Lecture notes of Analysis

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Chapter 1

Introduction

1.1 Lecture 1

- P1. 导论
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Chapter 2

L1-L10

2.1 Lecture 2

- P2. 最小上界、最大下界、Dedekind cut、数列极限的定义与性质
- Create Date: 26 Oct. 2023
- Last Update: 27 Oct. 2023

2.1.1 最小上界、最大下界、Dedekind cut

Definition 2.1.1 (Upper bound and lower bound)**.** Let $\mathcal{S} \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we say that:

1. r is an upper (resp. lower) bound of $\mathcal{S}$ if

$$\forall s \in \mathcal{S}, r \geq s \text{ (resp. } \forall s \in \mathcal{S}, r \leq s) . \tag{2.1}$$

2. r is the greatest (resp. least) element of $\mathcal{S}$ if

   (a) r is an upper (resp. lower) bound of $\mathcal{S}$; and

   (b) $r \in \mathcal{S}$.

   We can use the notation:

$$r = \max \mathcal{S} \text{ (resp. } r = \min \mathcal{S}) . \tag{2.2}$$

3. r is the least upper (greatest lower) bound of $\mathcal{S}$ if

$$\begin{gathered} r = \min \{u \in \mathbb{R} | u \text{ is an upper bound of } \mathcal{S}\} \\ \text{(resp. } r = \max \{u \in \mathbb{R} | u \text{ is a lower bound of } \mathcal{S}\}) . \end{gathered} \tag{2.3}$$

   We can use the notation:

$$r = \sup \mathcal{S} \text{ (resp. } r = \inf \mathcal{S}) . \tag{2.4}$$

Exe 0: 自己举例，任选一个集合 $\mathcal{S}$, 判断它是否有上下界，如果有分别是多少。

在 least upper (or greatest lower) bound，我们最常用到的定理是什么?

- “比 least upper bound 小的不是 upper bound, 比 greatest lower bound 大的不是 lower bound”. 这可以作为脑子里的一个反证工作。
- Every $r \in \mathbb{R}$ is an upper and lower bound of $\emptyset$. 我们可以从定义出发来想这件事，因此，我们一般不会谈论空集的上下界。

对于第二点，我们有如下 **convention**:

1. We write
$$\sup \mathcal{S} = \infty \text{ (resp. } \inf \mathcal{S} = -\infty) , \tag{2.5}$$
if and only if (iif.) $\mathcal{S}$ has no upper (resp. lower) bound. If this is the case, we say that $\sup \mathcal{S}$ (resp. $\inf \mathcal{S}$) doesn't exist.
2. $\mathcal{S}$ is bounded from above (resp. below) iff. $\mathcal{S}$ has an upper (resp. lower) bound.

Definition 2.1.2 (Dedekind cut)**.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$. We say that $(\mathcal{A}, \mathcal{B})$ is Dedekind cut (of $\mathbb{R}$) if

1. $\mathcal{A} \neq \emptyset \neq \mathcal{B}$, and
2. $\mathcal{A} \cup \mathcal{B} = \mathbb{R}$, and
3. $\forall a \in \mathcal{A}, b \in \mathcal{B}\ [a < b]$.

We usually call $\mathcal{A}$ (resp. $\mathcal{B}$) the lower (resp. upper) part of $(\mathcal{A}, \mathcal{B})$.

From now on (until Dr. Qi say stop), we assume that $\mathbb{R}$ has the following property:

(**Dedekind's gapless property**) If $(\mathcal{A}, \mathcal{B})$ is a D-cut of $\mathbb{R}$, then exactly one of the following happens:

1. $\max \mathcal{A}$ exists but $\min \mathcal{B}$ doesn't
2. $\min \mathcal{B}$ exists but $\max \mathcal{A}$ doesn't

We call $\max \mathcal{A}$ in (1) (resp. $\min \mathcal{B}$) is the cutting of $(\mathcal{A}, \mathcal{B})$.

Exe 1: We may define Dedekind cuts of $\mathbb{Q}$ (or $\mathbb{Z}$) similarly. Does the Dedekind gapless property still hold for $\mathbb{Q}$ (or $\mathbb{Z}$)?
Hint: Consider $\mathcal{B} = \left\{x \in \mathbb{Q} | x > 0, x^2 > 2\right\}$. You are allowed to use the fact $\forall r \in \mathbb{Q}[r \neq 2]$.
Hint: 假设 b 是 $\mathcal{B}$ 的最小数，我们需要在 $\mathcal{B}$ 里面找一个比 b 更小的数.

