3d Rotation with Quaternions

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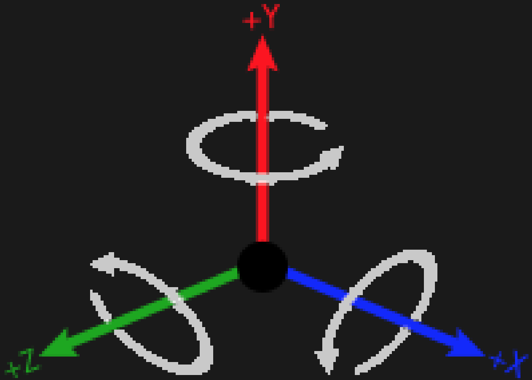
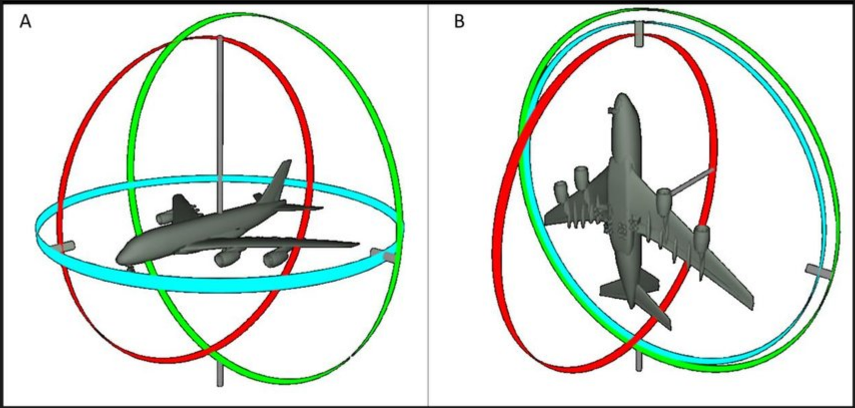
Why Calculating Rotation in 3d is Valuable:

- Physics Simulations.
- Animation.
- Mathematical Modeling.
- Navigation.
- And MUCH MORE!

