# Spin-Glass: Replica Approach and Its Applications

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Half of the Noble prize in Physics 2021 was awarded to Giorgio Parisi, "for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales". Parisi's theory on spin glasses deeply revealed the hidden symmetry in disordered systems. However, hard to find a clear physical understanding about the breaking replica symmetry in the spin glass theory. In this paper, we attempt to introduce spin glass theory from pre-requisite knowledge to broken replica symmetry, and appreciate its great power from its various applications.

## CONTENTS



## INTRODUCTION

The Nobel Prize in Physics 2021 was awarded to three scientists, including Syukuro Manabe, Klaus Hasselmann and Giorgio Parisi. Giorgio Parisi received half of the prize, " for the discovery of the interplay of disorder and fluctuations in physical systems from atomic to planetary scales". So we want to have a better understanding of his contribution.

Parisi is best known for his work giving a rigorous solution to The Sherington Kirkpatrick (SK) spin-glass model [15]. In this paper, we want to start with the introduction of the glass state and gradually understand the different models that describe the glass state. Meanwhile, we want to learn Parisi's work, understand the replica approach and replica symmetry breaking, and finally understand its application in different fields.

### Phase transition and vitrification

Phase transitions are the physical processes of transition between a state of a medium, identified by some parameters, and another one, with different values of the parameters. Water phase transition and ferromagnetic phase transition are typical examples.

Lev Davidovich Landau first raised his theory about phase transition in 1937 [9]. According to his theory, the process of phase transition must be accompanied by some "order" change. The generation of order change in the phase transition process corresponds to the breaking of some symmetry, such as translation symmetry and rotational symmetry. However, vitrification can not be described by Landau's theory.

The molten silica liquid at high temperatures undergoes rapid cooling to form quartz glass. Unlike crystal, when in a glassy state, the arrangement of molecules is disordered (FIG1,2). This process is called vitrification.

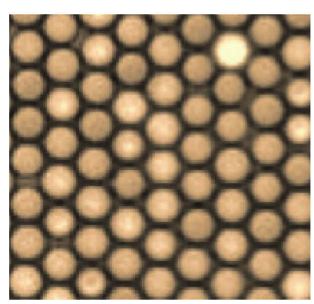

FIG. 1. Crystal

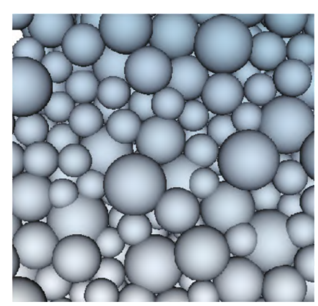

FIG. 2. Glass

Similarly, the paramagnetic phase can change to the spin-glass phase at low temperatures by means of a spin-glass phase transition (FIG3). Spin-glass does not have macroscopic spontaneous magnetism, yet almost all spins are "frozen" and do not flip in mesoscopic time (similar to ferromagnetism). As the spin freezing occurs, one can observe a cust at critical temperature $T_f$ in the magnetic susceptibility[2], which implies a phase transition at $T_f$. However, it's difficult to find out a correlating order parameter of the spin-glass phase transition. Many models have been established to explain the strange cust of magnetic susceptibility.Next, we will introduce these models initially

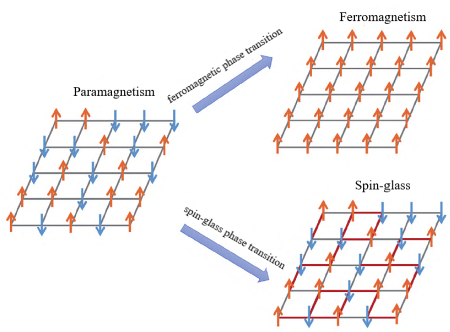


FIG. 3. spin-glass phase transition

### Model evolution and replica approach

From the above, we know that the focus of describing spin-glass systems is to describe the relationship between their different spins. This is the same model that describes ferromagnetism, so the first model applied to spin glass is the Ising model used in statistical mechanics to describe ferromagnetism.

Ising model was first raised by Wilhelm Lenz in 1920.[10] His student Ernst Ising solved the one-dimensional model of it.[8] Ising model describes the collective features of a set $\Lambda$ of lattice sites belonging to $\Lambda$, and for each adjacent site, spin variable $s_i$ can be $\pm 1$. In the Ising model, all sites interact with other sites equally. So the Hamiltonian of this model is written as equation(1).

$$\widehat{H}[s_i] = -J\sum_{\langle i,k\rangle} s_i s_k - \mu H \sum_{i=1}^{N} s_i \qquad (1)$$

where J denotes the exchange interaction constant, and H denotes the strength of the external magnetic field.

Ising found the 1-D Ising model had no phase transition. He mistakenly generalize this feature to higher dimensions and thus abandoned it. Later Lars Onsager gave the analytic description of the 2D square-lattice Ising model with no external magnetic field.[11] The analytical description of the 2-D Ising model with the external magnetic field was given by Chen-Ning Yang.[18] Different from the 1-D Ising model, the 2-D model includes the phase transition. Analytic solutions of the 3 and higher-dimensional Ising model haven't been figured out yet. Researchers often use other methods like mean-field theory to describe them. We will discuss the mean-field theory applied to the Ising model in the "METHOD" part.

To make the model more accurate, more modifications were added to the old model.

In 1975, S. F. Edwards and P. W. Anderson raised Edward-Anderson model to better describe the spin-glass state[5]. This model is a long-range model and the interaction between each two sites is not necessarily the same. For each pair of sites $i, k$, we can define an interaction coefficient $J_{i,k}$. In the Edwards-Anderson model, $J_{i,k}$ is $\pm 1$ of a random distribution with a fixed mean square deviation, indicating a description of antiferromagnetic interactions. For a particular spin configuration $s$, the Hamiltonian is given by:

$$\widehat{H} = -\sum_{i,k} J_{i,k} s_i s_k - \mu H \sum_{i=1}^{N} s_i \qquad (2)$$

Edward and Anderson's calculations are consistent with the general trend of experimental phenomena, but there is still a certain gap with the details, which we will cover in detail in the "RESULTS" section.

