# Introduction to Calculus

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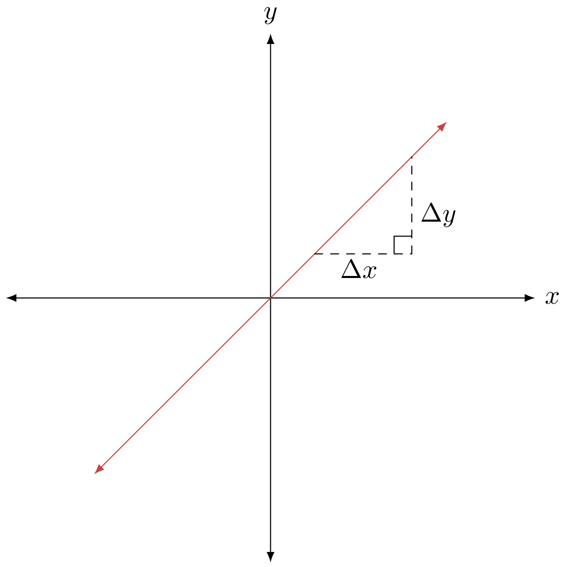
# 1 Gradients of Non-Linear Functions

## 1.1 Linear Gradients

We are already familiar with the gradient of a linear function. Lines always have a constant gradient ($m$) as their slope never changes.

$$m = \frac{rise}{\text{run}} = \frac{\Delta y}{\Delta x}$$




## 1.2 Non-Liner Gradients

### 1.2.1 Notation

The notation used to represent the gradient of a non-linear function is different to that of linear functions.
As a quick introduction, there are three main notations used.

$$y^{'} \quad \text{or} \quad \frac{dy}{dx} \quad \text{or} \quad f^{'}(x)$$

All these mean the same thing. To maintain consistency, the $f^{'}(x)$ notation will be used in these notes.

### 1.2.2 Concept of Tangents and Derivatives

We define the gradient of a non-linear function (curved function) by using the concept of a tangent. A tangent is a line that touches a curve at a single point.
Consider the interactive below.

cubic with tangent
slider

The gradient of the tangent is equal to the gradient of the curve at the point where the tangent touches the curve. This is a very important relationship to grasp your head around as it is the backbone to understanding derivatives. This is mathematically defined below.

For $m$ being the gradient of a tangent and $f(x)$ being the function of a curve:

$$m = f^{'}(x)$$

It may have been noticed that the gradient of the tangent changes for every point on the curve in the slider above. This shows that the gradient changes on every point of the curved function.

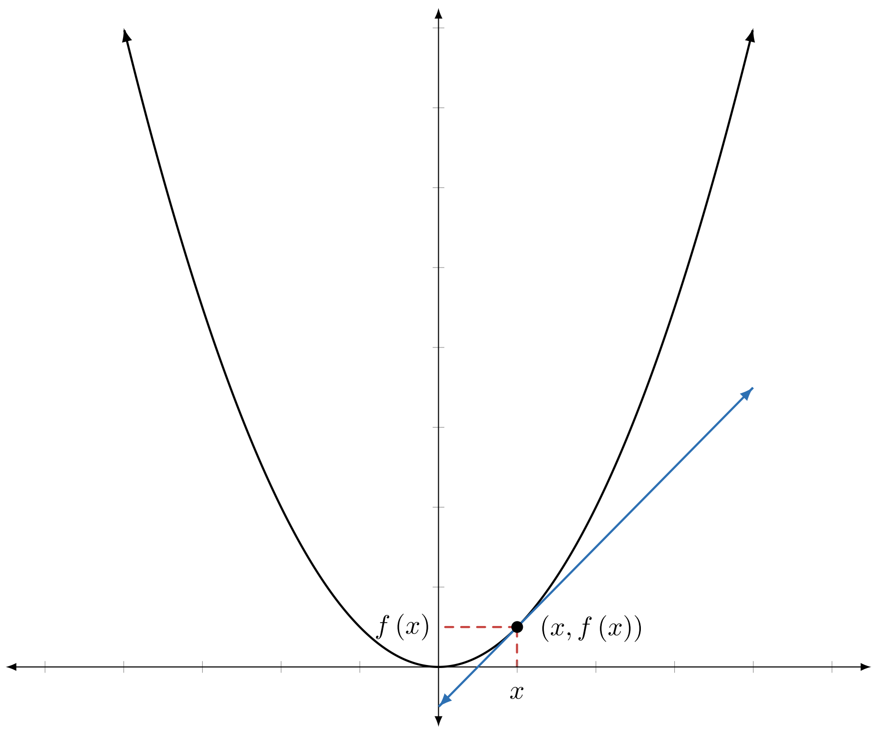
When considering curved functions, we are concerned with the gradient at a point rather than the gradient between an interval (like linear functions). This is because the gradient changes at every point on the curve. In fact, this is what a derivative is. A derivative is defined as the instantaneous rate of change of a function or the gradient of a function at a particular point.

