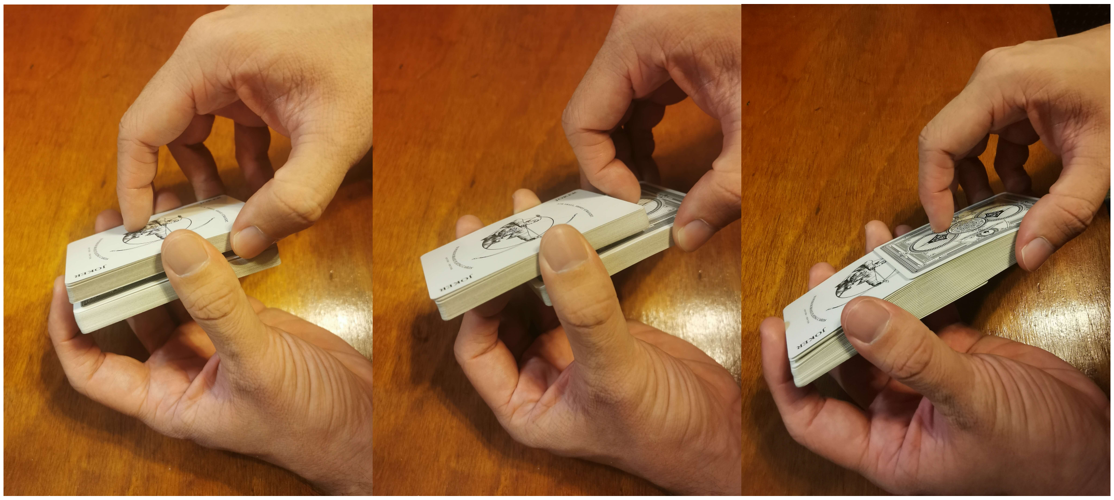


FIGURE 1. An in-faro being handled as described in the text. The bottom half of the deck is moved to the right hand, the left index helping it along. Note how the original top card, the joker, is no longer the top card due to the in-faro moving it into the deck.

2.2. **The Faro Jog.** The faro jog is achieved by not fully squaring the deck after a faro, as seen in Figure 2. When bridging the cards as usual for the faro shuffle, make sure to stop the cards with a knuckle or fingertip. The fingertip, knuckle, or whatever section of the relevant finger used should be placed lightly on the bottom of the deck, allowing the sprung cards to spring and rest against it. The fingertip/knuckle might move during the course of the spring in response to the greater pressure from the cards, but do not remove it from the bottom of the deck. Maintain a gentle pressure till the spring is complete. Once complete, push the out-jogged cards partially in without fully squaring the deck. We want a perfect alternation of cards, with the slight but perfectly alternating jogging seen in the lower right picture of Figure 2.

Due to the handling we have chosen for the in and out faros, a faro jog will always result in the final top card of the deck being jogged in the direction towards us, and the second card being jogged away from us, regardless of which faro we use to get into it.

The jogged arrangement is needed for semi-perfect shuffles. This author has found keeping the cards jogged out about 4 millimeters to be reliable for the upcoming shuffles, though this might differ depending on the card manipulator. The two halves of the deck can also be interweaved without the need for a card spring, simply pushing the two halves into the jogged arrangement.

2.3. **Straddled Double Faros.** Being the perfect shuffle, the faro has single cards precisely alternating from the two halves of the deck. It turns out you can get a perfect alternation of every two cards as well! Given the regularity of these kinds of shuffles (just with pairs rather than single cards) we have decided to call them semi-perfect.

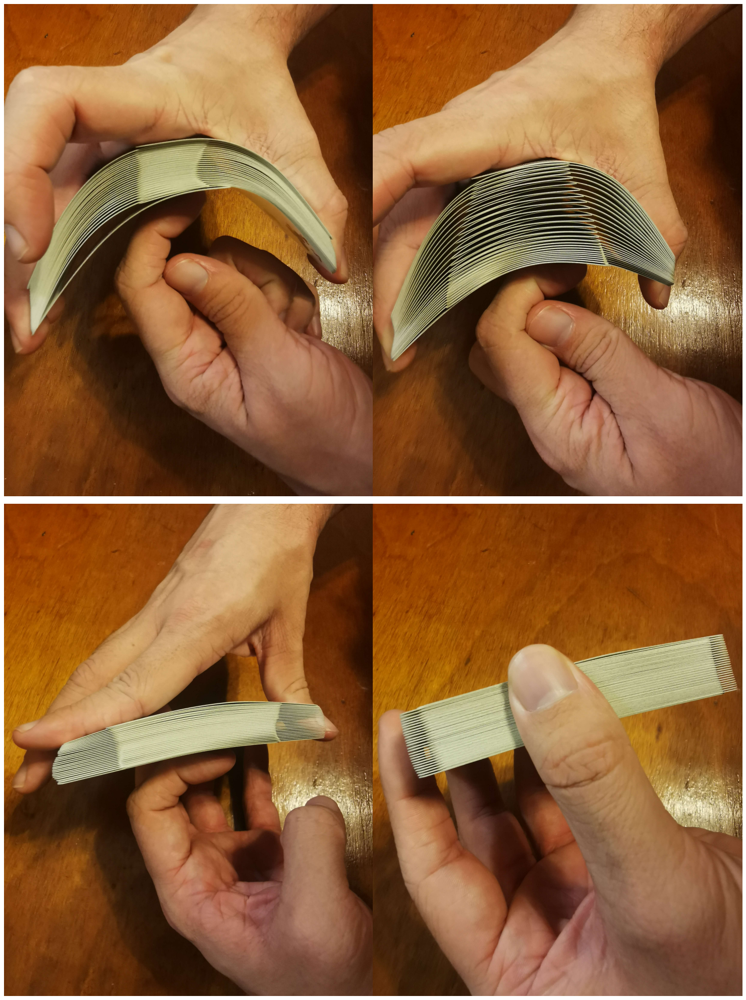


FIGURE 2. The faro jog.

The first of these semi-perfect shuffles is tentatively named the straddled double faro. To perform this shuffle the deck must first be in the perfectly jogged arrangement, achievable by a faro jog. The jogged deck is then split into two halves. After taking half of the deck in each hand, gently tap the two halves together at a somewhat perpendicular angle: this is identical to what is usually done for a regular faro, where the goal is to line up the out-jogged edges of the cards to make the interweave easier and more consistent. Be careful not to tap with too much force though, as losing the perfect jogging prevents a successful shuffle. Hold half of the deck stationary in your left hand, while taking the the other half of the deck in your right. Weave the right hand's half into the stationary left hand's half. When doing so, start the interweave at a tilted angle as seen in Figure 3. Let the top card of the right hand's half of the deck be on the top, but place the next two cards of the right hand's half under the top two cards of the left hand's half. Gently bring the two halves to the same level by keeping the hands where they are but by gradually altering the angle of the cards, allowing the pairs of cards to naturally and almost effortlessly weave together. To reiterate, it is a gentle movement and does not require any significant force at all, being almost identical to the weaving of a regular faro. This description misses out on some of the subtleties, such as how this author primarily uses pressure from the ring finger and thumb to hold the halves in place in each hand, with the middle fingers simply resting on the sides of the deck, straightening things out if needed. Or how the index finger of the right hand is used to help guide the first cards into the weave on occasion, though this is not always necessary. These details could differ from person to person, as there are many ways to perform a standard faro, and the straddle double faro is merely an extension. And these directions are of course given from the perspective of a right handed author. Nonetheless, Figure 3 illustrates how this shuffle can be done far more effectively than a wordy description.

