Math 124 Notes

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Contents

1 Lecture 1, 9/24

Beginning Topology by Sue E. Goodman will be closely followed, most resources are on their webpage, homework maybe physically submitted.

- Chapter 1: Point set topology - lecture notes

- Chapter 2: Surfaces

- Chapter 3: Euler characteristic, fundamental group, knot theory

1.1 Chapter 1, Point Set Topology

Def: A metric space X is a non empty set with a distance function $d, X * X \to \mathbb{R}_0^+$.

- There is a notion of symmetry, $d(x, y) = d(y, x)$

- Triangle inequality, $d(x, z) \leq d(x, y) + d(y, z)$

- $d(x, y) \longleftrightarrow x = y$

Some examples:

- Euclidean metric, aka $\mathbb{R}^n$ with the distance being the magnitude of the line between the points.

- p-metric, $\mathbb{R}^n$, but distance is $(\sum_i |x_i - y_i|^p)^{\frac{1}{p}}$. 2-metric is the same as Euclidean metric. Called a norm in Numerical Analysis.

- Discrete metric, for any set $d(x, y) = 1 \longleftrightarrow x \neq y$. Like the path length in a un-weighted graph.

ε-neighborhood in a metric space (X, d). The set of points $y \in X$, called $B_\varepsilon(x)$ near x such that $d(y, x) < \varepsilon$ and $0 < \varepsilon$.

2-metric gives circle neighborhoods, 1-metric gives diamond, ∞-metric gives a square.

For any $\emptyset \neq Y \subset X$ and (X, d) is a metric space then (Y, d) is a metric space.

A point $x \in X$ is called an interior point of $A \subseteq X$ if $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. The set of all interior points is A°

A point $x \in X$ is called a limit point (cluster point, accumulation point) of A if for $\forall \varepsilon > 0, \exists y \neq x$ such that $y \in B_\varepsilon(x) \cap A$, the set of all points is called A'.

A set is open if every point is interior, a set is closed if every limit point of the set belongs to A.

The closure of a set $\overline{A} = A \cup A'$

$$A \subseteq \overline{A}, A^\circ \subseteq A$$

2 Lecture 2 9/27

For (x, y) all points in the set are interior points, therefore it is a open set.

The set of points $A = \{\frac{1}{n} | n \in \mathbb{N}\}$ is a peculiar example. 0 is a limit point of the set. The set is not closed because $A' \not\subseteq A$. A also has no interior points because you can always pick a $\varepsilon < \frac{1}{n} - \frac{1}{n+1}$ around the point $\frac{1}{n}$.

If $0 \in A$ then the set would be closed.

$$\mathbb{Q}^\circ = \emptyset, \mathbb{Q}' = \mathbb{R}$$

3 Lecture 3 9/29

The product of closed sets are closed and the product of open sets are open.

Proof, suppose we have open points in S_1 and S_2. This implies that there are ε_1 and ε_2 neighborhoods around P_1, P_2. Then the point $P_1 \times P_2$ would have a ε of the smaller of $\varepsilon_1, \varepsilon_2$.

Proof2, we have two closed sets S_1, S_2 and we want to know if $S_1 \times S_2$ is closed. Consider a limit point of $S_1 \times S_2$. This means that there is a $P_1 \times P_2$ that has a point y that is arbitrarily close. Well that means that the components of y are arbitrarily close to P_1 and P_2 in their respective sets. This means that P_1 and P_2 are in their sets from the closure and thusly $P_1 \times P_2$ is in $S_1 \times S_2$.

The ε neighborhood around a open point is a open set. This can be done with triangle inequality to find a epsilon for each of the points.

4   Lecture 4 10/1

If A and B are open sets then $A \cap B$ and $A \cup B$ are open. Proof:

For $A \cap B$, consider a point x in A and B. Because these sets are open then there is a ε neighborhood in both A and B. Choosing the smaller of the two εs we can find equal neighborhoods in both A and B which then exists in the intersection. Repeating this for all points means that $A \cap B$ is open.

For $A \cup B$, any point belongs to either A or B, and there is a ε neighborhood in that respective set. This neighborhood is then in $A \cup B$.

A infinite union of open sets is still open, but a infinite intersection of open sets is not always open. For instance consider $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \emptyset$

For a metric space (X, d) with $A \subseteq X$. A is open iff $B = X \backslash A$ is closed. Proof:

Suppose that there was a point x in B' and not in B making it not closed. This would mean that $x \in A$ meaning that there is a radius around x not in B, making it not a limit point of B.

Suppose that there was a point x in A and not in A°, making it not open. This means that there is a point not in A that is arbitrarily close to x. But that point is in B and that makes it a limit point of B. B is closed therefore we have a contradiction.