In the next sections we will further build on this idea and quantify the concept of a derivative.

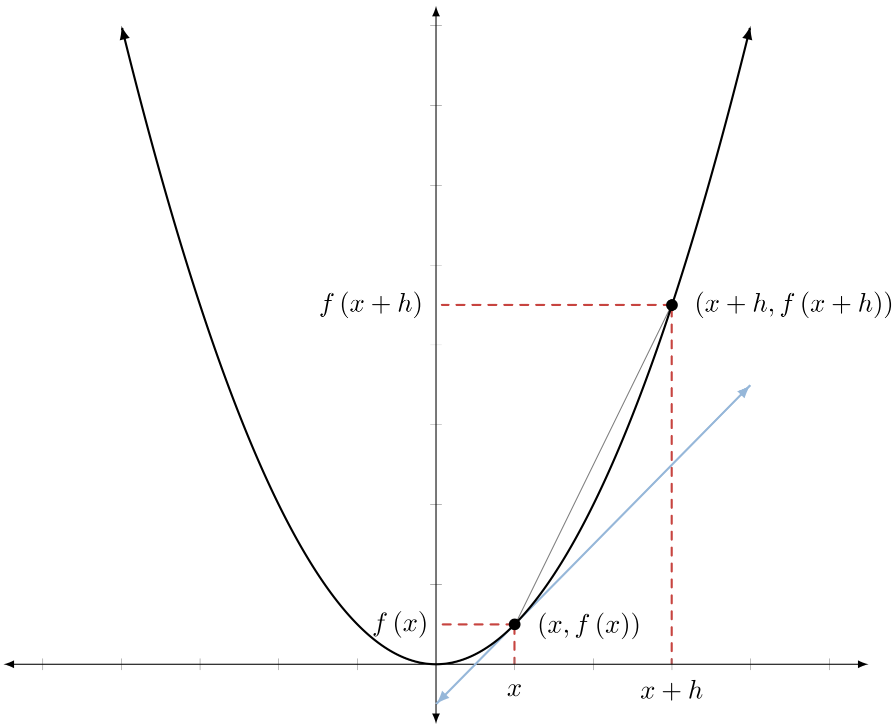
# 2 The Concept of the Derivative

## 2.1 The Difference Quotient

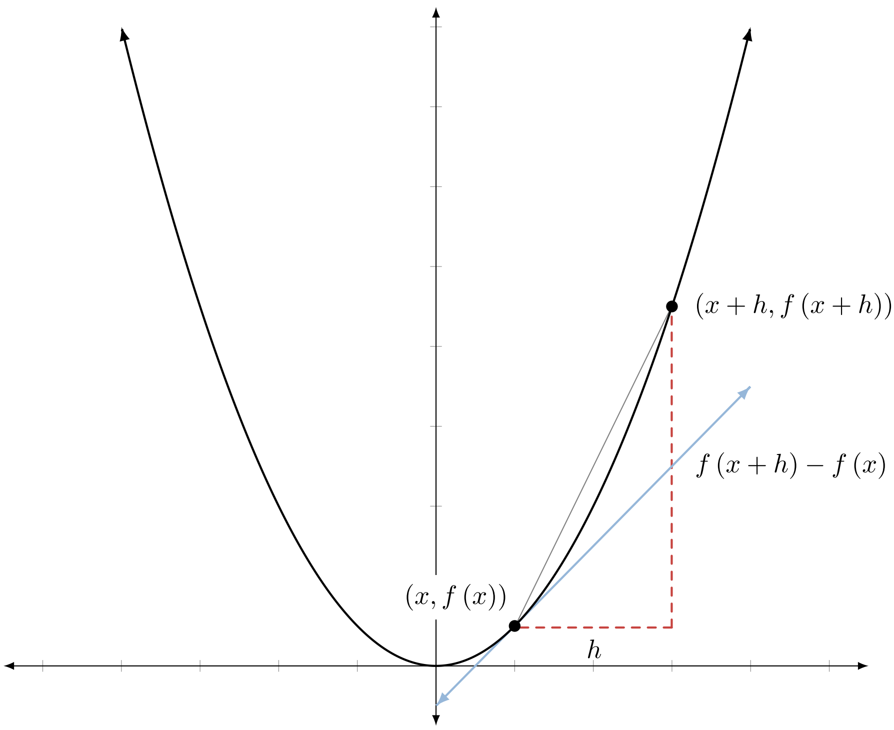
We are now going to mathematically define the concept of the derivative.
Consider the function $y = f(x)$ below. A fixed point $(x, f(x))$ is labeled. Our end goal is to find the gradient at this point $(x, f(x))$ i.e. $f^{'}(x)$. We will achieve this by finding an expression for the gradient of the tangent line at this point. Remember that the gradient of the tangent is equivalent to the gradient at the point the line contacts the curve.