Theorem 2.1.1 (Weierstrass)**.** Let $\emptyset \neq \mathcal{S} \subseteq \mathbb{R}$. If $\mathcal{S}$ has an upper bound, then $\sup \mathcal{S}$ exists.

Proof. Let $\mathcal{B} = \{b \in \mathbb{R} | b \text{ is an upper bound of } \mathcal{S}\}$ and $\mathcal{A} = \mathbb{R} \setminus \mathcal{B}$. It is sensed that $(\mathcal{A}, \mathcal{B})$ is a D-cut of $\mathbb{R}$.
Goal: We need to show that $\min \mathcal{B}$ exists.
Step 1: Prove $(\mathcal{A}, \mathcal{B})$ is a D-cut of $\mathbb{R}$. Our thoughts are:

1. $\mathcal{S}$ has upper bound $\Rightarrow \mathcal{S} \neq \emptyset \Rightarrow \mathcal{A} \neq \emptyset$.

2. $\mathcal{S}$ has upper bound $\Leftrightarrow \mathcal{B} \neq \emptyset$.

3. $\mathcal{A} = \mathbb{R} \setminus \mathcal{B} \Rightarrow \mathcal{A} \cup \mathcal{B} = \mathbb{R}$.

4. For $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we need to show that $a < b$. Were this false, then $a \geq b$, and hence a is a upper bound of $\mathcal{S}$, i.e., $a \in \mathcal{B}$. This is impossible. Therefore, $a < b$.

It is proved that $(\mathcal{A}, \mathcal{B})$ is a D-cut of $\mathbb{R}$.
Step 2: Prove $\min \mathcal{B}$ exists. Our thoughts are:

1. Were this false, according to Dedekind's gapless property, $\max A$ exists, denoted by a_0. We have
$$\begin{aligned} a_0 \in \mathcal{A} \Leftrightarrow & a_0 \notin \mathcal{B} \\ \Leftrightarrow & a_0 \text{ is not an upper bound of } \mathcal{S} . \end{aligned} \tag{2.6}$$
This means at least one element in $\mathcal{S}$ is larger than a_0, i.e.,
$$\Leftrightarrow \exists s_0 \in \mathcal{S} \ [a_0 < s_0] . \tag{2.7}$$

2. Since $a_0 < s_0$, s_0 cannot belong to $\mathcal{A}$,i.e.
$$\Leftrightarrow s_0 \notin \mathcal{A} \Leftrightarrow s_0 \in \mathcal{B} . \tag{2.8}$$

3. Choose x such that $a_0 < x < s_0$ (We can always find a x that satisfies this equation,e.g., $x = (a_0+s_0)/2$). In this case, $x \in \mathcal{B} \Leftrightarrow x$ is an upper bound of $\mathcal{S}$. However, $s_0 \in \mathcal{S}$. This is impossible.

Therefore, $\min \mathcal{B}$ exists.

$\square$

Exe 2. Prove the following statement:
(**The Archimedean property**) $\forall r \in \mathbb{R} \ [r > 0 \Rightarrow \exists n \in \mathbb{N} \ [1/n < r]]$.
直观理解：将整数 1 切为 n 等分，当 n 最够大的时候，每一份将比任意正实数小.
Hint: Rephrase this statement in a way linking it to the upper bounds of the set $\mathcal{S} = \mathbb{N} \subseteq \mathbb{R}$. You can consider to prove whether an positive integer has an upper bound in $\mathbb{R}$.

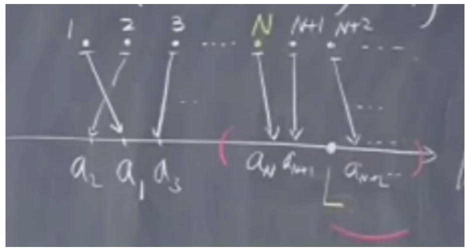

Figure 2.1: Enter Caption

2.1.2 数列极限的定义与性质

Definition 2.1.3 (Limit)**.** Let a_n $(n \in \mathbb{N})$ (or say $\{a_n\}_{n=1}^{n}$) be a sequence in $\mathbb{R}$ and $L \in \mathbb{R}$. We say that a_n converges to L (as $n \to \infty$) if

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; [n \geq N \Rightarrow |a_n - L| < \epsilon] \,. \tag{2.9}$$

This means that 第 N 项后所有的项都要落在 Fig.2.1红色区间内.

