武汉大学数学与统计学院

偏微分方程数值解实验报告

Homework 3

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目录

Problem1

用线性有限元方法求解一维 Strum-Liouville 问题

$$
\begin{aligned}
-(pu')' + qu = f, \quad 0 < x < 2\pi, \\
u(0) = u(2\pi) = 0.
\end{aligned}
$$

其中 $p(x) = 1 + x,\ q(x) = x,\ f(x) = (2x + 1)\sin x - \cos x$, 方程的真实解为 $u(x) = \sin x$.

1.1 理论推导

1. 将问题转化成弱形式，在等式的两边同时乘以一个试函数 $v(x)$，其满足 Dirichlet 边界条件 $v(0) = v(2\pi) = 0$，得到

$$
\begin{aligned}
\int_0^{2\pi} \left(-(pu')' + qu\right) v dx &= -pu'v\Big|_0^{2\pi} + \int_0^{2\pi} (pu'v' + quv) dx \\
&= \int_0^{2\pi} fv dx \\
\Longrightarrow \int_0^{2\pi} (pu'v' + quv) dx &= \int_0^{2\pi} fv dx, \quad \forall v \in H_0^1(0, 2\pi).
\end{aligned}
$$

2. 生成网格，在本问题的实现中使用的是笛卡尔网格 $x_i = ih,\ i = 0, ..., n$，因此网格尺寸为 $h = \frac{1}{n}$，并定义区间 $(x_{i-1}, x_i), i = 1, 2, ..., n.$

3. 构造基函数，选择分片线性具有紧支撑的 hat function.$(i = 1, 2, ..., n - 12)$

$$
\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{if } x_{i-1} \le x < x_i, \\ \frac{x_{i+1} - x}{h}, & \text{if } x_i \le x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}, \quad \phi_i'(x) = \begin{cases} \frac{1}{h}, & \text{if } x_{i-1} \le x < x_i, \\ -\frac{1}{h}, & \text{if } x_i \le x < x_{i+1}, \\ 0, & \text{otherwise.} \end{cases}
$$

4. 用基函数的线性组合表示有限元解

$$
u_h(x) = \sum_{j=1}^{n-1} c_j \phi_j(x)
$$

将 $u_h(x)$ 代入问题的弱形式 $\int_0^{2\pi} (pu'v' + quv) dx = \int_0^{2\pi} fv dx$ 中得到

$$
\int_0^{2\pi} (p \sum_{j=1}^{n-1} c_j \phi_j' v' + q \sum_{j=1}^{n-1} c_j \phi_j v) dx = \sum_{j=1}^{n-1} c_j \int_0^{2\pi} (p\phi_j' v' + q\phi_j v) dx = \int_0^{2\pi} fv dx
$$

将试函数 $v(x)$ 取为 $\phi_1, \phi_2, ..., \phi_{n-1}$，可以得到如下的线性方程组

$$
\begin{aligned}
\left(\int_0^{2\pi}(p\phi_1'\phi_1' + q\phi_1\phi_1)dx\right)c_1 + ... + \left(\int_0^{2\pi}(p\phi_1'\phi_{n-1}' + q\phi_1\phi_{n-1})dx\right)c_{n-1} &= \int_0^{2\pi} f\phi_1 dx \\
\left(\int_0^{2\pi}(p\phi_2'\phi_1' + q\phi_2\phi_1)dx\right)c_1 + ... + \left(\int_0^{2\pi}(p\phi_2'\phi_{n-1}' + q\phi_2\phi_{n-1})dx\right)c_{n-1} &= \int_0^{2\pi} f\phi_2 dx \\
... \quad ... \quad ... \quad ... \quad ... \quad ... \\
\left(\int_0^{2\pi}(p\phi_i'\phi_1' + q\phi_i\phi_1)dx\right)c_1 + ... + \left(\int_0^{2\pi}(p\phi_i'\phi_{n-1}' + q\phi_i\phi_{n-1})dx\right)c_{n-1} &= \int_0^{2\pi} f\phi_i dx \\
... \quad ... \quad ... \quad ... \quad ... \quad ... \\
\left(\int_0^{2\pi}(p\phi_{n-1}'\phi_1' + q\phi_{n-1}\phi_1)dx\right)c_1 + ... + \left(\int_0^{2\pi}(p\phi_{n-1}'\phi_{n-1}' + q\phi_{n-1}\phi_{n-1})dx\right)c_{n-1} &= \int_0^{2\pi} f\phi_{n-1} dx
\end{aligned}
$$

上述线性方程组可被表示为以下的形式

$$
\begin{bmatrix}
a(\phi_1,\phi_1) & a(\phi_1,\phi_2) & \cdots & a(\phi_1,\phi_{n-1}) \\
a(\phi_2,\phi_1) & a(\phi_2,\phi_2) & \cdots & a(\phi_2,\phi_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
a(\phi_{n-1},\phi_1) & a(\phi_{n-1},\phi_2) & \cdots & a(\phi_{n-1},\phi_{n-1})
\end{bmatrix}
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}
=
\begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_{n-1}) \end{bmatrix}
$$

$$
a(\phi_i,\phi_j) = \int_0^{2\pi}(p\phi_i'\phi_j' + q\phi_i\phi_j)dx,\ (f,\phi_j) = \int_0^{2\pi} f\phi_j dx
$$