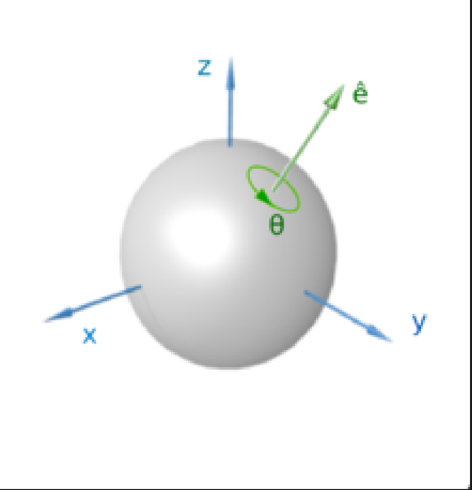
Why They Fail: Rotation Around Axes



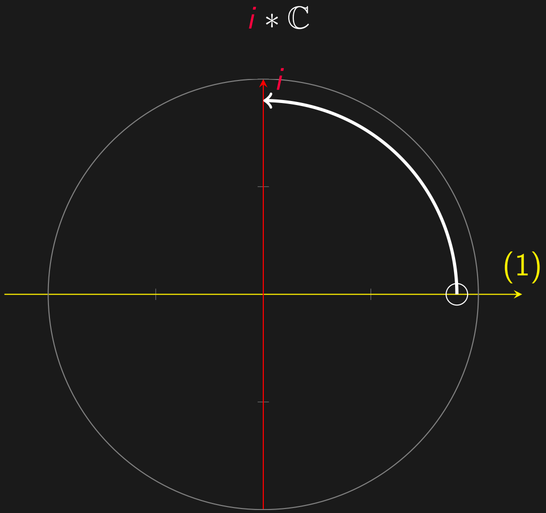
Why They Fail: Plane



What We Want

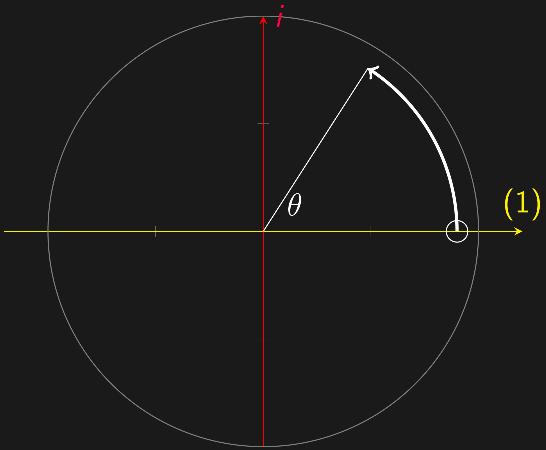


Complex Numbers



Complex Number Angles

$$(\cos\theta(1) + \sin\theta\, i) * \mathbb{C}$$



Introduce Quaternions

Complex Numbers

Quaternions

Introduce Quaternions

Complex Numbers

$$c_0(1) + c_1 i$$

Quaternions

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Complex Numbers

$$c_0(1) + c_1 i$$

Quaternions

$$c_0(1) + c_1 i + c_2 j + c_3 k$$

Introduce Quaternions

Complex Numbers

$$c_0(1) + c_1 i$$

$$i^2 = -1$$

Quaternions

$$c_0(1) + c_1 i + c_2 j + c_3 k$$

Introduce Quaternions

Complex Numbers

$$c_0(1) + c_1 i$$

$$i^2 = -1$$

Quaternions

$$c_0(1) + c_1 i + c_2 j + c_3 k$$

$$i^2 = j^2 = k^2 = -1$$

Introduce Quaternions

Complex Numbers	Quaternions
$c_0(1) + c_1 i$	$c_0(1) + c_1 i + c_2 j + c_3 k$
$i^2 = -1$	$i^2 = j^2 = k^2 = -1$

The product of any 2 different complex parts gives the third and any two different complex parts **anti-commute.**

Introduce Quaternions

Complex Numbers

$$c_0(1) + c_1 i$$

$$i^2 = -1$$

Quaternions

$$c_0(1) + c_1 i + c_2 j + c_3 k$$

$$i^2 = j^2 = k^2 = -1$$

The product of any 2 different complex parts gives the third and any two different complex parts **anti-commute.**