Terminology. If such L exists (resp. doesn't exist), we call it the limit of a_n and call a_n is a convergent (resp. divergent)sequence. We can use the notation:

$$\lim_{n \to \infty} a_n = L \,. \tag{2.10}$$

Some generalized notations.

$$\begin{aligned} \lim_{n \to \infty} a_n = \infty &\Rightarrow \forall M > 0 \; \exists N \in \mathbb{N} \; [n \geq N \Rightarrow a_n \geq M] \\ \lim_{n \to \infty} a_n = -\infty &\Rightarrow \forall M > 0 \; \exists N \in \mathbb{N} \; [n \geq N \Rightarrow a_n \leq M] \,. \end{aligned} \tag{2.11}$$

In these two cases, we don't say that a_n is convergent.

Exe 3.

1. Prove that $lim_{n \to \infty} a_n = L \; lim_{n \to \infty} a_n = M \Rightarrow L = M$.
   Hint. 分别在 L 和 M 处取一个小区间，让他们不相交，根据极限定义来证明 (反证)。

2. Prove that $a_n (n \in \mathbb{N})$ is convergent $\Rightarrow \{a_n | n \in \mathbb{N}\}$ is bounded.
   Hint. a_n 收敛说明某项（例如 10,000 项）后，所有项都被包在某个区间内，这说明 10,000 项后所有项都是有界的。我们只需要再证明 10,000 之前的所有项也是有界的即可。

3. Prove that if $a_n \leq b_n$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} b_n = M$, then $L \leq M$. What if $\leq$ is replaced by $<$?

Remark 2.1.1. Changing of removing finitely many terms in a_n does not affect $a_n (n \in \mathbb{N})$'s being convergent (and its limit)/divergent.

Proposition 2.1.1. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty} a_n \pm b_n = L \pm M$
2. $\lim_{n\to\infty} a_n b_n = LM$
3. if $M \neq 0$, then $b_n \neq 0$ for all but finitely many n, and $\lim_{n\to\infty} a_n/b_n = L/M$ (Hint: we can remove the terms with $b_n = 0$; removing finitely many terms in b_n does not affect b_n's limit).

Proof. (1) Consider $|(a_n \pm b_n) - (L \pm M)|$, we have

$$\begin{aligned} |(a_n \pm b_n) - (L \pm M)| &= |(a_n - L) \pm (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \end{aligned} \tag{2.12}$$

$$\begin{aligned} \forall \epsilon > 0\ \exists N_1, N_2 \in \mathbb{N}\ &[n \geq N_1 \Rightarrow |a_n - L| < \epsilon/2] \\ \text{and } &[n \geq N_2 \Rightarrow |b_n - M| < \epsilon/2]\,. \end{aligned} \tag{2.13}$$

Let $N = \max\{N_1, N_2\}$. Then $n \geq N \Rightarrow |a_n - L| + |b_n - M| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $\forall \epsilon > 0\ \ |(a_n \pm b_n) - (L \pm M)| < \epsilon$.

(2) Consider $|a_n b_n - LM|$, we have

$$\begin{aligned} |a_n b_n - LM| &= |a_n b_n - L b_n + L b_n LM| \\ &= |(a_n - L) b_n + L(b_n - M)| \\ &\leq |a_n - L|\,|b_n| + |L||b_n - M|\,. \end{aligned} \tag{2.14}$$

Choose $C > 0$ such that $|b_n| \leq C$ and $|L| \leq C$ for all $n \in \mathbb{N}$ (PS: $|b_n| \leq C$ 的存在性可以用 Ex.3(2) 证明). Then we have

$$|a_n - L|\,|b_n| + |L|\,|b_n - M| \leq C\,|a_n - L| + C\,|b_n - M|\,. \tag{2.15}$$

(三角不等式)

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \left[n \geq \mathbb{N} \Rightarrow |a_n - L| < \frac{\epsilon}{2C} \text{ and } |b_n - M| < \frac{\epsilon}{2C} \right]. \tag{2.16}$$

Therefore,

$$|a_n b_n - LM| < C \times \frac{\epsilon}{2C} + C \times \frac{\epsilon}{2C} = \epsilon\,. \tag{2.17}$$

□

Exe 4.