由于 ϕ_i 是具有紧支撑的基函数，易知 $p\phi_i'\phi_j' + q\phi_i\phi_j$ 也是具有紧支撑的，因为

$$
\phi_j'(x)\phi_i'(x) \equiv 0,\ \phi_j(x)\phi_i(x) \equiv 0,
\begin{cases}
\forall x \notin (x_{j-1}, x_j) & \text{if } i = j-1, \\
\forall x \notin (x_{j-1}, x_{j+1}) & \text{if } i = j, \\
\forall x \notin (x_j, x_{j+1}) & \text{if } i = j+1, \\
\forall x \in (0, 2\pi) & \text{otherwise.}
\end{cases}
$$

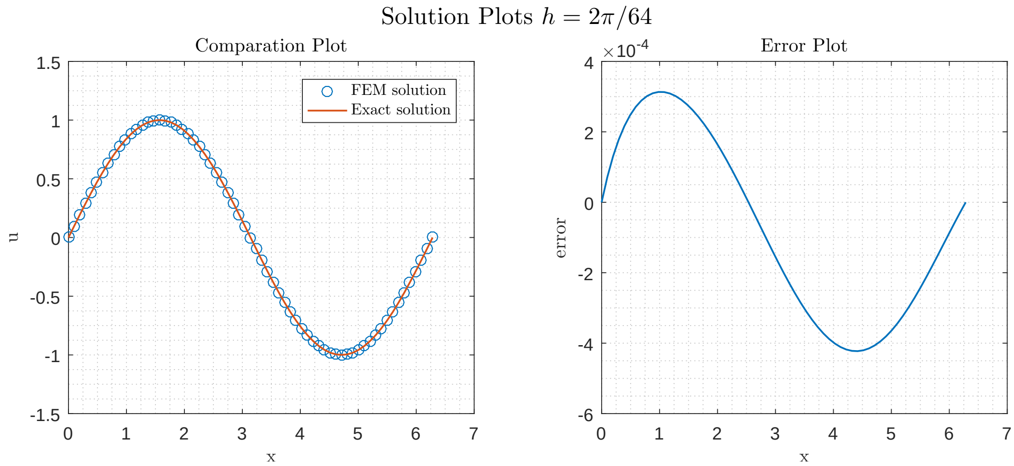
那么全局刚度矩阵应该是一个三对角矩阵，且全局刚度矩阵和全局载荷向量分别可以用以下的局部刚度矩阵和局部载荷向量逐元素组装.

$$
K_i^e = \begin{bmatrix}
\int_{x_i}^{x_{i+1}} p\phi_i'^2 dx & \int_{x_i}^{x_{i+1}} p\phi_i'\phi_{i+1}' dx \\
\int_{x_i}^{x_{i+1}} p\phi_{i+1}'\phi_i' dx & \int_{x_i}^{x_{i+1}} p\phi_{i+1}'^2 dx
\end{bmatrix}
+
\begin{bmatrix}
\int_{x_i}^{x_{i+1}} q\phi_i^2 dx & \int_{x_i}^{x_{i+1}} q\phi_i\phi_{i+1} dx \\
\int_{x_i}^{x_{i+1}} q\phi_{i+1}\phi_i dx & \int_{x_i}^{x_{i+1}} q\phi_{i+1}^2 dx
\end{bmatrix}
$$

$$
F_i^e = \begin{bmatrix}
\int_{x_i}^{x_{i+1}} f\phi_i dx \\
\int_{x_i}^{x_{i+1}} f\phi_{i+1} dx
\end{bmatrix}
$$

1.2 数值结果

令 $h = \pi/32$，计算得到如下结果，其中 $\|u - u_h\|_\infty = 0.00042261$.



1.3 误差分析

根据教材 P178 *Remark7.1.3* 的结论，在一维 Strum-Liouville 问题中，线性有限元方法有如下的误差估计

$$\|u - u_h\|_\infty \leq Ch^2 \|u''\|_\infty$$

改变网格尺度，再次计算得到如下结果

h	Error
$2\pi/32$	0.0016872
$2\pi/64$	0.00042261
$2\pi/128$	0.00010566
$2\pi/256$	2.642e-05
$2\pi/512$	6.6049e-06
$2\pi/1024$	1.6512e-06

h_1	h_2	Order
$2\pi/32$	$2\pi/64$	1.9973
$2\pi/64$	$2\pi/128$	2
$2\pi/128$	$2\pi/256$	1.9997
$2\pi/256$	$2\pi/512$	2
$2\pi/512$	$2\pi/1024$	2

从网格细化分析的结果可以看出此问题中误差是二阶的.