Also in 1975, David Sherrington and Scott Kirkpatrick proposed a further model "SK model".[15]In the Sherrington-Kirkpatrick model(SK model), the correlation coefficient $J_{ij}$ follows the Gaussian distribution, i.e., the probability density of each $J_{i,k}$ is

$$p(J_{i,k}) = \sqrt{\frac{N}{2\pi}} \exp \frac{-NJ_{i,k}^2}{2} \qquad (3)$$

,which means the random distribution after annealing.

An effective method in solving the partition function $Z$ in the SK model is the replica method: introduce $n$ numbers of replicates to replace $z^n$. Along with the way of the replica method, the hidden order parameter was found to be related to the replicate number $n$.

However, David Sherrington and Scott Kirkpatrick failed to get the equilibrium solution of SK model. The equilibrium solution is found by Giorgio Parisi.[13]

In 1979, Giorgio Parisi proposed the overlapping order parameter between replicates to characterize the phase transition of spin glass[12]. Later, the concept of replica symmetry breaking is proposed in order to solve the problem of negative zero temperature entropy in theory.

In fact, The reason why the replica symmetry is broken is still unknown. In this paper, we will give the meaning of the replica standing from mathematical derivation and provide a reason for the source of replica symmetry breaking.

## METHODS

### Symmetry and order parameter

First, we look at the phase transitions initially mentioned earlier in a more quantitative and profound perspective.

When there is a critical point, a continuous transition can be effected between any two states of the substance. Whereas, the critical point can exist only for phases such that the difference between them is purely quantitative. For example, a liquid and a gas differ only in the degree of interaction between the molecules.

However, we can say only that a particular symmetry property exists or does not exist, so there is only a sudden change in symmetry. Such phases as a liquid and a solid (crystal), or different crystal modifications of a substance, are qualitatively different since they have different internal symmetry.

This phase transition where the symmetry has changed is called phase transition of the second kind.

The transition of ferromagnetic or antiferromagnetic substances, the transition of metal to the superconducting state, and that of liquid helium to the superfluid state are examples of phase transition of the second kind. The states of the two phases are the same at the transition point, but the substance acquires a qualitatively new property at the transition point.

In 1937, Lev Landau proposed a phenomenological theory to describe the phase transition of the second kind[9].

To quantitatively describe the change in the structure of the substance when it passes through the phase transition point, Landau defined a quantity $\eta$, called the order parameter, in such a way that it takes non-zero values in the unsymmetrical phase and is zero in the symmetrical phase.

TABLE II lists some order parameter in phase transition of the second kind.

| Phase transition | Order parameter |
|---|---|
| gas-liquid | $\rho_l - \rho_g$ |
| ferromagnetic | magnetization |
| antiferromagnetic | sublattice magnetization |
| ferroelectric | polarization |
| binary solution | $\rho_1 - \rho_2$ |
| $\dots$ | $\dots$ |

TABLE I. Order parameter in phase transitions

As we already mentioned earlier, the system transitions to a spin-glass phase with the lowest potential energy when undergoing rapid cooling. In such a case, $\langle s_i \rangle = 0$, the system is neither ferro- nor antiferromagnetic on any scale. We need some new formalism to express this physical picture. So Edwards and Anderson [5] introduce an order parameter $q$ to describe the "frozen spin". If on one observation a particular spin is $s_i^{(1)}$, then it is observed after a long time and the spin is $s_i^{(2)}$, then we can define $q$ as follow:

$$q = \langle s_i^{(1)} \cdot s_i^{(2)} \rangle$$

.

However, using such a formalism still does not eliminate the difficulty caused by the complexity of $J_{ij}$. So we use the replica approach to transform the integral of a random variable into a summation of multiple interacting replicas. With the assistance of the replica approach, the free energy expression is transformed into the sum of the spin couplings between the different replicas:

$$f \sim \exp\Big(\sum_{\alpha,\beta} \sum_{i,j} s_i^{(\alpha)} \cdot s_i^{(\beta)} \cdot s_j^{(\alpha)} \cdot s_j^{(\beta)}\Big),$$

where $\alpha$, $\beta$ are replica indices. Since there is an interrelationship between replicas, Edward and Anderson take

$$q_{\alpha\beta} = \langle s_i^{(\alpha)} \cdot s_i^{(\beta)} \rangle,$$

the probability of the same spin in the same direction in different replicas $\alpha$, $\beta$, as the new order parameter.

In 1975, Sherrington and Kirkpatrick defined the order parameter in another way. In the process of derivation, they found mathematical substitutions $q$, $m$ and demonstrated their physical significance:

$$\begin{aligned} q &= 1 - (2\pi)^{-\frac{1}{2}} \int dz \exp\left(-\frac{z^2}{2}\right) \cosh^{-2}[(\tilde{J} q^{\frac{1}{2}}/kT) z + \tilde{J}_0 m/kT] \\ &= \langle \langle S_i \rangle^2 \rangle_J \end{aligned} \tag{4}$$

$$\begin{aligned} m &= (2\pi)^{-\frac{1}{2}} \int dz \exp\left(-\frac{z^2}{2}\right) \tanh[(\tilde{J} q^{\frac{1}{2}}/kT) z + \tilde{J}_0 m/kT] \\ &= \langle \langle S_i \rangle \rangle_J \end{aligned} \tag{5}$$

At this point, we have already had all the preliminaries for quantitative analysis of phase transitions in spin-glass.