We are now going to fix another point $h$ units to the right of the first point. A grey line joins these two coordinates together.




Our first step to finding the gradient of the tangent is to find the gradient of the grey line connecting the two points.




We can use rise over run to find the gradient of the line.

$$
\begin{aligned}
m &= \frac{\textit{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} \\
&= \frac{f(x+h) - f(x)}{x+h-x} \\
&= \frac{f(x+h) - f(x)}{h}
\end{aligned}
$$

$$m = \frac{f(x+h) - f(x)}{h}$$

This result is called the difference quotient. The difference quotient gives an approximation to the gradient of the tangent. This approximation improves as the distance $h$ gets smaller and smaller.

### 2.2 The Limiting Chord Process

Consider we now shorten the length of $h$. This is achieved by moving the far coordinate down the curve towards the tangent point.

Drag this point in the interactive below and investigate what happens to the grey line.

- interactive same as graphic above with no labels and no red
- grey line and washed out tangent line are visible
- drag coordinate down the curve
- when points are on top of each other, tangent in blue (not washed out) is shown.

You should have noticed the following:

- The difference quotient better approximates the gradient of the tangent when the distance $h$ is decreased.
- The difference quotient is equal to the gradient of the tangent when $h$ is zero (the two coordinates are on top of each other).

We now know that when $h$ approaches zero, the gradient of the difference quotient approaches the gradient of the tangent.

We must mathematically represent the idea of $h$ approaching zero to complete the mathematical representation of a derivative. This is achieved by introducing the concept of a limit.

$$\lim_{h \to 0}$$

This notation is read as "the limit as $h$ approaches zero" and simulates the idea of decreasing the distance $h$ to zero.

We can now complete our definition of a derivative by applying this limit notation to the difference quotient.

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This result is called the first principle definition of a derivative. Any functions derivative can be found using this definition.

## 3 Computation of Derivatives

### 3.1 Example 1

So far we have derived the first principle definition of a derivative. Let us now investigate how to use this result with an example.

Consider we wanted to find the derivative ($f^{'}(x)$) of $f(x) = x^2$.

We can begin with our derivative definition.

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sub in the relevant information.

- $f(x) = x^2$
- $f(x+h) = (x+h)^2$

$$f^{'}(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Our aim is to eliminate the limit. We achieve this by making the $h$'s zero in the difference quotient. Notice that if we eliminated the limit now, we would be dividing by zero which is not allowed. To solve this problem, we must simplify the difference quotient in a way to cancel out the $h$ on the bottom of the fraction. It will then be 'legal' to eliminate the limit.

Let us expand and simplify.

$$\begin{aligned} f^{'}(x) &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h \to 0} (2x+h) \end{aligned}$$

The $h$ on the bottom of the fraction has canceled and it is now 'legal' to eliminate the limit. Remember we eliminate the limit by making all the $h$'s zero.