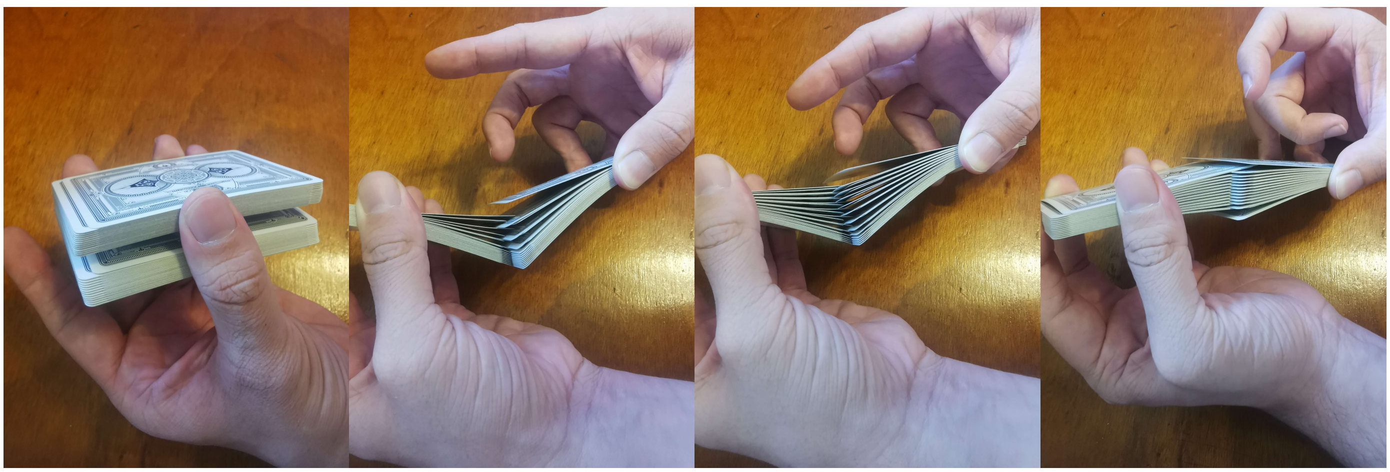


FIGURE 3. The straddled double faro.

If you are successful in doing a straddled double faro, all the cards of the deck will alternate in perfect pairs with the exception of the top and bottom cards. The top and bottom cards will be singular, straddling the half of the deck they are being interwoven with. A side-view of this can be seen in the final picture of Figure 3. The partial fan in Figure 4 more clearly shows the alternating pairs, with the exception of the singular top card (bottom card not visible). The cards can then be sprung as per usual, to add the vital dramatic flair. As the cards are falling in pairs, they even make a different sound than the regular faro.

The straddled double faro has both an in and out variety. If the card that was originally on the top of the deck stays on the top, we would call that an *out-straddle*. If the top card does not stay on the top, it means it has been moved into the deck, and so we call that an *in-straddle*. Consider a deck that is already in the jogged position. To do the out-straddle split the deck in half, take the top half of the deck in your right hand and leave the bottom half in your left: continue to do the weave as outlined above. To do the in-straddle take the bottom half of the deck in your right hand instead, leaving the initial top half in your left.

FIGURE 4. The deck after a straddled double faro. Note how the top card is singular, as the bottom card will be as well, but all other cards alternate in perfect pairs.

2.4. **Double Faros.** The next semi-perfect shuffle on the table is the double faro. This results in cards alternating in pairs, with no exceptions at the top or bottom. Despite the final outcome being seemingly simpler, the double faro is trickier than its straddled cousin. To do the double faro, we once again assume the deck is in the necessary jogged position. Take the top half of the deck in your right hand and rotate it. In other words, holding the top half spin it 180 degrees and place it back on the rest of the deck in its rotated orientation. Readjust your right hand to a more comfortable position and pick up the top half once more. All of this is to ensure the top card of the top half is out-jogged away from yourself, and the top card of the bottom half is jogged toward yourself. If the out-jogged cards aren't all flush you might need to tap the edges of the two halves together at a perpendicular angle, though the tap must be gentle and should not disturb the jogged arrangement.

Next you tilt the two halves at an angle to one another, and allow the top two cards of the top half of the deck to slot either above or below the top two cards of the bottom half. To weave the two halves together, keep your hands where they are but flatten the angle between the halves to let the cards naturally interweave. Phrased another way, you gradually move the ends of the two deck halves up and into each other. If you know how to do a regular faro, this will already be a familiar motion to you, and our poor description shouldn't hinder you. Remember, the weave should be unforced and almost effortless. After inserting the first two cards the deck will do the rest of the work. Figure 5 shows the steps of the technique more clearly than our words convey.

Like its straddled cousin, the double faro has both an in and out variety. The *out-double* has the original top card stay on the top, and the *in-double* has the original top card move into the deck (moving to the third position). Unlike the straddle, whether the double is an in or out does not affect which half of the deck you take in your right hand. Rather, you *always* take the top half of the deck in your right hand, and simply guide the original top two cards to either below or above the left half's top pair as needed.

Figure 5. The double faro, specifically an out-double. Without disturbing the jogged arrangement, the top half of the deck is rotated before starting the weave. The partial fan in the final frame shows how the deck perfectly alternates in pairs, with no exceptions.

The double faro as we have presented above is incomplete. When placing text or drawings on the side of a deck, we wish for all the inked edges of the cards to remain on the same side. During the double faro half the deck is rotated though, which would mix inked and blank edges in undesirable ways. A simple solution is to rotate the appropriate half of the deck back to its original orientation before squaring the deck. The process of flipping is illustrated in Figure 6.

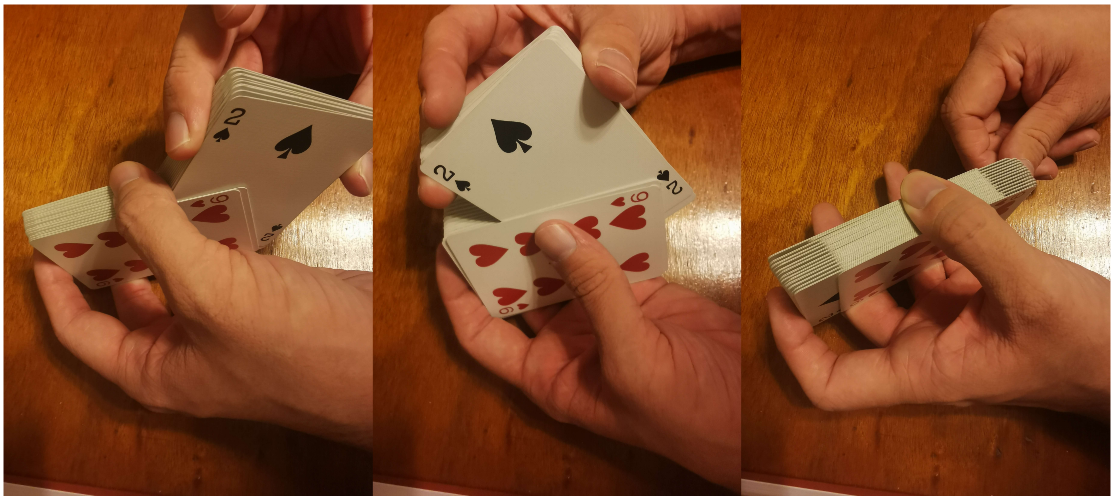


FIGURE 6. After performing the double faro as we've seen, the original top half of the deck should be rotated back. This returns the edges of the cards to their original orientation, and can still leave the alternating pairs of cards out-jogged if desired.

Further variations are possible for semi-perfect shuffles, in both handling and outcome. We can even have shuffles with sets of four cards alternating perfectly, or have alternating sets of eights, sixteens, etc (though this quickly becomes physically infeasible and usually requires unreasonable setups). With all of this said, the shuffles we have already learnt are more than enough for our purposes.