### Mean-field theory
### – Taking Ising model as an example

Since it is difficult to accurately calculate these models, we generally consider approximate methods to solve these problems. One of the most important methods is the mean-field theory.

Pierre Curie and Pierre Weiss first used mean-field theory (MFT) to describe phase transitions in 1907[17]. The main idea of MFT is to replace all interactions with any one body with an average or effective interaction. This reduces any many-body problem into an effective one-body problem.

MFT can be applied to a number of physical systems so as to study phenomena such as phase transitions. As mentioned in the INTRODUCTION section, the Ising model can also apply the MFT to get an approximate solution. We rewrite equation(1):

$$\begin{aligned}\widehat{H}[s_i] &= -J\sum_{\langle i,k\rangle} s_i s_k - \mu H\sum_{i=1}^{N} s_i \\ &= -\sum_i \mu s_i (H + \frac{J}{\mu}\sum_j{}' s_j) \\ &= -\sum_i \mu s_i (H + h_i) \\ &= -\sum_i \mu s_i H_{eff}\end{aligned} \tag{6}$$

Where,

$$H_{eff} = H + h_i \tag{7}$$

$$h_i = \frac{J}{\mu}\sum_j{}' s_j \tag{8}$$

$H_{eff}$ represents the effective magnetic field, including the external field $H$ and the internal field $h_i$ generated by the interaction between spins. ($\sum_j{}'$ means that only the nearest neighbor lattice J of lattice I is summed ).

MFT replaces $s_j$ with its average value $\overline{s_j}$ in equation(8), that is

$$h_i = \frac{J}{\mu}\sum_j{}' s_j \rightarrow \overline{h_i} = \frac{J}{\mu}\sum_j{}' \overline{s_j} \tag{9}$$

When the fluctuation is completely ignored, the average spin of each lattice point should be the same, which means $\overline{s_j} = \bar{s}; \overline{h_i} = \bar{h}$. So

$$\overline{h_i} = \bar{h} = \frac{zJ}{\mu}\bar{s},$$

where z denotes the number of nearest neighbor of lattice points, called coordination number. Therefore, the Hamiltonian under the mean field approximation is

$$\widehat{H_{MF}} = -\sum_{i=1}^{N} \mu (H + \bar{h}) s_i,$$

Which is a one-body problem.

We can easily get the partition function $Z_N$ under the mean-field approximation:

$$Z_N = [2\cosh(\frac{\mu H}{kT} + \frac{zJ}{kT}\bar{s})]^N \tag{10}$$

At the same time, $\bar{s}$ should satisfy the equation below:

$$\bar{s} = \tanh(\frac{\mu H}{kT} + \frac{zJ}{kT}\bar{s}) \tag{11}$$

According to MFT, the critical temperature of one-dimensional Ising model is $T_c = \frac{zJ}{k} = \frac{2J}{k}$. However, the exact solution proves that the 1-D Ising model has no phase transition at finite temperature ($T_c = 0$), so the mean-field theory is completely wrong for the one-dimensional case.

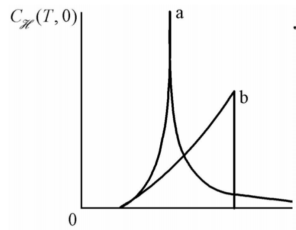


FIG. 4. Thermal capacity

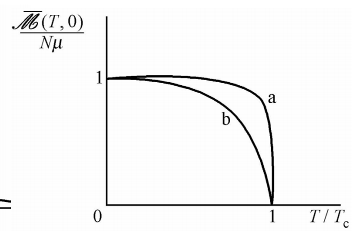


FIG. 5. Spontaneous magnetization

As for the 2-D case, the differences between the exact solution and the MFT solution are shown above (curve a is the exact solution, curve b is the MFT solution). The Divergence between two curves in Fig4 and Fig5 indicas that the MFT solution is still very problematic, so we need to develop a more accurate theoretical calculation.

## The replica approach — Taking SK model as an example

The replica approach is a mathematical trick in statistical physics. It is used in EA model[5] and SK model[15]. We can understand the role of the replica method in solving the SK model. In this model, any two sides are adjacent. Thus the energy of the set is:

$$H_J[s] = -\sum_{1\le i<k\le N} J_{i,k} s_i s_k - h_i \sum_{i=1}^{n} s_i \tag{12}$$

where $N$ is the amount of the sites. And the subscript $J$ of $H$ denotes the particular configuration of the N number of $s$.

Similarly, the configuration probability and the partition function can be derived from it.

$$Z_J = \sum_{\{S\}} e^{-\beta H_J[s]} \tag{13}$$

Where $\beta = \dfrac{1}{k_B T}$

It's convenient to figure out the various information of a system if we know its partition function. For example, given the possibility of each interaction coefficient $J$, the free energy density of the system can be derived as equation(15):

$$f_J = -\frac{1}{\beta N} \times \ln Z_J \tag{14}$$

$$f = \sum_J P[J] f_J \tag{15}$$

However, the index summation in equation(13) is difficult to calculate directly. Here we introduce a special mathematical trick to overcome the difficulty, which is the core of the replica approach:

$$\ln Z = \lim_{n\to 0}\frac{Z^n - 1}{n}, \tag{16}$$

$$\overline{Z^n} = Z_n = \sum_{[J]} P_{[J]}\{Z_J\}^n \tag{17}$$

When $n$ is an integer, the exponentiation of $n$ order is equivalent to the operation of introducing $n$ replicas same as the initial system, hence the name.

The advantage of the replica approach appears when calculating $\overline{\ln Z}$ because it reduces the problem to computing $\overline{Z^n}$. The latter can be easily calculated in some conditions.