Problem2

求以下问题的弱形式

$$-a^2\frac{\partial^2 u(x,y)}{\partial x^2}-b^2\frac{\partial^2 u(x,y)}{\partial y^2}+q(x,y)u(x,y)=f(x,y),$$

$$(x,y)\in\Omega=\{(x,y):x^2+y^2\le 1\},\quad u(x,y)|_{\partial\Omega}=0,\quad q(x,y)\ge 0.$$

2.1 理论推导

考虑坐标变换 $\bar{x}=x/a;\ \bar{y}=y/b$，则 $(\bar{x},\bar{y})\in\bar{\Omega}=\{(\bar{x},\bar{y}):a^2\bar{x}^2+b^2\bar{y}^2\le 1\}$ 并定义

$$\bar{u}(\bar{x},\bar{y})=u(a\bar{x},b\bar{y});\ \bar{q}(\bar{x},\bar{y})=q(a\bar{x},b\bar{y});\ \bar{f}(\bar{x},\bar{y})=f(a\bar{x},b\bar{y}).$$

显然有 $\bar{u}(\bar{x},\bar{y})=u(x,y),\ \bar{q}(\bar{x},\bar{y})=q(x,y),\ \bar{f}(\bar{x},\bar{y})=f(x,y)$，代入原方程得到

$$-a^2\frac{\partial^2\bar{u}}{\partial x^2}-b^2\frac{\partial^2\bar{u}}{\partial y^2}+\bar{q}(\bar{x},\bar{y})\bar{u}(\bar{x},\bar{y})=\bar{f}(\bar{x},\bar{y}),$$

又因为

$$\frac{\partial\bar{u}}{\partial x}=\frac{\partial\bar{u}}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x}=\frac{1}{a}\frac{\partial\bar{u}}{\partial\bar{x}},\quad \frac{\partial^2\bar{u}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1}{a}\frac{\partial\bar{u}}{\partial\bar{x}}\right)=\frac{1}{a^2}\frac{\partial^2\bar{u}}{\partial\bar{x}^2},$$
$$\frac{\partial\bar{u}}{\partial y}=\frac{\partial\bar{u}}{\partial\bar{y}}\frac{\partial\bar{y}}{\partial y}=\frac{1}{b}\frac{\partial\bar{u}}{\partial\bar{y}},\quad \frac{\partial^2\bar{u}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{1}{b}\frac{\partial\bar{u}}{\partial\bar{y}}\right)=\frac{1}{b^2}\frac{\partial^2\bar{u}}{\partial\bar{y}^2},$$

可将方程化为

$$-\frac{\partial^2\bar{u}(\bar{x},\bar{y})}{\partial\bar{x}^2}-\frac{\partial^2\bar{u}(\bar{x},\bar{y})}{\partial\bar{y}^2}+\bar{q}(\bar{x},\bar{y})\bar{u}(\bar{x},\bar{y})=\bar{f}(\bar{x},\bar{y}),$$

取试函数 $\bar{v}(\bar{x},\bar{y})=v(a\bar{x},b\bar{y})=v(x,y)$，且满足 $v(x,y)|_{\partial\Omega}=0$，故显然有 $\bar{v}(\bar{x},\bar{y})|_{\partial\bar{\Omega}}=0$，在上式两端同时乘以 $\bar{v}$ 并在 $\bar{\Omega}$ 上积分有

$$\iint_{\bar{\Omega}}\left(-\Delta\bar{u}+\bar{q}\bar{u}\right)\bar{v}d\bar{x}d\bar{y}=\iint_{\bar{\Omega}}\bar{f}\bar{v}d\bar{x}d\bar{y}$$

由散度定理

$$\iint_{\bar{\Omega}}\bar{v}\Delta\bar{u}d\bar{x}d\bar{y}=\int_{\partial\bar{\Omega}}\bar{v}\frac{\partial\bar{u}}{\partial\bar{n}}d\bar{s}-\iint_{\bar{\Omega}}\nabla\bar{u}\cdot\nabla\bar{v}d\bar{x}d\bar{y}=-\iint_{\bar{\Omega}}\nabla\bar{u}\cdot\nabla\bar{v}d\bar{x}d\bar{y}$$

代入前面的结果有

$$\iint_{\bar{\Omega}}\left(\nabla\bar{u}\cdot\nabla\bar{v}+\bar{q}\bar{u}\bar{v}\right)d\bar{x}d\bar{y}=\iint_{\bar{\Omega}}\bar{f}\bar{v}d\bar{x}d\bar{y},\quad \forall\bar{v}\in H_0^1(\bar{\Omega}).$$

将坐标还原得到原问题的弱形式

$$\iint_{\Omega}\left(a^2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+b^2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+quv\right)dxdy=\iint_{\Omega}fvdxdy,$$
$$\forall v\in H_0^1(\Omega)=\{v(x,y),\ v(x,y)|_{\partial\Omega}=0,\ v(x,y)\in H^1(\Omega)\}.$$