3. Mathematical Descriptions

For those interested in card manipulation alone and who might not be comfortable with the mathematics, please skip directly to Section 4 to see how our new shuffles can mix up and bring back a drawn scene on the side of a deck. If you are comfortable with modular arithmetic though, this section looks at how to define our shuffles and what properties such definitions reveal.

A convenient way to see how cards move after a shuffle is to index the positions in a deck, and see how the index of individual cards change. If we have a deck of N cards, the card at the top of the deck is in position 0, the next card is in position 1 and so on until position $N-1$ for the final card. Starting our indices from 0 rather than 1 might seem unusual, but it makes the mathematics significantly tidier and is a common convention among computer scientists. With this in mind, we can consider a deck of N cards where N is even. If a card is originally in position p, an out-faro will move it to the position $F^{out}(p)$ where

$$F^{out}(p) \equiv \begin{cases} 2p \mod (N-1) & \text{if } p < (N-1) \\ N-1 & \text{if } p = (N-1) \end{cases}$$

If a card is originally in position p, an in-faro will move it to the position $F^{in}(p)$ where

$$F^{in}(p) \equiv 2p + 1 \mod (N+1)$$

Explanations for these equations can be found elsewhere[1][2]. A rough intuition can be given though. Consider a card in the original top half of the deck at position p: because the out-faro results in a perfect weave of cards from two halves of the deck, not only will the card previously at position p have all the same cards above it as it did before, it will have twice as many due to the perfect weaving of new cards between the old ones, doubling its position to $2p$. Similar lines of reasoning for the other positions, and for the in-faro, lead to the equations above.

The straddled faro can be similarly defined, though we will only look at its definitions for decks of size $N = 4n$ where n is some positive integer. In other words, the deck's length must be a multiple of four. An out-straddle will move a card at position p to position $S^{out}(p)$ where

$$S^{out}(p) \equiv \begin{cases} 2p \mod (N-1) & \text{if } p \text{ is even} \\ 2p+1 \mod (N+1) & \text{if } p \text{ is odd} \end{cases}$$

The equation for S^{out} tells us that a card will be moved in the same way as an out-faro if p is even, and will be moved in the same way as an in-faro if p is odd. If you have some cards on you at home, you can verify why this is the case with a pack as little as 8 or 12 cards. Split the deck into two equal halves and weave them as you would for an out-faro, without squaring up the deck. Starting from the top, the cards should follow a $near, far, near, far, \cdots$ pattern with the original top card being closer to your body ($near$), and the next card being away (far). Do the same with an in-faro, and without squaring up the deck, flip it so that the card originally on the top is nearer to you and not far: you should then notice a $far, near, far, near, \cdots$ pattern. If we were to do the same for an out-straddle though, keeping the card originally on the top of the jogged deck towards us, we will see a $near, far, far, near, \cdots$ repeating pattern. In other words, for every packet of four cards after the out-straddle, the first two obey the out-faro pattern, and the second two obey the in-faro pattern. It can be verified that the first two of every four such cards originally had even p indexes before the out-straddle though, and the second two cards initially had odd p indexes before the shuffle. Hence the rule for S^{out} above. The reasoning here is handwavy and less than rigorous, but it can be explained more formally.

We can similarly define an in-straddle as

$$S^{in}(p) \equiv \begin{cases} 2p+1 \mod (N+1) & \text{if } p \text{ is even} \\ 2p \mod (N-1) & \text{if } p \text{ is odd and } p < (N-1) \\ N-1 & \text{if } p \text{ is odd and } p = (N-1) \end{cases}$$

which moves a card at position p to position $S^{in}(p)$. For this shuffle, a card's index changes according to the out-faro rule if p is odd, and changes according to the in-faro rule if p is even. This is the inverse situation of the out-straddle, and similar reasoning to what we've already gone through for the out-straddle can show us why this is the case.

Take the 12 cards again and do an out-faro without squaring up the deck, such that you get the $near, far, near, far, \cdots$ pattern once more. Looking at the top four cards, the first card's initial index would have been even and would have initially come from the top half of the deck (so $p < \frac{N}{2}$). The second card's previous index would be even and from the initial bottom half ($p \geq \frac{N}{2}$). The third card would have had an index that was previously odd, that was from the top half of the deck. The fourth card would have previously had an odd index as well, but would have come from the bottom half of the deck. These four statements will be true of the next four cards, and the next. Let's change the cards around a bit though. From the first four cards, swap the second and third cards. Do this for the next four cards, and then the next. What we are left with is an out-double faro! What this tells us is that the out-double is like the out-faro, but that every previously even card from the bottom half of the deck is moved up one more compared to the usual out-faro, and every previously odd card from the top half is moved down by one index wise. This gives a description of the out-double as

$$D^{out}(p) \equiv \begin{cases} 2p & \text{if } p \text{ is even and } p < \frac{N}{2} \\ 2p-1 & \text{if } p \text{ is odd and } p < \frac{N}{2} \\ 2p+1 \mod (N-1) & \text{if } p \text{ is even and } \frac{N}{2} \leq p < N \\ 2p \mod (N-1) & \text{if } p \text{ is odd and } \frac{N}{2} \leq p < N \\ N-1 & \text{if } p = (N-1) \end{cases}$$

which moves a card from position p to position $D^{out}(p)$. Similar reasoning (using an in-faro rather than an out-faro) gives the description of the in-double as

$$D^{in}(p) \equiv \begin{cases} 2p+2 & \text{if } p \text{ is even and } p < \frac{N}{2} \\ 2p+1 & \text{if } p \text{ is odd and } p < \frac{N}{2} \\ 2p+1 \mod (N+1) & \text{if } p \text{ is even and } \frac{N}{2} \leq p \leq N \\ 2p \mod (N+1) & \text{if } p \text{ is odd and } \frac{N}{2} \leq p \leq N \end{cases}$$

which moves a card from position p to position $D^{in}(p)$. The positions of cards from the initial top half of the deck (corresponding to $p < \frac{N}{2}$) do not need to be modded in these descriptions, but there is no harm in adding such mods in if desired.

In the world of card magic there is a move known as the *anti-faro*, the inverse of a regular faro. The version pioneered by Christian Engblom entails springing a deck from one hand to another in such a way that every even card goes one way, and every odd card goes the other. The even and odd cards are then separated into two halves in a single motion, and one half is placed atop the other. The anti-faro in this form is considered one of the most difficult card sleights known, and only a handful of individuals in the world seem to be able to do it reliably. Other handlings are possible that differ from Christian's, though most come with difficulties. Of these the easiest method is still a significant performance challenge, involving the slow process of going through the deck one card at a time to take out every second card. If multiple anti-faros are needed though, one method that is both easy and fairly time efficient comes from Juan Tamariz. A magician deals the deck into multiple piles of cards and places them atop each other in a specific order: the number of piles dictates how many anti-faros are being performed (4 piles being equal to 2 anti-faros, 8 piles being equal to 3 anti-faros and so on), while the order in which the piles are placed atop each other dictates the sequence of in and out anti-faros.[3] Returning to single shuffles though, for even sized decks the description of a single anti-faro is

$$A^{out}(p) \equiv \begin{cases} \frac{N}{2}p \mod (N-1) & \text{if } p < (N-1) \\ N-1 & \text{if } p = (N-1) \end{cases}$$

where it moves a card in position p to position $A^{out}(p)$. This description is specifically for the inverse of the out-faro, keeping the top card of the deck at the top during the shuffle. The inverse of the in-faro can be described by

$$A^{in}(p) \equiv \frac{N+2}{2}(p-1) \mod (N+1)$$

for even sized decks. If you intend to use the perfect and semi-perfect shuffles for card control as seen in the coming sections, you do not need to know how to do any anti-faros. The above descriptions will make the upcoming mathematics a little simpler though.