Take the SK model as an example. In equation(12), we assume that the $J$'s are independent variables. The average free energy of the system is:

$$\begin{aligned} f &= \overline{f_J} \\ &= \overline{-\frac{1}{\beta N}\ln Z_J} \\ &= -\frac{1}{\beta N}\sum_J P[J]\ln Z_J \end{aligned} \tag{18}$$

where $P[J]$ is the probability of a certain $J$ configuration. And $\sum_J$ denotes the sum over all possible $J$ configurations.

Using equation(16) twice we can transform the formula above:

$$\begin{aligned} f &= -\frac{1}{\beta n N}\sum_J nP[J]\ln Z_J \\ &= -\lim_{n\to 0}\frac{1}{\beta n N}\ln\left(1 + n\sum_J P[J]\ln Z_J\right) \\ &= -\lim_{n\to 0}\frac{1}{\beta n N}\ln\left(\sum_J P[J](1 + n\ln Z_J)\right) \\ &= -\lim_{n\to 0}\frac{1}{\beta n N}\ln\sum_J (P[J]Z_J^n) \\ &= -\lim_{n\to 0}\frac{1}{\beta n N}\ln\overline{Z_J^n} \end{aligned} \tag{19}$$

denote $\overline{Z_J^n}$ by $Z_n$, and $-\frac{1}{\beta n N}\ln\overline{Z_J^n}$ by $f_n$:

$$f = \lim_{n\to 0} f_n \tag{20}$$

$$= -\lim_{n\to 0}\frac{1}{\beta n N}\ln Z_n \tag{21}$$

Based on the definition of $Z_J$ (formula 13), we can rewrite $Z_J^n$:

$$\begin{aligned} Z_J^n &= \left(\sum_{\{s\}} e^{-\beta H_J[s]}\right)^n \\ &= \prod_{i=1}^n \sum_{\{s^{(i)}\}} e^{-\beta H_J[s^{(i)}]} \\ &= \sum_{\{s^{(1)}\}}\sum_{\{s^{(2)}\}}\cdots\sum_{\{s^{(n)}\}}\exp\left(-\beta\sum_{i=1}^n H_J[s^{(i)}]\right) \end{aligned} \tag{22}$$

The final form of $Z_J$ can be explained as that there are $n$ same spin configuration $s$. In this way we introduced $n$ replicas $s^{(1)},...,s^{(n)}$. For a particular $s^{(i)}$, there is totally $2^N$ spin configurations. Equation (22) includes $2^{nN}$ terms.

We assume that $J$'s follow Gaussian distribution.

$$\begin{cases} \overline{J_{i,k}} &= 0 \\ \overline{J_{i,k}^2} &= \frac{1}{N} \\ J_{i,k} &= J_{k,i} \end{cases} \tag{23}$$

i.e., the probability density of each $J_{i,k}$ is

$$p(J_{i,k}) = \sqrt{\frac{N}{2\pi}}\exp\frac{-NJ_{i,k}^2}{2} \tag{24}$$

Using equation(22) and (24) to transform $Z_n$:

$$\begin{aligned} Z_n &= \sum_J (P[J]Z_J^n) \\ &= \sum_J \prod_{i<k} p(J_{i,k})\mathrm{d}J_{i,k}\sum_{\{s\}}\exp\left(-\beta\sum_{i=1}^n H_J[s^{(i)}]\right) \\ &= \left(\frac{N}{2\pi}\right)^{\frac{N(N-1)}{4}}\sum_{\{s\}}\int_{R^{\frac{N(N-1)}{2}}}\mathrm{d}J\exp\Bigg(\sum_{i<k}\frac{-NJ_{i,k}^2}{2} + \\ &\quad \beta\sum_{1\le i<k} J_{i,k}\sum_{a=1}^n s_i^{(a)}s_k^{(a)} + \beta h\sum_i\sum_a s_i^{(a)}\Bigg) \end{aligned} \tag{25}$$

where $\sum\limits_{\{s\}}$ represents $\sum\limits_{\{s^{(1)}\}}\sum\limits_{\{s^{(2)}\}}...\sum\limits_{\{s^{(n)}\}}$, i.e., the sum over all replicas. And $\mathrm{d}J$ represents $\prod\limits_{i<k}\mathrm{d}J_{i,k}$. $\sum\limits_i$ and $\sum\limits_a$ are shorthands of $\sum\limits_{i=1}^N$ and $\sum\limits_{a=1}^n$ respectively. When the amount of the spin $N$ is much larger than the replicate number $n$, we have:

$$\begin{aligned} &\sum_{i<k}\left(\sum_a s_i^{(a)}s_k^{(a)}\right)^2 \\ &= -\frac{Nn^2}{2} + \frac{nN^2}{2} + \sum_{a<b}\left(\sum_i s_i^{(a)}s_i^{(b)}\right)^2 \\ &\approx \frac{nN^2}{2} + \sum_{a<b}\left(\sum_i s_i^{(a)}s_i^{(b)}\right)^2 \end{aligned} \tag{26}$$

Using equation (26) we can rewrite formula (25):

$$
\begin{aligned}
&Z_n \qquad (27)\\
&=\left(\frac{N}{2\pi}\right)^{\frac{1}{4}N(N-1)}\sum_{\{s\}}\int_{R^{\frac{1}{2}N(N-1)}} dJ\exp\left(\sum_{i<k}\left(\frac{\beta^2\left(\sum_a s_i^{(a)}s_k^{(a)}\right)^2}{2N}\right.\right.\\
&\left.\left.-\frac{1}{2}N\left(J_{i,k}-\frac{\beta\sum_{a=1}^n s_i^{(a)}s_k^{(a)}}{N}\right)^2\right)+\beta h\sum_i\sum_{a=1}^n s_i^{(a)}\right)\\
&=\left(\frac{N}{2\pi}\right)^{\frac{1}{4}N(N-1)}\sum_{\{s\}}\int_{R^{\frac{1}{2}N(N-1)}} dJ\exp\left(\sum_{i<k}\frac{1}{2}(-N)\left(J_{i,k}\right.\right.\\
&\qquad\left.-\frac{\beta\sum_{a=1}^n s_i^{(a)}s_k^{(a)}}{N}\right)^2+\beta h\left(\sum_i\sum_{a=1}^n s_i^{(a)}\right)\\
&\qquad\left.+\frac{\beta^2\left(\sum_{a<b}\left(\sum_i s_i^{(a)}s_i^{(b)}\right)^2-\frac{n^2N}{2}+\frac{nN^2}{2}\right)}{2N}\right)\\
&=\sum_{\{s\}}\exp\left(\frac{\beta^2Nn}{4}+\frac{\beta^2N}{2}\sum_{a<b}\left(\sum_i\frac{s_i^{(a)}s_i{}^{(b)}}{N}\right)^2+\beta h\sum_i\sum_{a=1}^n s_i^{(a)}\right)
\end{aligned}
$$

