Assessment 1:

Question 1. Let $f : X \to Y$ be a function and take $A \subseteq X, B \subseteq Y$.

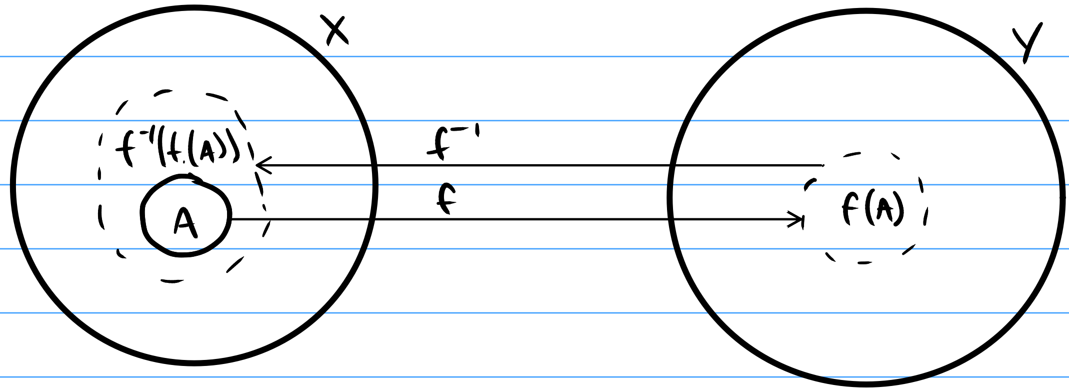
Prove the following statements.

(a) $A \subseteq f^{-1}(f(A))$.

(b) In general, equality need not hold in (a).

(c) $G = f^{-1}(f(G))$ for every subset G of X if and only if f is injective (1–1).

Q1 a)



case: f is injective

f injective $:\Leftrightarrow \forall x, x' \in X, \ (f(x) = f(x')) \to (x = x')$

this condition means that $f^{-1}(f(A)) = A$

case: f is not injective

f ~(injective) $:\Leftrightarrow \exists x, x' \in X, \ (f(x) = f(x') \nrightarrow (x = x'))$

this condition means several elements in A map onto the same element in f(A)

$\therefore A \subset f^{-1}(f(A))$

taking both cases together, $A \subseteq f^{-1}(f(A))$

Q1 b) I think the equality is $A = f^{-1}(f(A))$ but as demonstrated, it is only true if f is injective

Let G be any subset $G \subseteq X$

Q1 c) The claim:

$\forall G \subseteq X,\ G = f^{-1}(f(G))$ <u>iff</u> f is injective

We want to show:

f injective $:\Leftrightarrow \forall x, x' \in X,\ (f(x) = f(x')) \to (x = x')$

($\Rightarrow$) Assume

$$G = f^{-1}(f(G))$$

We can rewrite this as a left inverse

$g = f^{-1}$

$g \circ f = id_X$, $g \circ f : X \to X$

$x \to x$

So, $\forall x, x' \in X,\ \left(g(f(x)) = g(f(x'))\right) \to (x = x')$

($\Leftarrow$) Assume:

f injective $:\Leftrightarrow \forall x, x' \in X,\ (f(x) = f(x')) \to (x = x')$

then $f(x) = f(x')$

$x = x'$

and $g \circ f(x) = g \circ f(x')$

$x = x'$

$g \circ f(x)$ can be written as

$f^{-1}(f(x))$

Since it was specified for all $x \in G$

then $\therefore G = f^{-1}(f(x))$

Question 2.

Given functions $f : X \to Y$ and $g : Y \to Z$, prove the following statements.

(a) If f and g are both surjective, then so is $g \circ f$.

(b) If $g \circ f$ is surjective, then so is g, but not necessarily f.

(c) If f and g are bijective, so is $g \circ f$.

(d) If $g \circ f$ is bijective, then neither f nor g need be bijective.

Q2 a)

- (f is surjective) $:\Leftrightarrow \forall y \in Y, \exists x \in X : f(x) = y$
- (g is surjective) $:\Leftrightarrow \forall z \in Z, \exists y \in Y : g(y) = z$

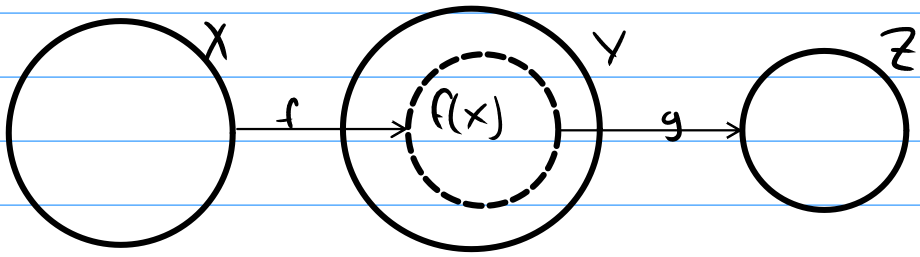
$g \circ f$ is a composition such that:

$\forall z \in Z, \exists x \in X : g(f(x)) = z$

$\therefore g \circ f : X \to Z$, $x \to z$ meets the criteria for surjectivity

Q2 b)

$g \circ f$ is surjective $:\Leftrightarrow \forall z \in Z, \exists x \in X : g(f(x)) = z$



- g must be surjective so that every element in Z, $z = g(f(x))$
- if g was not surjective, then $\exists z \in Z, \forall y \in Y : g(y) \neq z$
  then $\exists z \neq g(f(x))$, so $g \circ f$ would not be surjective.

- but f could be not surjective. This would be true where the image $f(x) \subset Y$

Q2 c)

Assume f is bijective :<=> injective and surjective
g " " " " "

(surjectivity) f satisfies $(\forall y \in Y, \exists x \in X : f(x) = y)$
g satisfies $(\forall z \in Z, \exists y \in Y : g(y) = z)$

then $g \circ f : X \to Z$
$x \to g(f(x))$
implies $(\forall z \in Z, \exists x \in X : g(f(x)) = z)$
then $g \circ f$ exhibits the surjective property

(injectivity) f satisfies $\forall x, x' \in X, (f(x) = f(x')) \to (x = x')$
g satisfies $\forall y, y' \in Y, (g(y) = g(y')) \to (y = y')$

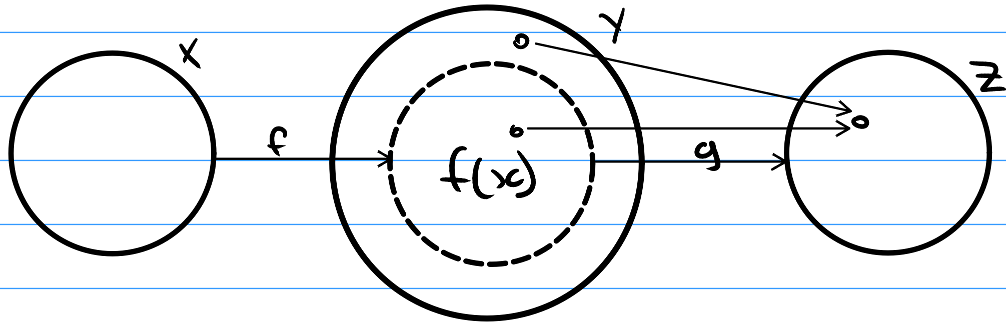
then $g \circ f := g(f(x))$
satisfies $\forall x, x' \in X, (g(f(x)) = g(f(x')) \to (x = x'))$
so $g \circ f$ is injective

since $g \circ f$ is injective and surjective,
then it is also bijective.

Q2 d)

If $g \circ f$ is bijective then:

it is injective $\forall x, x' \in X, (g(f(x)) = g(f(x')) \rightarrow (x = x'))$
it is surjective $\forall z \in Z, \exists x \in X : g(f(x)) = z$



- then g is also surjective because

$\forall z \in Z, \exists y \in Y : g(y) = z$
but if $f(x) \subset Y$ then g is not injective
strictly less than

- f is injective because

$\forall x, x' \in X, (f(x) = f(x') \rightarrow (x = x'))$

but if $f(x) \subset Y$ the f is not surjective.

Q3 (GR2) The group has a right neutral:

① $a * e_R = a$

Suppose it has a left inverse

② $a = e_L * a$

Using eq ①, substitute $a = e_L$

so: $e_L * e_R = e_L$

$e_R = e_L * e_R = e_L$

$\therefore e_R = e_L$

let it be e

so, the axiom G2 is satisfied

(GR3) The group has a right inverse element:

$a * \bar{a}_R = e$

suppose there is also a left inverse

$e = \bar{a}_L * a$

$(\bar{a}_L * a) * \bar{a}_R = e\,\bar{a}_R = \bar{a}_R$

by associativity:

$\bar{a}_L\, e = \bar{a}_L * (a * \bar{a}_R) = \bar{a}_L$

$\therefore \bar{a}_L = \bar{a}_L * a * \bar{a}_R = \bar{a}_R$

$\bar{a}_L = \bar{a}_R$

G1, G2, G3 are all satisfied

$\therefore (G, *)$ is a group