The perfect and semi-perfect shuffles are interesting when considered alone, but their strengths truly shine when we look at sequences of them. To this end we will introduce the idea of the shuffle sequence. Let f^o represent the out-faro, f^i the in-faro, s^o the out-straddle, s^i the in-straddle, d^o the out-double, d^i the in-double, a^o the out anti-faro and a^i the in anti-faro. A shuffle sequence is (unsurprisingly) a sequence of shuffles to be applied to a deck, where the order is read from left to right. An example

$$[f^i, f^o, s^i]$$

would have us apply an in-faro first, followed by an out-faro and finally an in-straddle. It is important to note that a sequence does not specify the size of the deck upon which it is acting, and different sized decks can act very differently given the same shuffle sequence.

Recall that an out-faro moves a card from position p to position $2p \mod (N-1)$, unless it is in position $N-1$. This means k such shuffles will move a card from position p to position $2^k p \mod (N-1)$, where the exception of card $N-1$ stays in its position no matter how many out-faros are applied. The smallest positive integer k for which $2^k \mod (N-1) = 1$ equals the minimum number of out-faros we'd need to apply to a given deck to return it to its original order. For a deck of 52 cards this is true when $k = 8$, giving rise to the well known secret amongst magicians that 8 out-faros will leave a standard deck of cards entirely unaltered. The shuffle sequence this corresponds to would be

$$[f^o, f^o, f^o, f^o, f^o, f^o, f^o, f^o]$$

The surprising part is that the deck is genuinely mixed during the eight shuffles, even looking randomly ordered at points in between, but on the eighth shuffle the deck is returned to its original order. For decks of size $N = 2^n$ we have the same property but for n out-faros. In general, any sequence that doesn't alter the order of a deck can be called an identity sequence.

If the deck is returned to order after k out-faros, we can then see that $k-1$ out-faros will be equivalent to an out anti-faro. In both cases, all that is needed to return the deck to its previous order is a single out-faro. For decks of size $N = 2^n$ this is even more apparent, since with the exception of the stationary card at $N-1$, we have that $n-1$ out-faros will move a card from position p to position $2^{n-1}p \mod (N-1)$. This is identical to the factor used in the definition of the out anti-faro, since $2^{n-1} = \frac{N}{2}$. In general, rather than explicitly writing $k-1$ out-faros, we will represent any such sequence with a^o due to this anti-faro equivalence.

The in-faro can also be reversed by a sequence of shuffles, without explicitly invoking the in anti-faro. The in-straddle and out-straddle additionally have sequences to reverse them.

Theorem 3.1. *The shuffle sequence* $[f^i, a^o, f^i, a^o]$ *is an identity sequence for any even sized deck.*

Proof. As always, let N be even and represent the size of our deck. If we apply the first half of the shuffle sequence we end up with the transformation

$$A^{out}(F^{in}(p)) = \begin{cases} \frac{N}{2}\left[2p+1 \mod (N+1)\right] \mod (N-1) & \text{if } p \neq \frac{N}{2}-1 \\ N-1 & \text{if } p = \frac{N}{2}-1 \end{cases}$$

If we substitute the value $p = N-1$, we have

$$\begin{aligned} A^{out}\left(F^{in}(N-1)\right) &= \frac{N}{2}\left[N-2 \mod (N+1)\right] \mod (N-1) \\ &= \frac{N}{2}(N-2) \mod (N-1) \\ &= -\frac{N}{2} \mod (N-1) \\ &= -\frac{N}{2} + (N-1) \mod (N-1) \\ &= \frac{N}{2} - 1 \mod (N-1) \end{aligned}$$

which tells us the values $N-1$ and $N/2-1$ are inverses of one another under this transformation. The full sequence $[f^i, a^0, f^i, a^o]$ leaves these two cards unmoved then, as an identity should.

What about the other positions? We will tackle the trickier case of $\frac{N}{2}-1 < p < N-1$. This means we can rewrite our index as $p = \frac{N}{2} + h$ for some integer h, where $0 \leq h < \frac{N}{2} - 1$. Applying the first in-faro moves our card to

$$\begin{aligned} F^{in}(p) &= 2p+1 \mod (N+1) \\ &= 2\left[\frac{N}{2} + h\right] + 1 \mod (N+1) \\ &= N+1+2h \mod (N+1) \\ &= 2h \end{aligned}$$

Applying the first anti-faro moves this to

$$\begin{aligned} A^{out}(2h) &= \frac{N}{2}(2h) \mod (N-1) \\ &= hN \mod (N-1) \\ &= h \end{aligned}$$

Applying the second in-faro gives

$$\begin{aligned} F^{in}(h) &= 2h + 1 \mod (N+1) \\ &= 2h+1 \end{aligned}$$

And applying the final anti-faro gives

$$\begin{aligned} A^{out}(2h+1) &= \frac{N}{2}(2h+1) \mod (N-1) \\ &= hN + \frac{N}{2} \mod (N-1) \\ &= h(N-1) + h + \frac{N}{2} \mod (N-1) \\ &= h + \frac{N}{2} \mod (N-1) \end{aligned}$$

But we know that $p = \frac{N}{2} + h$, and so the shuffle sequence has left the card in position p unmoved! We leave the slightly tidier case of $p < \frac{N}{2} - 1$ to the reader, though the logic is the same to that used above simply without needing to make the $p = \frac{N}{2} + h$ substitution. This all shows that the sequence $[f^i, a^0, f^i, a^o]$ is the same as an identity sequence. □

Theorem 3.2. *For decks of size $N = 4n$, where n is a positive integer, the shuffle sequences $[s^i, a^o, s^i, a^o]$ and $[s^o, a^o, s^o, a^o]$ are both identity sequences.*

Proof. The description of the out-straddle can be rephrased as

$$S^{out}(p) \equiv \begin{cases} F^{out}(p) & \text{if } p \text{ is even} \\ F^{in}(p) & \text{if } p \text{ is odd} \end{cases}$$

If p is even we can see that $A^{out}(S^{out}(p))$ will be equivalent to applying an out-faro and an out anti-faro, returning p to position p. Applying the full sequence of $[s^o, a^o, s^o, a^o]$ will then leave any even positioned card unmoved.

If the position p is odd, applying $A^{out}(S^{out}(p))$ will leave the card in an odd position. We can see this because the the same two shuffles moved all the initially even cards to their same even positions. We can't have two cards in the same position, but if a single initially odd card was moved to an even position we'd have exactly that (due to the Pigeonhole Principle). We can also note that, since p is odd, the two shuffles would be equivalent to applying an in-faro and then an out anti-faro. As this leaves the card odd, applying the two shuffles again will do the same, leaving us with all odd cards effectively obeying the shuffle sequence $[f^i, a^o, f^i, a^o]$, which is an identity sequence. All of this means, whether we talk of cards at even or odd positions, $[s^o, a^o, s^o, a^o]$ is an identity sequence.