Provided the property of Gaussian integral

$$
e^{\lambda a^2}=\sqrt{\frac{1}{2\pi}}\int_R\exp\left(-\frac{1}{2}x^2+\sqrt{2\lambda}ax\right)\mathrm{d}x \qquad (28)
$$

The cross term in equation(27) can be transformed into an integral term of parameter $Q_{a,b}$:

$$
\begin{aligned}
&\exp\left(\frac{\beta^2N}{2}\times\left(\sum_i\frac{s_i^{(a)}s_i{}^{(b)}}{N}\right)^2\right)\\
&=\sqrt{\frac{N\beta^2}{2\pi}}\int_R dQ_{a,b}\exp\left(-\frac{N\beta^2}{2}Q_{a,b}^2+\beta^2Q_{a,b}\sum_i s_i^{(a)}s_i^{(b)}\right)
\end{aligned}
\qquad (29)
$$

Note that in equation(29) new parameter $Q_{a,b}$ is introduced to rewrite the mathematical form of the equation, $Q_{a,b}$ has nothing to do with $a$ and $b$ replicates. Up till now, the introduction of $n$ provides a way to replace the $Z_J^n$, in other words, the n-th power of the interaction in an N-sites system is equivalent to the interaction in n number of N-site system. Nevertheless, if we change the order of integral symbols, summation symbols, and multiplication symbols, equation(29) would be converted to an expression related to matrix Q(n-order symmetric matrix with $Q_{a,b}$ as elements).

$$
\begin{aligned}
Z_n=&\int\prod_{a<b}\left(\sqrt{\frac{N\beta^2}{2\pi}}dQ_{a,b}\right)\exp\left(-\frac{N\beta^2}{2}\sum_{a<b}Q_{a,b}^2+\frac{\beta^2Nn}{4}\right.\\
&\left.+\ln\sum_{\{s\}}\exp\left(\beta^2\sum_{a<b}\sum_i Q_{a,b}s_i^{(a)}s_i^{(b)}+\beta h\sum_i\sum_{a=1}^n s_i^{(a)}\right)\right)\\
=&\int\prod_{a<b}\left(\sqrt{\frac{N\beta^2}{2\pi}}dQ_{a,b}\right)\exp\left(-NA[Q]\right)
\end{aligned}
\qquad (30)
$$

Where $\int$ means $\int_{\frac{n(n-1)}{2}}$, and

$$
\begin{aligned}
A[Q]=&\frac{\beta^2}{2}\sum_{a<b}Q_{a,b}^2-\frac{\beta^2n}{4}\\
&-\frac{1}{N}\ln\sum_{\{s\}}\exp\left(\beta^2\sum_{a<b}\sum_i Q_{a,b}s_i^{(a)}s_i^{(b)}\right.\\
&\left.+\beta h\sum_i\sum_{a=1}^n s_i^{(a)}\right)
\end{aligned}
\qquad (31)
$$

We can rewrite the exponential part of $A[Q]$ as the summation of each replicate is independent of the summation of the location of a site.

$$
\begin{aligned}
&\sum_{\{s\}}\exp\left(\beta^2\sum_{a<b}\sum_i Q_{a,b}s_i^{(a)}s_i^{(b)}+\beta h\sum_i\sum_{a=1}^n s_i^{(a)}\right)\\
&=\prod_i\sum_{\{s_i\}}\exp\left(\beta^2\sum_{a<b}Q_{a,b}s_i^{(a)}s_i^{(b)}+\beta h\sum_{a=1}^n s_i^{(a)}\right)\\
&=\left(\sum_{\{s_i\}}\exp\left(\beta^2\sum_{a<b}Q_{a,b}s_i^{(a)}s_i^{(b)}+\beta h\sum_{a=1}^n s_i^{(a)}\right)\right)^N
\end{aligned}
\qquad (32)
$$

It should be noted that $\sum_{\{s_i\}}$ is different from $\sum_{\{s\}}$. Each $\{s_i\}$ is an array of n spin sites. It means that n replicates of the N-site system are equivalent to the N-th power of an n-site system. In beginning, we introduce $Z_n$ as $Z_J^n$, after a set of mathematical deductions, we find that $Z_n$ can be considered as a product of the interaction of N systems. Finally, we derive the expression of $Z_n$ as below:

$$
H[Q,S]=-\beta\sum_{a<b}Q_{a,b}S_aS_b-\beta h\sum_{a=1}^n S_a \qquad (33)
$$

$$
Z[Q]=\sum_{\{S\}}\exp\left(-\beta H[Q,S]\right) \qquad (34)
$$

$$
A[Q]=\frac{\beta^2}{2}\sum_{a<b}Q_{a,b}^2-\frac{\beta^2n}{4}-\ln Z[Q] \qquad (35)
$$

$$
Z_n=\int\prod_{a<b}\sqrt{\frac{N\beta^2}{2\pi}}dQ_{a,b}\exp\left(-NA[Q]\right) \qquad (36)
$$

