Homework #1

1. Program the central FD method for the self-adjoint BVP

$$\begin{aligned}(\beta(x)u')' - \gamma(x)u(x) &= f(x), \quad 0 < x < 1, \\ u(0) = u_a, \qquad & au(1) + bu'(1) = c \,,\end{aligned}$$

using a uniform grid and the central FD scheme

$$\frac{\beta_{i+\frac{1}{2}}(U_{i+1} - U_i)/h - \beta_{i-\frac{1}{2}}(U_i - U_{i-1})/h}{h} - \gamma(x_i)U_i = f(x_i) \,. \tag{1}$$

Test your code for the case where

$$\beta(x) = 1 + x^2, \quad \gamma(x) = x, \quad a = 2, \quad b = -3, \tag{2}$$

and the other functions or parameters are determined from the exact solution

$$u(x) = e^{-x}(x-1)^2 \,. \tag{3}$$

Plot the computed solution and the exact solution, and the errors for a particular grid $n = 80$. Do the grid refinement analysis, to determine the order of accuracy of the global solution. Also try to answer the following questions:

- Can your code handle the case when $a = 0$ or $b = 0$?
- If the central finite difference scheme is used for the equivalent differential equation

$$\beta u'' + \beta' u' - \gamma u = f \,, \tag{4}$$

what are the advantages or disadvantages?

2. Modify the Matlab code *non_tp.m* to apply the finite difference method and Newton's non-linear solver to find a numerical solution to the non-linear pendulum model

$$\begin{aligned}\frac{d^2\theta}{dt^2} + K\sin\theta = 0, \qquad & 0 < \theta < 2\pi, \\ \theta(0) = \theta_1, \qquad & \theta(2\pi) = \theta_2,\end{aligned} \tag{5}$$

where K, θ_1 and θ_2 are parameters. Compare the solution with the linearized model $\dfrac{d^2\theta}{dt^2} + K\theta = 0$.

3. Implement and compare the Gauss-Seidel, and the SOR (trying to find the best ω by testing), methods for the following elliptic problem:

$$u_{xx} + p(x,y)u_{yy} + r(x,y)u(x,y) = f(x,y)$$

$$a < x < b; \qquad c < y < d,$$

with the following boundary conditions:

$$u(a,y) = 0, \quad u(x,c) = 0, \quad u(x,d) = 0, \quad \frac{\partial u}{\partial x}(b,y) = -\pi \sin(\pi y).$$

Test and debug your code for the case $0 \le x, y \le 1$, and

$$p(x,y) = (1 + x^2 + y^2), \qquad r(x,y) = -xy.$$

The source term $f(x,y)$ is determined from the exact solution

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

Do the grid refinement analysis for $n = 16, n = 32$, and $n = 64$ (if possible) in the infinity norm (**Hint:** In Matlab, use $\max(\max(abs(e)))$. Take the tolerance as 10^{-5}. Does the method behave like a second order method? Compare also the number of iterations and test the optimal relaxation factor ω. Plot the solution and the error for $n = 32$.

Having made sure that you code is working correctly, try your code with a point source $f(x,y) = \delta(c - 0.5)\delta(y - 0.5)$ and $u_x = -1$ at $x = 1$, with $p(x,y) = 1$ and $r(x,y) = 0$. Note that the $u(x,y)$ can be interpreted as the steady state temperature distribution of a room with insulated wall on three sides, a constant heat flow in from one side, and a point source (a heater for example) in the room. Note that the heat source can be expressed as $f(n/2, n/2) = 1/h^2$ and $f(i,j) = 0$ for other grid points. Use the mesh and contour plots to visualize the solution for $n = 36(mesh(x,y,u), contour(x,y,u,30))$.