We can rephrase the description of the in-straddle to the equivalent

$$S^{in}(p) \equiv \begin{cases} F^{in}(p) & \text{if } p \text{ is even} \\ F^{out}(p) & \text{if } p \text{ is odd} \end{cases}$$

Applying identical logic to what we have done for the out-straddle will show that $[s^i, a^o, s^i, a^o]$ is an identity sequence as well, though this will be left as an exercise for the reader (that most dreaded of terms). □

This means we can write the in anti-faro as $a^i = [a^o, f^i, a^o]$. We can also write the inverse of the in-straddle as $[a^o, s^i, a^o]$, and the inverse of the out-straddle as $[a^o, s^o, a^o]$. The eagle-eyed reader might have noticed that we haven't proven $[a^o, s^o, a^o, s^o]$ to be an identity sequence though, something we'd need to show for $[a^o, s^o, a^o]$ to be considered a true inverse of the out-straddle. The same could be said about the sequences given for the inverses of the in-straddle and in-faro. We will not prove it rigorously here, but simply encourage you dear reader to verify this for yourself. Grab a pack of cards, as long as the length is a multiple of four, and jog the first card toward you, the next two away from you, the next two toward you and so on till you get to the final single card jogged toward you. Grab all of the cards jogged towards you, square them up in a pile, and place them on the top of all the cards that were jogged away from you. Well done, you've just done an anti out-straddle! You can then manually jog the cards one by one to get into the physical setup needed for doing an out-straddle, do said shuffle, and find that the deck has returned to its original order. However if you do the out-straddle first followed by the anti out-straddle, you will find yourself in the exact same situation. Regardless of the order of application the deck's order is unchanged. We know that the shuffle sequence $[a^o, s^o, a^o]$ is equivalent to an anti out-straddle when applied after an out-straddle, meaning it is a good description of the manual anti out-straddle we have just performed. Since it is equivalent to our manual handling though, and since the manual handling stays the same whether applied before or after the out-straddle (thus being an inverse itself), we can see that our sequence represents a true inverse. The same can be checked for the in-straddle and in-faro as well.

The double faros give rise to a messier situation. To save both the reader and ourselves endless case bashing, we forego some rigour by simply claiming that $[d^i, a^o, d^o, a^o, f^i, a^o]$ acts as an identity sequence for any deck of size $N = 4n$. Both simulations and physical card shuffling bears this out, though a proof can be made available if demanded. What this tells us is that $[a^o, d^o, a^o, f^i, a^o]$ is the inverse of an in-double. The related shuffle sequence $[d^o, a^o, f^i, a^o, d^i, a^o]$ is also an identity sequence for any deck of size $N = 4n$, which tells us that $[a^o, f^i, a^o, d^i, a^o]$ is a valid inverse of the out-double. These inverses are not necessarily minimal in length though. For example, the sequence

$$[f^o, f^o, f^o, f^o, f^o, d^i, f^o, f^o, f^o, f^o, f^i]$$

is an inverse of the out-double for a deck of 64 cards. Keeping in mind that anti-faros represent sequences of multiple faros in how we have used them, this turns out to be six shuffles shorter than the inverse we initially gave. Regardless of efficiency though, what is important to note is that the in-double and out-double do in fact have inverses.

One last comment on inverses is that every shuffle sequence has an inverse, since the six perfect/semi-perfect shuffles all individually do so. Let

$$[z^1, z^2, ..., z^{k-1}, z^k]$$

be an arbitrary shuffle sequence, where z^i represents the perfect or semi-perfect shuffle being performed at the *ith* position in the sequence. If we let $\bar{z}^i$ represent the inverse of the single shuffle z^i, the sequence

$$[\bar{z}^k, \bar{z}^{k-1}, ..., \bar{z}^2, \bar{z}^1]$$

acts as an inverse to our original sequence. Here the individual $\bar{z}^i$ do not represent single shuffles, but rather represent the shuffle sequences we are familiar with as inverses. What is important is that applying these two shuffle sequences to a deck, with either sequence being applied first, will result in no change to the deck's order. To see this we note that adding two sequences together is as simple as appending the one onto the other. Since $[z^i, \bar{z}^i] = [\bar{z}^i, z^i] = \mathbb{1}$ for all i, where $\mathbb{1}$ is used here to represent the identity sequence, we can note that all the shuffles will iteratively cancel out when we append the two sequences together. This is true regardless of which of the two sequences is performed first.

If we look at the set of all possible shuffle sequences for a given sized deck, constructed from the perfect and semi-perfect shuffles we've been studying, we can note a few details. When we add two sequences together a new and valid sequence is created, meaning that the set is closed under the addition of any two sequences. We know that there is an identity element that can be achieved with multiple shuffle sequences or simply by leaving the deck alone. We also know that every sequence has an inverse. Unless there are issues with associativity that this author has been unable to see, this all seems to imply that our perfect and semi-perfect shuffles form a group!

Unlike the shuffle group formed by in and out faros alone, this new group does not preserve central symmetry. What is central symmetry? Consider the cards originally at positions 0 and $N-1$, at 1 and $N-2$, at 2 and $N-3$, etc, where the first card is as many cards from the top as the second is from the bottom. We can call these mirrored partners. Central symmetry is the fact that no matter how many in or out faros are performed, even though both partners can be moved around, mirrored partners remain mirrored partners.[2] Amongst magic circles this is known as the *stay stack principle*, and has been used to great effect.[1] One immediate consequence is if you take a 52 card deck with the ace through king of hearts, ace through king of clubs, king through ace of diamonds and king through ace of spades and give the deck any number of in and out faros, the top and bottom halves of the deck will always have an equal number of aces, an equal number of twos, etc.

For our semi-perfect shuffle group the faros and double faros seem to retain central symmetry, but the straddled double faros do not. If we consider a deck of size 32 we can look at the initial mirrored partners 0 and 31. If we apply an out-straddle the card at 0 would stay at 0 but the card at 31 would move to 30, breaking the symmetry. Though lacking this trait, the new shuffle group gives rise to other novel behaviours.

4. Card Control

For those who skipped the last section, the important takeaway is the idea of the shuffle sequence. We can let f^o represent the out-faro, f^i the in-faro, s^o the out-straddle, s^i the in-straddle, d^o the out-double and d^i the in-double. A shuffle sequence would then be a list of perfect and semi-perfect shuffles, read from left to right, to be applied to a deck. For example the sequence

$$[f^i, f^o, s^i]$$

would have us apply an in-faro first, followed by an out-faro and then an in-straddle. You will also need to be familiar with permutations and cycle notation. If these are new concepts to you do not worry, as an appendix has been attached that goes over the necessary details.

Consider a deck of 64 cards. To make such a deck you'd need to add 12 cards to a standard 52 deck, but that is hardly a problem for magicians or card manipulators (who are rarely without multiple decks scattered around the house). A deck of this size has some interesting properties. One such property is that 6 out-faros will leave the deck's order unaltered; every card will be moved during the course of the 6 shuffles and yet return to their original positions by the end. A similar property has 6 in-faros perfectly reverse the deck's order. Both results are extremely promising for performance purposes, but other equally remarkable full deck manipulations are possible.

We are going to give a quick spoiler for the coming results. For a deck of 64 cards, we can consider a shuffle sequence of length 6. The shuffle in the first position will be able to control the order of the deck's halves. The shuffle in the second position will be able to control the order of the deck's quarters. The shuffle in the third position will control the order of the eighths, and so on. The shuffle in the last position will be able to control the order of individual cards. This is not the complete story, as there are strict limitations on how our techniques can move the components of the deck around, but it does give us the broad strokes of what we are working towards.

To get a deck of the correct size simply add three additional cards to each suit of a standard deck. Adding an ace, two and three to the end of each suit would serve this purpose, and we might label these cards as the 14^{th}, 15^{th} and 16^{th} of their respective suits (the jack, queen and king of course serving as the 11^{th}, 12^{th} and 13^{th}). Markings can be placed on the faces of the cards to distinguish them from the original aces, twos and threes. We also recommend arranging each of the four suits as ace through to 16 to keep things simple. Have the spades on the bottom, followed by the diamonds, the clubs, and ending with the hearts on top. The bottom facing card of the deck should then be an ace of spades with this setup.

SHUFFLE	PERMUTATION
f^o	$(1)(2)(3)(4)$
f^i	$(1\ 2)(3\ 4)$
s^o	$(1)(2)(3\ 4)$
s^i	$(1\ 2)(3)(4)$
d^o	$(1)(2\ 3)(4)$
d^i	$(1\ 3\ 4\ 2)$

Table 1. Each one of the six perfect and semi-perfect shuffles is associated with a permutation.