$$\begin{aligned} f^{'}(x) &= 2x + 0 \\ &= 2x \end{aligned}$$

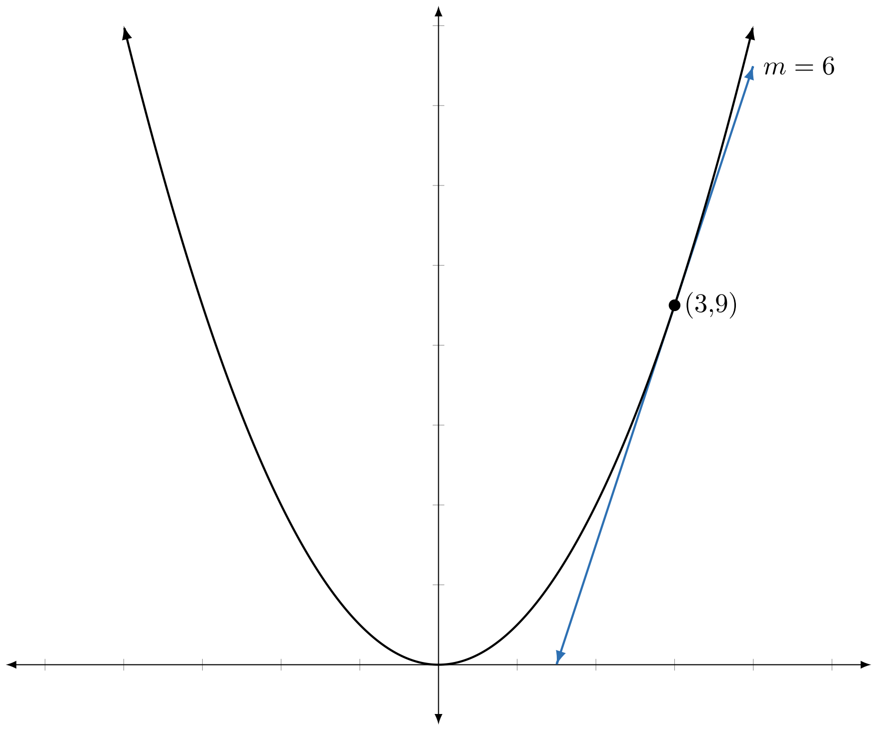
For $f(x) = x^2$,
$f^{'}(x) = 2x$

It may be unusual at first to see the derivative of the function having an $x$ in it. However this makes a lot of sense when we realise that all curved functions have a different gradient at every point on the curve.

This result can be thought of as a formula to find the gradient at any point on the curve of $f(x) = x^2$. For example, consider we wanted to find the gradient at $x = 3$ i.e. $f^{'}(3)$.

$$\begin{aligned} f^{'}(x) &= 2x \\ f^{'}(3) &= 2(3) \\ &= 6 \end{aligned}$$

This means that the tangent on the curve $f(x) = x^2$ at $x = 3$ has a gradient of $m = 6$.




## 3.2 Example 2

Consider another example. We want to find the derivative of $f(x) = x^3$.

We can begin with our derivative definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sub in the relevant information.

- $f(x) = x^3$
- $f(x+h) = (x+h)^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

Expand and simplify until it is 'legal' to cancel out the limit.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} \\ &= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) \end{aligned}$$

The $h$ on the bottom of the fraction has canceled and it is now 'legal' to eliminate the limit. Remember we eliminate the limit by making all the $h$'s zero.

$$\begin{aligned} f'(x) &= 3x^2 + 3x(0) + (0)^2 \\ &= 3x^2 \end{aligned}$$

$$\text{For } f(x) = x^3,$$
$$f^{'}(x) = 3x^2$$

### 3.3 The Power Rule

In this unit we are only concerned with finding the derivatives of polynomials. A polynomial is any function of the form $x^n$ where $n$ is any number.

It is possible to find the derivatives of polynomials by using the first principle definition. This strategy is very strenuous as you may have realised in the above examples.

There exists another method useful for quickly finding the derivative of any polynomial. Consider the two results from the previous examples.

$$\text{For } f(x) = x^2, \quad f^{'}(x) = 2x$$

$$\text{For } f(x) = x^3, \quad f^{'}(x) = 3x^2$$

Can you identify any patterns?

- The power becomes the coefficient in the derivative.
- The power decreases by one.

We can generalise these patterns to create the power rule.

$$\text{For } f(x) = x^n,$$
$$f^{'}(x) = nx^{n-1}$$

This is known as the power rule. It is important to note that the power rule only finds the derivative of a subset of functions (polynomials). This differs from the first principle definition that defines the derivative of any function (including polynomials).

### 3.4 Derivation of the Power Rule

#### 3.4.1 Derivation of the Power Rule 1

The power rule can be derived by the use of binomial expansion. More information about binomial expansion can be found in the combinations playlist. The formula is stated below.

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{r} x^{n-r} y^r + \cdots + y^n$$

The derivation begins with finding $f^{'}(x)$ of $x^n$ using the first principle definition of a derivative.

$$f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sub in the relevant information.