Some basic set properties. $X \backslash (A \cup B) = X \backslash A \cap X \backslash B$ and $X \backslash (A \cap B) = X \backslash A \cup X \backslash B$ which inherit the properties of infinite unions of open sets and finite intersection of open sets. This means that the finite union of closed sets is closed and the infinite intersection of closed sets is closed.

$(X \backslash A)^{\circ} = X \backslash \overline{A}$
$\overline{X \backslash A} = X \backslash A^{\circ}$

Definition of limit of a sequence. Limits are unique.

5 Lecture 5 10/4

$$x \in A' \longleftrightarrow \exists \{x_n\}_{n=1}^{\infty} x_n \in A, x_n \neq x, x_n \longrightarrow x$$
$$x \in \overline{A} \longleftrightarrow \exists \{x_n\}_{n=1}^{\infty} x_n \in A, x_n \longrightarrow x$$

x_n can equal x

A is closed if for every sequnce the limit of that sequence is in A.

Two metrics, $d1$ and $d2$ are equivilent if $\forall x, \varepsilon > 0, \exists \delta > 0$ such that $B_\delta^2(x) \subseteq B_\varepsilon^1(x)$ and vice versa. This makes one set being open means the other is open.

6   Lecture 6 10/6

If two metric spaces are equivalent then the ideas of openness, closedness, limit points and convergence.

A topological space is a set $X \neq \emptyset$ togehter with a collection $\tau \subseteq P(X)$ called the topology of X.

$$\emptyset, X \in \tau$$

$$A, B \in \tau \longrightarrow A \cap B \in \tau$$

$$A_n \in \tau \longrightarrow \bigcup A_n \in \tau$$

The sets in τ are called the open sets. X need not be a metric space

$\tau = \{\emptyset, X\}$ is called the trivial topology.

b is called the basis of a topology space if:
i) if for $B_1, B_2 \in b$, and $z \in B_1 \cap B_2$, then there exists $B \in b$ with $z \in B \subseteq B_1 \cap B_2$
ii) for every $x \in X$ there exists $B \in b$ with $x \in B$

THM: the union of all $B \in b$ is τ
NOTE: I believe that the basis is generated from the metric space, and it generates a topology. It is possible to have basies made from the elements of the topology that do not generate the topology, like $b = \{X\}$.

7   Lecture 7, 10/8

We are going over some properties of basies. If a basis generates a topology for X and there is an $A \in \tau$, then there is a basis element that is a subset of A that contains a point in A for every point in A and vice versa. If my understanding of basies is correct this is a pretty trivial result. A is generated via the union of B therefore every point is in one B and are all subsets.

If we have a topological space and we intersect it with a subset of its domain then it is still a topological space. And a basis under this intersection will also generate a topology for this subset. These are also trivial because properties like unions and intersections don't concern themselves with points outside of the set.

Hausdorff Property: first degree seperation.

This property is that if for every two distince points, there exists a pair sets in the topology that do not intersect and repectfully contain one of the points each. Examples are the discrete topology with the singular sets. Another one would be if the basis for the topology was all rows and columns on the chess board. Metric spaces are Hausdorff because we can choose a small enough radius around each points as to exclude any point.

Topologically a point is called a limit point of A if for all $x \in U \in \tau$, $A \cap U \backslash x \neq \emptyset$, meaning that it can be arbitarially approached

Topologically a point is called a interior point of A if there exists $U \in \tau$ such that $x \in U \subseteq A$

Infinite intersections and finite unions of closed sets are still closed. $X \backslash A$ is open if A is closed

8   Lecture 8, 10/11

Definition of convergence on a topological space is, given a sequence of $\{x_n\}$, for any neighborhood of x, there exists a N such that for $n \geq N$ all of those points in the sequence are in that set.

Definition of continuity on a metric space. For $\forall \varepsilon, \exists \delta, d_1(x, y) < \delta \longrightarrow d_2(f(x), f(y)) < \varepsilon$ across possibly two metric spaces with metrics d_1 and d_2.

You can say it is continuous on a point if you fix x.

Continuity on a topological space: $\forall U \in \tau_y$ with $f(a) \in U$, $\exists V \in \tau_x$ with $a \in V, V \subseteq f^{-1}(U)$

If it is continuous on the basis then it is continuous on the topology.

9 Lecture 9, 10/13

Continuity is equiv to the limit being equal to the value.

$f^{-1}(u)$ is the preimage of u. For a topological space with a continuous function, the preimage of all open sets are open. And as a reminder, if a set is in the topology then it is open.

$$\forall U \in \tau_y, f(a) \in U, \exists V \in \tau_x, a \in V \longrightarrow V \subseteq f^{-1}(u)$$

All closed sets in the range are closed in the domain and all open sets in the range are open in the domian.

A function f from X to Y is a homeomorphisim if f is continuous, bijective and f^{-1} is also continuous.