Say we wish to swap the clubs and the hearts. We could separate the deck into quarters and switch the suits before putting the quarters all back together. If we were then to apply two out-faros the deck will be in a new order with the cards repeating the pattern ♠, ♦, ♥, ♣. All the aces will be together, all the twos together and so on. Let's call this arrangement of the deck **S**. We know four more out-faros will take the deck from **S** and return it to how it was before the first two out-faros (though after the swapping of the clubs and hearts, as that occurred before any faro was applied). Alternatively we could take the deck in its original order, where the hearts and clubs have yet to be swapped, and perform an out-faro followed by an in-straddle to get to the same arrangement of **S**. If we were to follow this up with four out-faros the deck will be in an arrangement where the suits are separated and ordered, but the clubs and hearts have been swapped. This despite us never explicitly swapping them before applying the perfect and semi-perfect shuffles.

If we had performed an out-straddle in the sequence above rather than the in-straddle, we would have swapped the diamonds and spades instead. If we'd done an out-double we would have swapped the clubs and diamonds. To describe what's going on more compactly we can label the quarters of the deck. The hearts will correspond to quarter 1, clubs to quarter 2, diamonds to quarter 3 and spades to quarter 4. With this labelling, we can associate a permutation with each of the perfect and semi-perfect shuffles as seen in Table 1. The shuffle sequence

$$[f^o, x^p, f^o, f^o, f^o, f^o]$$

will then move the suits/quarters of the deck around according to the permutation associated with x^p, where x^p can be any one of our perfect or semi-perfect shuffles.

What if we are curious about having all out-faros, but of varying the shuffle in the third position? That is to say we look at the sequence

$$[f^o, f^o, x^p, f^o, f^o, f^o]$$

where x^p is any perfect or semi-perfect shuffle. We've already hinted that this will give us control over the eighths of the deck, so let's break the deck up into eighths. Another way of thinking about this is breaking the deck up into halves, and breaking each half up into quarters. It turns out doing a perfect or semi-perfect shuffle in the third position will apply its associated permutation to each half of the deck separately. The spades and diamonds form one half, and the clubs and hearts form the second. If we wish to swap the ace through eight of hearts with the nine through sixteen of hearts we would need to do an in-straddle as the third shuffle. An unavoidable consequence is that this will also swap the ace through eights of diamonds with the nine through sixteen of diamonds. As said, the associated permutation will be applied to both halves of the deck separately. When figuring out how the eighths will be moved around yourself, remember to label the quarters of each half with 1 to 4: the quarter closest to the top of the deck in each half should be labeled as 1, and the quarter closest to the bottom as 4.

For the magic effect we are building to we will not need control over sections smaller than eighths (with the exception of reflections). Nonetheless we can generalise this progressively more fine-grained control of the deck with the statement

Claim. 4.1. Assume we have a deck of 64 cards and a shuffle sequence of length 6. Break the deck up into non-overlapping packets of 2^{8-k} cards, and further break each of these packets into four quarters. A perfect or semi-perfect shuffle at position k in the sequence (where $2 \leq k \leq 6$) will move the quarters within each of the separate packets according to the shuffle's associated permutation.

For $k = 2$ this leads to breaking the deck up into quarters, for $k = 3$ it leads to breaking the deck into eighths, for $k = 3$ it leads to sixteenths, etc. We do not include the value $k = 1$ though as it would lead to packet sizes of 2^7, which are larger than the deck itself. We hinted that the first shuffle in the sequence can control the halves of the deck though, something that holds true if we look at perfect shuffles alone. To swap the halves of the deck such that the final order has $[clubs, hearts, spades, diamonds]$ rather than the original order of $[spades, diamonds, clubs, hearts]$, we simply perform an in-faro followed by five out-faros. In relation to this we can make our second claim

Claim. 4.2. Assume we have a deck of 64 cards and a shuffle sequence of length 6. Break the deck up into non-overlapping packets of 2^{7-k} cards, and further break each of these packets into two halves. An in-faro at position k in the sequence (where $1 \leq k \leq 6$) will swap the halves within each of the separate packets. An out-faro at the same position will leave the halves unmoved.

This gives identical results to the first claim when considered for the in-faro and out-faro, with the important exception of having extended the range to the first shuffle. It also gives us a slightly more intuitive way of thinking of the in-faro: whereas the semi-perfect shuffles must be viewed as applying to quarters, the in-faro can more organically be thought of as working on halves.

An exciting corollary of the second claim is that repeated applications of an in-faro can result in reflections! We've already mentioned that 6 in-faros will perfectly reverse the order of a 64 card deck. If we were to instead apply the sequence

$$[f^o, f^i, f^i, f^i, f^i, f^i]$$

the clubs and hearts will stay in the same half of the deck they started in, but that half will be perfectly reversed. The same reversal of order will happen to the half containing the the spades and diamonds. A third sequence

$$[f^o, f^o, f^i, f^i, f^i, f^i]$$

will keep every suit where it was originally, but perfectly reverse the order of each individual suit. Performing 3 out-faros followed by 3 in-faros will keep every eighth of the deck in its original position, but perfectly reverse the internal order of each of those individual eighths. And so on, as long as the tail-end of the sequence is a series of consecutive in-faros.

These properties can all be extended to decks of size 2^n in general, but a deck of 64 is the ideal compromise between a deck size that is small enough to handle yet large enough to draw on reliably.

5. Camping Under the Stars

Strong magical effects create a sense of wonder and surprise. The card manipulations we have touched on are interesting in themselves, but can hardly be called magical. Perfect shuffles have been used for great magical effect though, with a particularly relevant example being *Unshuffled* by Paul Gertner. In this effect a scrambled deck with seemingly arbitrary markings on its side is shuffled using three consecutive out-faros. After the first faro the word 'unshuffled' appears four times on the side of the deck. After the second faro those four words are combined into two. After the third shuffle only a single 'unshuffled' will remain. The deck is then spread to show that everything has returned to perfect order! Usually this is followed up by the revelation of a spectator's chosen card written on the other side of the deck, but as that uses a magician's force rather than mathematics we will not touch on that here. What we will do is try build on the idea of drawing manipulation, and to that end we ask the reader to think of starry nights and campsites.

The outdoors scene shown in Figure 7 was drawn specifically to take advantage of our shuffle techniques. The ground is a continuous and (mostly) steady line drawn along the side deck. Each tree is drawn individually on a packet of 8 cards, this being the size of an eighth of the deck. The moon in the sky is confined to the same eighth as the tree below it. The individual fire logs and their smoke plumes are similarly confined to an eighth of the deck each. The tent covers a quarter of the deck in total, but the triangle acting as the entrance is strictly confined to an eighth. The lowest branches of each tree are in-line with the roof of the tent. Though our campsite is rough and crudely drawn, it should suffice to show what is possible with these shuffles.

FIGURE 7. The camp scene, with a tent and burning logs amidst a forest of trees. This is the default arrangement of our deck that all altered scenes shall be generated from.

When handling a shuffle sequence we will never perform two semi-perfect shuffles in a row. To get into the jogged state needed for a semi-perfect shuffle we first need to perform a regular faro (as illustrated by the faro jog of earlier sections), and so there will always be an in or out faro preceding a semi-perfect shuffle. This need for an initial regular faro is avoidable if the card manipulator can perform Christian Engblom's anti-faro, or at least the springing component thereof, but it is a rare and difficult move to perfect.