## The saddle-point method — Taking SK model as an example

The saddle-point method is a powerful approximate way to process the exponential integral that locally holds a larger value compared to other regions. Consider the integral(37), where $N$ is a large number and $x_0$ is the maximum point.

$$I(N)=\int_R dx\exp\left(Nf(x)\right) \tag{37}$$

It can be Taylor expanded to the second order as equation(38).

$$\begin{aligned} I(N)=\int_R df\exp\Big(N\Big(f(x_0)+f'(x_0)(x-x_0)\\ +\frac{f''(x_0)}{2}(x-x_0)^2+\dots\Big)\Big)\end{aligned} \tag{38}$$

$f^{'}(x)$ equals 0 at the maximum point, so with the property of the Gaussian integral we derive the results of saddle point approximation(equation(39)).

$$\int_R dx\exp\left(Nf(x)\right)=\sqrt{\frac{2\pi}{N\left|f''(x_0)\right|}}\exp Nf(x_0)\times\left(1+O\left(\frac{1}{N}\right)\right) \tag{39}$$

Multivariate functions $f(x_1,x_2,\cdots,x_n)$ can be processed by the saddle-point method similarly.

$$\begin{aligned}\int_R dx_n\int_R dx_{n-1}\dots\int_R dx_2\int_R dx_1\exp\left(Nf(x_1,x_2,\dots x_n)\right)\\ =\exp Nf(x_{10},x_{20},\dots x_{n0})\left(\frac{2\pi}{N}\right)^{\frac{n}{2}}\sqrt{\frac{1}{|\det H|}}\end{aligned} \tag{40}$$

Where H is the Hessen matrix of $f(x_1,x_2,\cdots,x_n)$.

Using the saddle-point method, the n-th power of partition function $Z_n$ and the average density of free energy $f_n$ can be transformed into a more simple expression(equation(41) and equation(42)).

$$Z_n=\sqrt{\frac{\beta^{n(n+1)}}{\det(\Lambda)}}\exp(-N\min(A[Q])) \tag{41}$$

$$f_n\approx\frac{\min(A[Q])}{\beta n} \tag{42}$$

$$Q_{a,b}=\sum_{\{S\}}\frac{S_aS_be^{-\beta H[Q,S]}}{Z[Q]}=\langle S_aS_b\rangle_Q$$

where $\langle\rangle_Q$ denotes the average of all possible $Q$ configurations.

## The TAP method

In this section we will introduce a method, call the TAP method, to figure out the distribution of the spin site without using the replica approach method. Through its result conflicts with the replica method, it is helpful to take the main suggestion of the method to understand the physical meaning of the replica concept.

The TAP method is named for the originator of the method Thouless, Anderson, and Palmer[16]. It's based on the usual mean-field equations in a magnet:

$$\langle s_i\rangle\equiv m_i=\tanh(\beta\tilde{h_i});\quad \tilde{h_i}=h+\sum_{k=1}^{N}J_{i,k}m_k \tag{43}$$

Here we should denote that $\langle\rangle$ is the average quantity of the iþs in a specific coupling $J$. It's the construction of $J$ that makes $J$ different from zero when there is no external magnetic field $h$.

The TAP method said that: in the limit $N\to\infty$, the effective magnetization $m_i$ is determined by the system, not including the iþspin site. Therefore, we should correct equation(43) by subtracting the "reaction term".

Finally we obtain[14]

$$\begin{aligned} m_i&=th(\beta\tilde{h_i})+O(1/N)\\ \tilde{h_i}&=h+\sum_{k=1}^{N}J_{i,k}(m_k-m_iJ_{i,k}\chi_{k,k})\\ &\approx\sum_{k=1}^{N}J_{i,k}m_k-\sum_{k=1}^{N}m_iJ_{i,k}^2\beta(1-m_k^2)\end{aligned} \tag{44}$$

where $\chi_{k,k}$ is the magnetic susceptibility of site $k$.

$$\chi_{k,k}=\partial m_k/\partial\tilde{h}_k=\beta(1-m_k^2) \tag{45}$$

$$\tilde{h_i}=h+\sum_{k=1}^{N}J_{i,k}m_k-m_i(1-q_{EA}) \tag{46}$$

Equation(46) is in the limit $N\to\infty$, and $q_{EA}$ denotes $(1/N)\sum_{k=1}^{N}m_i^2$. Equation is called TAP equation. Provided the relatively small value of $m_i$ in high temperature(higher than $T_c$), we can expand equation(46) to the first degree. And we get

$$m_i=\beta\sum_j J_{ij}-\beta m_i-O(m^3) \tag{47}$$

Equation(47) can be written as

$$m_\lambda=\beta(J_\lambda-1)m_\lambda-O(m^3) \tag{48}$$

We call $m_\lambda$ the projection of the magnetization onto the vector eigenvector of the matrix $J$ with the eigenvalue $J_\lambda$. It's evident that $m_\lambda$ is different from zero only if $\beta(J_\lambda-1)$ is larger than one. Actually the largest $J_\lambda$ is two, which indicates a phase transition at $\beta = 1/T_c = 1$.

Let's go back to equation(44), this equation tells us that its solution number increases exponentially with $N$. Naturally, we think that the expectation of $m_i$ is the linear combination of the solutions. If we mark each solution with the unique order $\alpha$, we can obtain

$$\langle s_i \rangle \equiv m_i = \sum_\alpha \omega_\alpha m_i^\alpha$$
$$\sum_\alpha \omega_\alpha = 1 \tag{49}$$

Here we need to distinguish the difference between the expected value and a single solution. Firstly, a pure state is corresponding to a single solution, which can't be decomposed to the combination of the other solutions. Secondly, for intensive quantities $A_i$, $\frac{1}{N}\sum_i A_i$ doesn't fluctuate when $N \to \infty$, but the mixed state doesnt usually equip this property. For instance, at 0 Celsius, $H_2O$ has two pure states: the water and the ice. When the phase transition happens part of the water gets to condense into ice (or the ice melts into the water), and the system is in a state of the combination of the two pure states. A pure state but not the mixture state better meets our expectation of an equilibrium state.