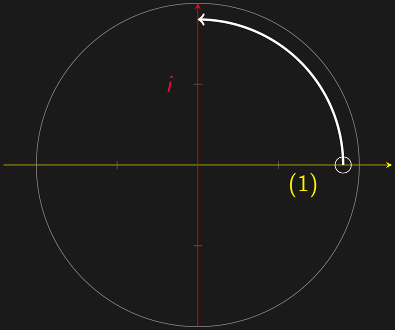
$$i * j = -j * i = k$$

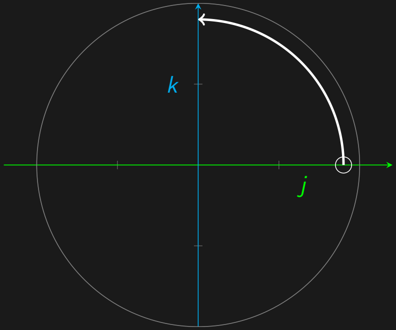
Times Tables

*	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

But What About Rotation

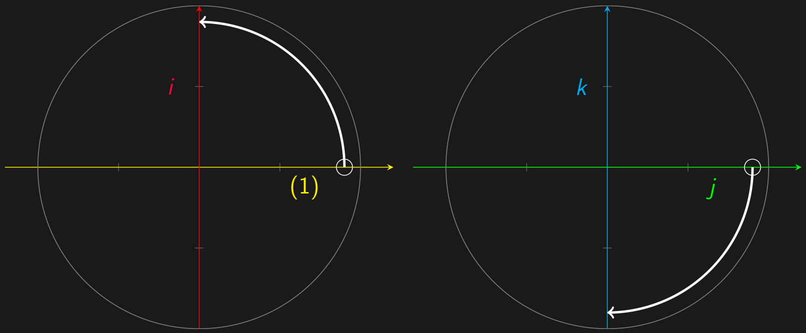
$$i * \mathbb{H}$$





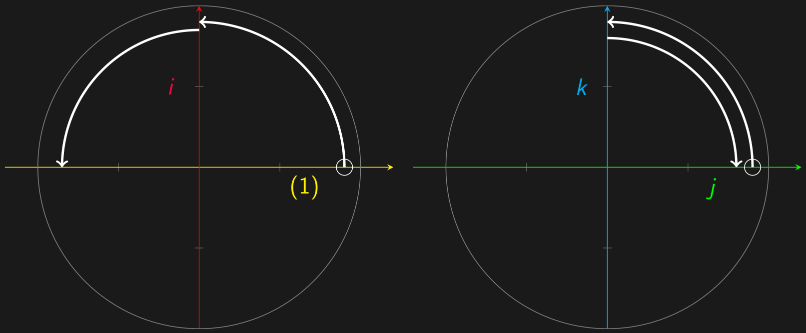
But What About Rotation

$$\mathbb{H} * i$$



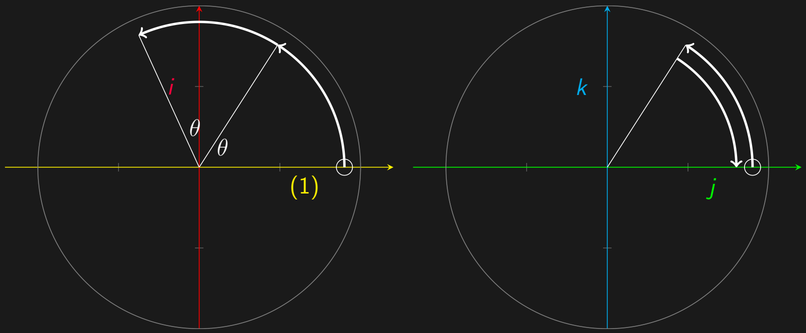
The Big Idea

$$i * \mathbb{H} * i$$



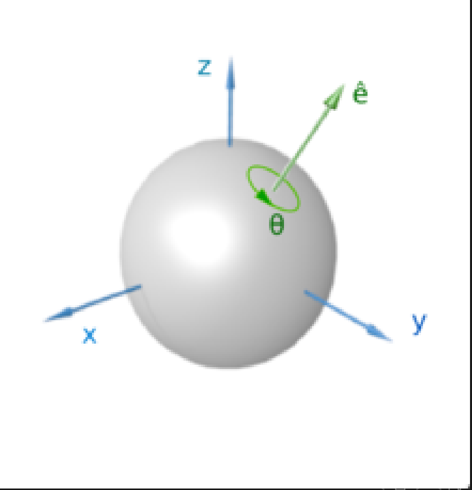
Rotation!

$$(\cos\theta(1) + \sin\theta i) * \mathbb{H} * (\cos\theta(1) + \sin\theta i)$$



3d Rotation

$$(\cos\frac{\theta}{2}(1) + \sin\frac{\theta}{2}\vec{v}) * \mathbb{H} * (\cos\frac{-\theta}{2}(1) + \sin\frac{-\theta}{2}\vec{v})$$



References

Images:

- https://upload.wikimedia.org/wikipedia/commons/thumb/5/51 /Euler_AxisAngle.png/220px-Euler_AxisAngle.png
- https://cdn.kastatic.org/ka-perseus-images/d24dd08a0ea7aaeeaa90d84f642e12998df3ffe7.svg
- https://www.researchgate.net/profile/Halim-Tannous/publication/331745225/figure/fig14/AS:736430066245632 lock-problem-for-Euler-angles-A-no-gimbal-lock-B-yaw-and-roll-angles-are.jpg

Work Cited:

- J. M. Chappel, A. Iqbal, J. G. Hartnett, and D. Abbott, *The Vector Algebra War: A Historial Perspective* arXiv, 2015
- J. B. Kuipers. *Quaternions and Rotation Sequences*. Princeton University Press, 1999.