Given that final constraint, consider the shuffle sequence

$$[f^i, f^i, d^o, f^o, f^o, f^o]$$

to be applied to our deck. Figure 8 shows the state of our camp scene after each perfect or semi-perfect shuffle. At step 3 particularly, after three shuffles and exactly halfway through the sequence, the scene is truly unrecognizable and mixed. This seemingly chaotic mess is then restored by the next three shuffles, though our heroic voyage through chaos has left things changed. Hopefully you can forgive us this dramatic use of language, as the intent is to encourage performers to think on how they might present the effect. We will not crush creativity by dictating performance details, but given the mixed-and-restored nature of this effect it might be worth taking cues from torn-and-restored or cut-and-restored magic routines.

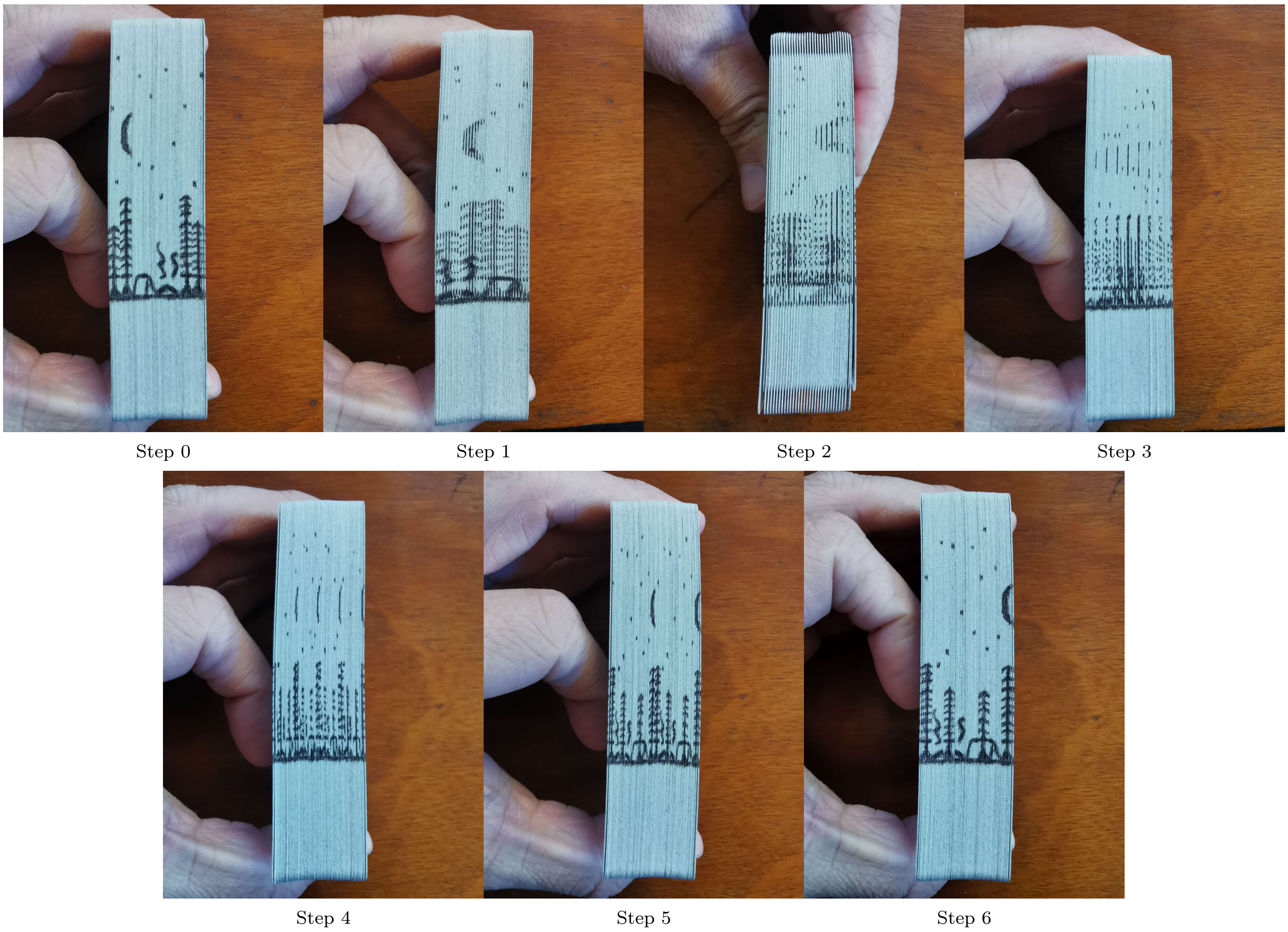
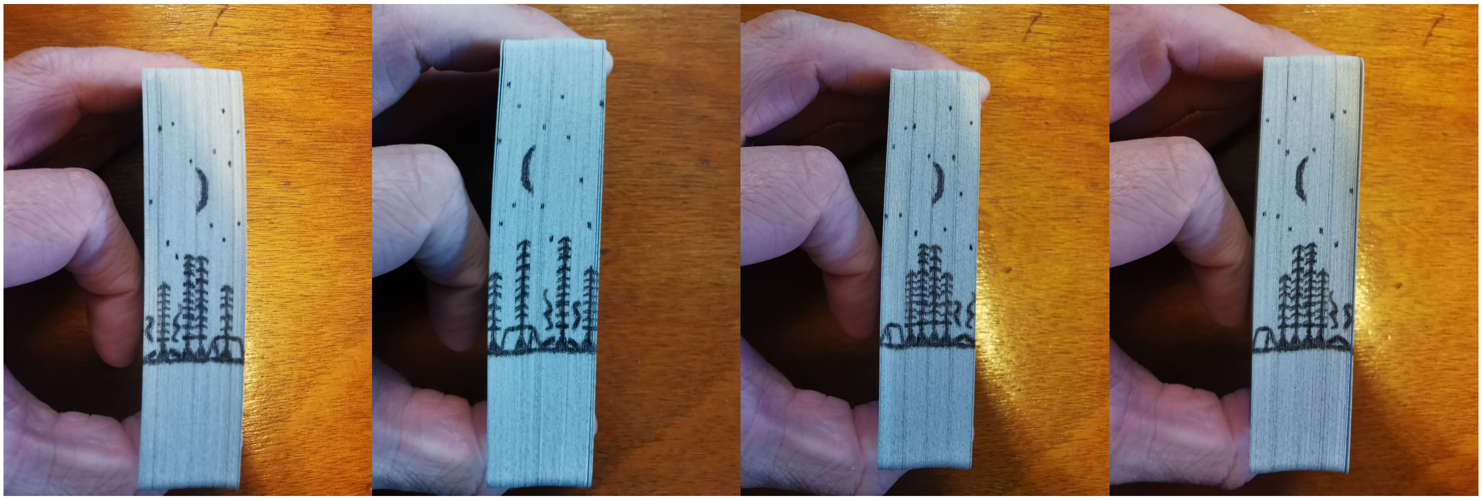


FIGURE 8. Here we show the state of the camp scene after each individual perfect or semi-perfect shuffle for the sequence $[f^i, f^i, d^o, f^o, f^o, f^o]$. The deck is not squared up at stage 3 as the jog is needed to perform the out-double. The inverse sequence to return the deck from stage 6 back to the original campsite seen in stage 0 would be $[f^i, f^o, d^i, f^o, f^o, f^o]$.