## RESULTS

### The replica symmetric solution of EA model

In "Method" part we have introduced the new order parameter $q_{\alpha\beta} = \langle s_i^{(\alpha)} \cdot s_i^{(\beta)} \rangle$ Edward and Anderson used. And they obtained some rough properties about the specific heat and susceptibility of spin glass by introducing the new order parameter, but there were some discrepancies with experimental facts.

We take the simplest possible probability distribution for $J_{ij}$, that is

$$P(J_{ij}) = \exp\left(-\frac{J_{ij}^2}{2J^2\rho_0^2}\right)$$

where $J^2 = \sum_{ij} J_{ij}^2$ and $\rho_0$ is the density of occupation.

The free energy $f$ of a particular spin-glass can be defined as

$$\exp\left[-\frac{f(J)}{kT}\right] = \int \exp\left[\sum J_{ij} s_i \cdot s_j\right] \prod (ds_i) \tag{50}$$

The ensemble free energy $F$ can be defined as

$$F = \int f(J)P(J)dJ \tag{51}$$

where $P(J)$ is the probability of J. However, the direct integration of s and J is too hard to perform, so we rewrite the equation so that the order of the integration can be altered, considering $m$ systems ($m$ is small) and defining an $\tilde{F}(m)$ by

$$\exp\left[-\frac{\tilde{F}(m)}{kT}\right] = \int \prod_{\alpha=1}^{m} \left(\prod ds^{(\alpha)}\right) \exp\left(\sum_{\alpha=1}^{m} \sum_{ij} J_{ij} s_i^\alpha \cdot s_j^\beta\right) P(J)dJ \tag{52}$$
$$= \int \exp\left[-\frac{mf(j)}{kT}\right] P(J)dJ \tag{53}$$
$$= 1 - \frac{mF}{kT} + O(m^2) \tag{54}$$

Hense

$$\tilde{F}(m) = mF + O(m^2)$$

The introduction of replicas allows us to have

$$\int \exp\left(\sum_{ij\alpha} J_{ij} s_i^{(\alpha)} s_j^{(\alpha)}\right) = \exp\left[\frac{3\rho}{2\rho_0} \sum_{\alpha\beta ij} \left(s_i^{(\alpha)} s_j^{(\alpha)} s_i^{(\beta)} s_j^{(\beta)}\right)\right] \tag{55}$$

where

$$\rho = \frac{2J^2\rho_0^2}{3(kT)^2}$$

We apply a variational principle of the Feynman type, replacing the quartic form with a best quadratic.

We introduce another parameter $\eta$ which minimizes

$$\int \prod ds \exp\left(-\frac{1}{2}\sum \eta \mathbf{ss} + C\right)$$

where C is chosen so that

$$\int \prod ds \exp\left(\frac{1}{2}\sum \eta \mathbf{ss} + C\right)$$
$$\left[\frac{1}{2}\sum \eta \mathbf{ss} - C - \frac{3\rho}{2\rho_0} \sum_{\alpha\beta ij} \left(s_i^{(\alpha)} s_j^{(\alpha)} s_i^{(\beta)} s_j^{(\beta)}\right)\right] = 0$$

From the perspective of accuracy and ease of calculation, we also note that

$$\sum_{\alpha \neq \beta} s_i^{(\alpha)} \cdot s_i^{(\beta)} = \left(\sum s_i^{(\alpha)}\right) \cdot \left(\sum s_i^{(\beta)}\right) - m \tag{56}$$

Additionally, we introduce another symbol $\epsilon$ to simplify our equation

$$\epsilon(m,n) = \int \exp\left(\frac{1}{2}\eta \sum_{i,\alpha\neq\beta} s_i^{(\alpha)} s_i^{(\beta)}\right) \prod (ds) \tag{57}$$

Through the above process, we have successfully replaced complex summations of $s_i^{(\alpha)}, s_j^{(\alpha)}, s_i^{(\beta)}, s_j^{(\beta)}$ with parameters $q, m, \eta, \epsilon$:

$$\langle s_i^{(\alpha)}\cdot s_j^{(\alpha)}\cdot s_i^{(\beta)}\cdot s_j^{(\beta)}\rangle = \frac{m}{3}+\frac{1}{3}(\frac{2\partial\epsilon}{\epsilon\partial\eta})^2\frac{1}{m(m-1)} \tag{58}$$
$$= \frac{m}{3}[1-q^2(1-m)] \tag{59}$$

Finally, we can combined these terms together and obtain the ensemble free energy $\tilde{F}$

$$-\frac{\tilde{F}}{kT} = \log\epsilon - \frac{\eta\partial\epsilon}{\epsilon\partial\eta}+\frac{\rho}{2\rho_0}[1-q^2(1-m)] \tag{60}$$
$$\eta = (\frac{\rho}{\rho_0})q \tag{61}$$

Then, we can get internal energy from $F=-Tf$:

$$E = T^2\frac{\partial f}{\partial T} \tag{62}$$
$$= -T(\frac{\rho}{\rho_0}(1-q^2)) \tag{63}$$

and then the specific heat $C_v$ ($\frac{\rho}{\rho_0}=\lambda$)

$$C_v = \frac{\partial E}{\partial T} \tag{64}$$
$$= -\lambda(1-q^2) - T\frac{\partial\lambda}{\partial T}\frac{\partial}{\partial\lambda}[\lambda(1-q^2)] \tag{65}$$
$$= \lambda(1-q^2) - 4\lambda^2 q\frac{\partial q}{\partial\lambda}, \tag{66}$$

This implies a cusp in $C_v$, which part satisfies the observations of the experiment (de Nobel and du Chatenier 1959, Zimmerman and Hoare 1960, Zimmerman and Crane 1961)[5].

