Homework #2

1. Show that the Crank-Nicholson method for 2D heat equation, $u_t = \beta(u_{xx} + u_{yy}) + f$, is consistent and stable.

2. The solution of the initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 \le x \le 1, \quad t > 0, \\ u(x,0) &= \begin{cases} 2x, & 0 \le x \le 1/2, \\ 2(1-x), & 1/2 \le x \le 1, \end{cases} \\ u(0,t) &= u(1,t) = 0 \end{aligned}$$

is

$$u(x,t) = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-k^2\pi^2 t} \sin(k\pi/2) \sin k\pi x.$$

Use the implicit scheme, take $h = 0.1,\ 0.05,\ 0.025,\ 0.0125$ and $\tau = h/2$, compare the results to the exact solution for $t = 0.1$ (take 5 terms in the summation of the exact solution).

3. Show that the box scheme:

$$\frac{(U_i^{k+1} + U_{i+1}^{k+1}) - (U_i^k + U_{i+1}^k)}{2\tau} + a\frac{(U_{i+1}^{k+1} - U_i^{k+1}) + (U_{i+1}^k - U_i^k)}{2h} = f_{i+\frac{1}{2}}^{k+\frac{1}{2}}$$

is consistent with the one-way wave equation with a source term ($u_t + au_x = f$) and is unconditionally stable. What is the order of the local truncation error?

4. Use Lax-wendroff scheme to solve

$$u_t + 3u_x = 0,$$

$$u(x,0) = x^2.$$

Let $h = 0.05$ and $\tau = 0.01$, compare the approximate solution to the exact solution at $x = 1, \quad t = 1$.