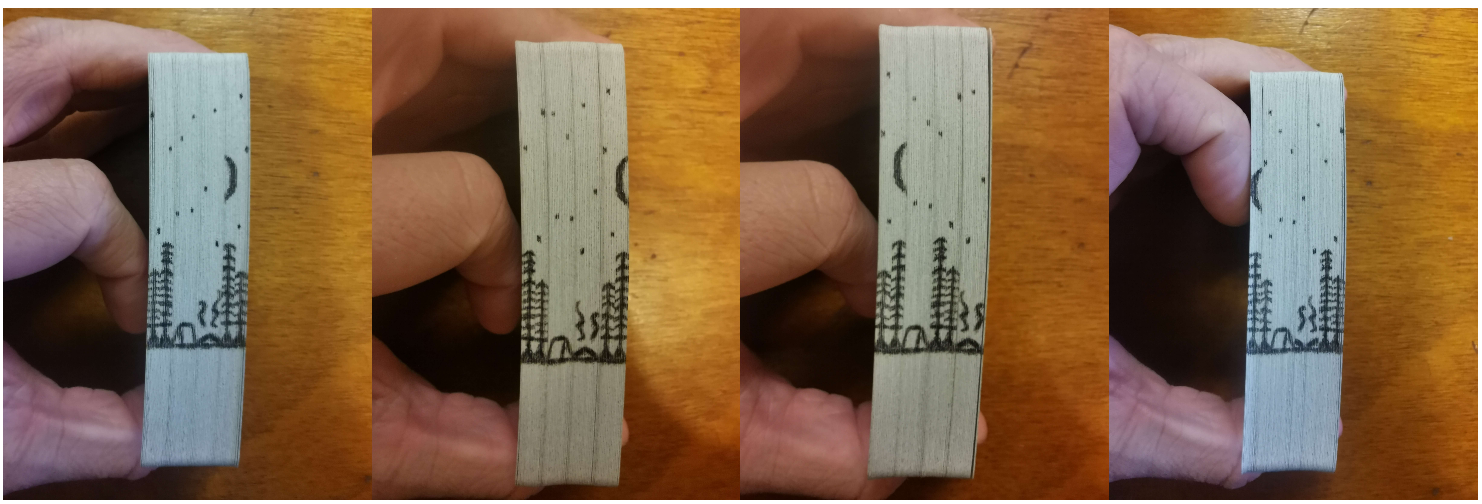


Sequence: $[f^i, f^o, d^i, f^i, f^i, f^i]$
Inverse: $[f^i, f^i, d^o, f^i, f^i, f^i]$

Sequence: $[f^o, f^o, d^o, f^o, f^o, f^o]$
Inverse: $[f^o, f^o, d^o, f^o, f^o, f^o]$

Sequence: $[f^o, d^i, f^i, f^i, f^i, f^i]$
Inverse: $[f^i, d^o, f^i, f^i, f^i, f^i]$

Sequence: $[f^o, f^i, f^o, f^o, f^o, f^o]$
Inverse: $[f^o, f^i, f^o, f^o, f^o, f^o]$



Sequence: $[f^i, d^i, f^i, f^i, f^i, f^i]$
Inverse: $[f^i, d^i, f^i, f^i, f^i, f^i]$

Sequence: $[f^i, d^i, f^o, f^o, f^o, f^o]$
Inverse: $[f^i, d^i, f^o, f^o, f^o, f^o]$

Sequence: $[f^o, s^i, f^o, f^o, f^o, f^o]$
Inverse: $[f^o, s^i, f^o, f^o, f^o, f^o]$

Sequence: $[f^o, f^o, s^o, f^o, f^o, f^o]$
Inverse: $[f^o, f^o, s^o, f^o, f^o, f^o]$

FIGURE 9. Each shuffle sequence here is applied to a deck initially in the order seen in Figure 7 to achieve the shown final state. The inverse sequence will return the deck to that original order.

When we bring the campsite back we can do so in multiple ways. Figure 9 shows a handful of such options and the respective shuffle sequences needed to achieve them, though many more are possible. The positions of the trees, the tent and the fire can all be adjusted within certain bounds, even to the level of moving single trees or individual halves of the tent. In addition to these coarse grained movements we can also reflect parts of the deck. This is best taken advantage of with the moon, where it can be decided whether the crescent will face left or right as desired. If you are comfortable enough with the shuffles you will be able to figure out what sequence is needed to get what particular outcome (if it is possible). You will even be able to let the spectator choose details of the final image while you are still shuffling, though the choices at that point will of course be restricted.

Different illustrations other than a campsite are of course possible. A simple example would have a smiley face on the side of a 32 card deck with the smile on the bottom 16 cards and two circles as eyes on the top 16 cards. An out-faro followed by four in-faros will transform the smiling face into a frowning one, and applying the same sequence once more will turn the frown back into a smile. The circle eyes will remain whether there is a frown or smile because a reflected circle is still a circle. As before, the deck will be mixed and seemingly disordered at stages during the shuffling. This is of course much simpler than the camp scene, but it does illustrate that both portrait and landscape images can benefit from the drawing manipulations of our shuffle techniques. We ourselves are not artistically trained, and can only imagine what a more skilled individual would be able to draw and create.

6. Conclusions

The shuffling techniques we have looked at in this paper are capable of novel forms of card manipulation, giving more control over a deck of cards than possible with perfect shuffles alone. These techniques are particularly well suited for drawing manipulation, as seen in our handling of the camp scene. The shuffles themselves can be mathematically described, and shuffle sequences made from perfect and semi-perfect shuffles seem to form a group.

Acknowledgements and Additional Remarks

Thanks to Brandon du Preez for pointing out how to proceed with Theorem 3.1, specifically in needing to tackle it in cases! If anyone wishes to see a video performance of the shuffles being applied to the camp site, please feel free to contact the author.

Appendix: Permutations

To understand what our shuffles can do, knowing how permutations work will be immensely helpful. If we have a collection of objects we can arrange said objects in a particular order. One example would be $\{a, b, c, d\}$, a collection of four letters in standard alphabetical order. If we were to rearrange the letters though, we could get a new order like $\{b, d, a, c\}$. Mathematicians would call this moving of objects a permutation, and one way to represent the above permutation would be

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

So what's going on here? The first row represents the original collection's order, and the second row represents where everything has been moved. Reading the above, we can look at the first column and see that whatever was originally in position 1 is moved to position 3. The second column tells us that whatever was originally in position 2 is moved to position 1. What was in the third position is moved to the fourth, and what was in the fourth is moved to the second. To illustrate with our letter example a little more explicitly, this would mean we move a to the third position, b to the first, c to the fourth and d to the second, giving the final arrangement $\{b, d, a, c\}$.

A more elegant and compact way of describing the above moving of pieces can be given by cycle notation. Consider

$$(1\ 3\ 4\ 2)$$

where we read the positions from left to right. This one line tells us that what was in position 1 moves to position 3, that what was in position 3 moves to position 4, and that what was in position 4 moves to position 2. It also tells us that what was at position 2 moves to position 1, as the end cycles back to the beginning. In this way it describes the same movement of letters as our earlier two line description. Because of the cyclical nature of the notation, the permutation $(1\ 3\ 4\ 2)$ could also be given by $(2\ 1\ 3\ 4)$ or others, all describing the same moving of items but just changing where we start the cycle.

Having more than one example usually helps to understand things. The line

$$(1)(2)(3\ 4)$$

tells us that whatever was in position 1 stays in position 1, and that whatever was in position 2 stays put as well. The $(3\ 4)$ tells us that what was third moves to the fourth position, and what was fourth moves to the third. Another example could have

$$(1)(2\ 3\ 4)$$

which tells us that whatever was second moves up to the third position, what was third moves up to the fourth, and what was fourth moves down to the second. This is because $(2\ 3\ 4)$ is a complete cycle by itself, represented by the brackets around it, and so the 4 cycles back to 2. The separate (1) tells us that whatever was in the first position stays put, being its own cycle of length one. A final example

$$(1\ 2)(3\ 4)$$

has the objects in positions 1 and 2 swap, and the objects in positions 3 and 4 swap. Many other permutations of four elements are possible, and permutations of larger collections are possible, but we hope this serves as a sufficient introduction.

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