1. Prove (3) of Proposition 2.1.1.
   Hint: $a_n/b_n = L/M$ can be written as $a_n \frac{1}{n_n} = L \frac{1}{M}$. You only need to prove $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$ when $\lim_{n\to\infty} a_n/b_n = L/M$. Nevertheless, you are wellcome to prove this Proposition following the above process.
2. (Optional) What if $L = \pm\infty$ or $M = \pm\infty$ in Proposition 2.1.1.
3. If $a_n = \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$, prove that $\lim_{n\to\infty} a_n = \frac{1}{1-\frac{1}{2}}$. You can use knowledge learned so far.
   Hint: 阿基米德性质.

2.2 Lecture 3

- P3. 單調數列的收斂性、區間套定理、Cauchy 數列的概念
- Create Date: 7 Nov. 2023
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2.2.1 單調數列的收斂性

Example If a>1, then $\lim_{n\to\infty} \frac{1}{a^n} = 0$.

$$\frac{1}{a^n} = \frac{1}{(1+(a-1))^n} \tag{2.18}$$

Let $b = a - 1 > 0$, 利用二项式定理展开分母可得,

$$\begin{aligned}(1+b)^n \geq& 1 + nb \\ \lim_{n\to\infty} \frac{1}{a^n} \leq& \lim_{n\to\infty} \frac{1}{1+nb} = 0\,.\end{aligned} \tag{2.19}$$

Since $\frac{1}{a^n} \geq 0$, we have $\lim_{n\to\infty} \frac{1}{a^n} = 0$.

Exe 1(Squzzing therome). If $a_n \leq c_n \leq b_n$ $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$, prove that $\lim_{n\to\infty} c_n = L$.
Hint: Use definition of limit.

Definition 2.2.1 (monotone)**.** A seq. a_n $n \in \mathbb{N}$ in $\mathbb{R}$ is

- nondecreasing monotone/ increasing (resp. nonincreasing monotone/ decreasing)if $\forall n \in \mathbb{N}$ $a - N \leq a_{n+1}$ (resp. $\forall n \in \mathbb{N}$ $a - N \geq a_{n+1}$), we use **notation**:

$$a_n \uparrow \text{ (resp. } a_n \downarrow \text{)} \tag{2.20}$$

- strictly increasing (resp. strictly decreasing) if $\forall n \in \mathbb{N}$ $a - N < a_{n+1}$ (resp. $\forall n \in \mathbb{N}$ $a - N > a_{n+1}$), we use **notation**:

$$a_n \upuparrows \text{ (resp. } a_n \downdownarrows \text{)} \tag{2.21}$$

Theorem 2.2.1 (??)**.** If $a_n \uparrow$ and $\{a_n | n \in \mathbb{N}\}$ has a upper bound, then a_n converges (to $\sup\{a_n | n \in \mathbb{N}\}$).

Proof. Our thinking are:

1. (use the limit property) $\{a_n | n \in \mathbb{N}\}$ has an upper bound $\Rightarrow L = \sup\{a_n | n \in \mathbb{N}\}$ exists. (ref. convention 2 in 2.1.1)

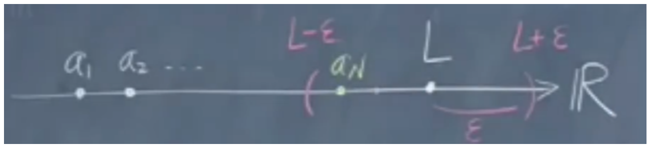

Figure 2.2: Illustrating proof of Them2.2.1.

2. (use the monotone property) We claim that $\lim_{n\to\infty} a_n = L$. Now we need to prove it:
   $\forall\epsilon\ L-\epsilon < L$ (which means $L-\epsilon$ is not an upper bound) and hence $\exists N \in \mathbb{N}\,[L-\epsilon < a_N]$, as shown in Figure 2.2. Therefore, we have

$$\forall n > N\,[a_N \le a_n]\ . \tag{2.22}$$

   Since $a_N > L-\epsilon$ and $a_n \le L+\epsilon$, we have

$$|a_n - L| \le \epsilon\,. \tag{2.23}$$

   This is the proof of $\forall\epsilon\ L-\epsilon < L$.

□

Examples

1. A decimal expression (小数表示法) gives a real number:
   $0.d_1d_2d_3\ldots$

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