- $f(x) = x^n$
- $f(x+h) = (x+h)^n$

$$f^{'}(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Use binomial expansion and simplify until it is 'legal' to cancel out the limit.

$$
\begin{aligned}
f^{'}(x) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \cdots + \binom{n}{r}x^{n-r}h^r + \cdots + h^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h + \cdots + \binom{n}{r}x^{n-r}h^r + \cdots + h^n}{h} \\
&= \lim_{h\to 0}\frac{h\left[\binom{n}{1}x^{n-1} + \cdots + \binom{n}{r}x^{n-r}h^{r-1} + \cdots + h^{n-1}\right]}{h} \\
&= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \cdots + \binom{n}{r}x^{n-r}h^{r-1} + \cdots + h^{n-1}\right]
\end{aligned}
$$

The $h$ on the bottom of the fraction has canceled and it is now 'legal' to eliminate the limit. Remember we eliminate the limit by making all the $h$'s zero.

$$
\begin{aligned}
f^{'}(x) &= \binom{n}{1}x^{n-1} + \cdots + \binom{n}{r}x^{n-r}0^{r-1} + \cdots + 0^{n-1} \\
&= \binom{n}{1}x^{n-1} \\
&= nx^{n-1}
\end{aligned}
$$

$$f^{'}(x) = nx^{n-1}$$

### 3.4.2 Derivation of the Power Rule 2

A general rule exists that factorises the difference of two $n^{th}$ powers.

If $a$ and $b$ are real numbers, and $n$ is a positive integer, then

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}\right)$$

This result can be used to derive the power rule.

The derivation begins with finding $f^{'}(x)$ of $x^n$ using the first principle definition of a derivative.

$$f^{'}(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Sub in the relevant information.

- $f(x) = x^n$
- $f(x+h) = (x+h)^n$

$$f^{'}(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

Use the difference of two $n^{th}$ powers identity to simplify. Replace $'a'$ with the quantity $(x+h)$ and $'b'$ by $x$.

$$
\begin{aligned}
f^{'}(x) &= \lim_{h\to 0}\frac{((x+h)-x)\left((x+h)^{n-1}+(x+h)^{n-2}x+(x+h)^{n-3}x^2+\cdots+(x+h)x^{n-2}+x^{n-1}\right)}{h}\\
&= \lim_{h\to 0}\frac{(h)\left((x+h)^{n-1}+(x+h)^{n-2}x+(x+h)^{n-3}x^2+\cdots+(x+h)x^{n-2}+x^{n-1}\right)}{h}\\
&= \lim_{h\to 0}\left((x+h)^{n-1}+(x+h)^{n-2}x+(x+h)^{n-3}x^2+\cdots+(x+h)x^{n-2}+x^{n-1}\right)
\end{aligned}
$$

The $h$ on the bottom of the fraction has canceled and it is now 'legal' to eliminate the limit. Remember we eliminate the limit by making all the $h$'s zero.

$$
\begin{aligned}
f^{'}(x) &= (x+0)^{n-1}+(x+0)^{n-2}x+(x+0)^{n-3}x^2+\cdots+(x+0)x^{n-2}+x^{n-1}\\
&= x^{n-1}+x^{n-2}x+x^{n-3}x^2+\cdots+xx^{n-2}+x^{n-1}\\
&= x^{n-1}+x^{n-1}+x^{n-1}+\cdots+x^{n-1}+x^{n-1}
\end{aligned}
$$

There are $n$ terms in the above line. This is a consequence of the identity used. This means that the value $x^{n-1}$ is repeated $n$ times. We can accumulate these terms to get the power rule.

$$f^{'}(x) = nx^{n-1}$$

## 3.5 Linearity Properties of the Derivative

There exists two linearity properties of a derivative.

### 3.5.1 Property 1

We can find the derivative of the sum or difference of two polynomials as shown below.

$$(f(x) \pm g(x))^{'} = f^{'}(x) \pm g^{'}(x)$$

An example follows.

Find the derivative of $f(x) = x^2 + x^7$.

$$f^{'}(x) = 2x + 7x^6$$

The idea of this linearity property is to use the power rule on every separate term.

### 3.5.2 Property 2

We can find the derivative of a polynomial with a coefficient as shown below.

$$(cf(x))^{'} = cf^{'}(x)$$

An example follows.

Find the derivative of $f(x) = 3x^4$.

$$
\begin{aligned}
f^{'}(x) &= 3\left(4x^3\right)\\
&= 12x^3
\end{aligned}
$$

The idea of this linearity property is to ignore the coefficient and use the power rule on the rest of the function. Simplify by multiplying in the coefficient.