## The replica symmetric solution of SK model and its physical properties

In the "Methods" part we have introduced the replica approach and other mathematical tricks in solving SK models. We finally get equation(42).

According to equation(31),it is apparent that $A[Q]$ is rotating symmetric about $Q$. That is, if we exchange the values of a certain $Q_{a,b}$ and $Q_{c,d}$, the value of $A[Q]$ never changes. This property indicates that

$$\begin{aligned} &\forall a,b(a\neq b): && Q_{a,b}=q\\ &\forall a: && Q_{a,a}=0 \end{aligned} \tag{67}$$

$$\frac{\mathrm{d}A[Q]}{\mathrm{d}q}=0 \tag{68}$$

may be a stationary point of $A[Q]$. Now we assume them the conditions of minimum $A[Q]$, which is just what we need in the equation(42).

According to equation(43), in this condition, all $\langle S_aS_b\rangle_Q$ have the same value $q$, thus all replicas are equivalent and their interactions are the same. This property is called the replica symmetry. And in this condition, we can get a solution, which was named the replica symmetric solution (RS solution).

We use the condition of RS solution (equation(67)) to rewrite $Z[Q]$ in equation(34):

$$Z[Q] = \sum_{\{S\}}\exp(\beta^2\sum_{a<b}qS_aS_b+\beta^2 h\sum_{a=1}^{n}S_a) \tag{69}$$
$$= \exp\frac{\beta^2 nq}{2}\sum_{a<b}(\frac{\beta^2 q}{2}\left(\sum_a S_a\right)^2+\beta h\sum_a S_a) \tag{70}$$
$$= \exp\frac{\beta^2 nq}{2}\sum_{\{S\}}\int_R \mathrm{d}z(\sqrt{\frac{1}{2\pi}}\exp(-\frac{1}{2}z^2) \tag{71}$$
$$\times\exp((\sum_a S_a)(\beta\sqrt{q}z+\beta h))) \tag{72}$$
$$= \frac{1}{\sqrt{2\pi}}\exp\frac{\beta^2 nq}{2}\int_R \mathrm{d}z\exp(-\frac{1}{2}z^2) \tag{73}$$
$$\times\sum_{S_1=\pm1}\sum_{S_2=\pm1}\dots\sum_{S_n=\pm1}\exp((\sum_a S_a)(\beta\sqrt{q}z+\beta h)) \tag{74}$$
$$= \frac{1}{\sqrt{2\pi}}\exp\frac{\beta^2 nq}{2}\int_R \mathrm{d}z\exp(-\frac{1}{2}z^2)(2\cosh(\beta\sqrt{q}z+\beta h))^n \tag{75}$$

In the third step we use the property of Gaussian integral (equation(28)) again and let $z$ to be $x$ and $\sum_a S_a$ to be $a$. In the forth step we use $\sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_n=\pm1}$ to replace $\sum_S$.

Use equation(75) to rewrite $A[Q]$ in equation(35):

$$A[Q] = -\frac{n\beta^2}{4}+\frac{\beta^2 n(n-1)q^2}{4}+\frac{\beta^2 nq}{2} + \ln\frac{1}{\sqrt{2\pi}}\int_R \mathrm{d}z\exp\left(-\frac{1}{2}z^2\right)(2\cosh(\beta\sqrt{q}z+\beta h))^n \tag{76}$$

We can apply the extremum condition (equation(68)) to equation(76).

$$\frac{\beta^2}{2}(n(n-1)q+n) = \frac{\int\mathrm{d}z\exp(-\frac{1}{2}z^2)(2\cosh(u))^n\tanh(u)\frac{n\beta z}{2\sqrt{q}}}{\int\mathrm{d}z\exp(-\frac{1}{2}z^2)(2\cosh(u))^n}$$

where $\int$ represents $\int_R$, $u=\beta\sqrt{q}z+\beta h$.

When $n\to 0$, we can rewrite equation(**??**):

$$q = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{d}z\exp\left(-\frac{z^2}{2}\right)\tanh^2(\beta h+\beta\sqrt{q}z) \tag{77}$$

Using this equation we can solve $q$ in specific $\beta$ and $h$. Use equation(77) to transform $f_n$ and let $n\to 0$ we can get:

$$f = \frac{1}{4}(-\beta)(q-1)^2 - \frac{1}{\beta}\int_R \mathrm{d}z \frac{\exp\left(-\frac{z^2}{2}\right)\ln\left(2\cosh\left(\beta h + \beta\sqrt{q}z\right)\right)}{\sqrt{2\pi}} \tag{78}$$

Solving equation(77) in the condition of $h = 0$ and we can find that in high temperature ($\beta > 1$), there is only one solution $q = 0$; but in low temperature ($\beta < 1$), there are two solutions, one zero and another nonzero.

If in the further calculation, the nonzero solution was proved to be the minimum, we will get a seemly perfect solution of the SK model. But unluckily, according to de Almeida and Thouless's paper[3], this solution is not stable. In low temperatures and external magnetic fields, the RS solution is not the real minimum.

To discuss whether the RS solution is stable or not, we need to investigate the Hessian matrix $M$ of function $A[Q]$.

$$\begin{aligned} M_{(a,b)(c,d)} &= \frac{\partial^2 A[Q]}{\partial Q_{a,b}\partial Q_{c,d}} \\ &= \frac{\partial^2}{\partial Q_{a,b}\partial Q_{c,d}}\left(-\ln\sum_{\{S\}}\exp\left(\beta^2 h\sum_{e=1}^{n} S_e + \beta^2\sum_{e<f} S_e S_f Q_{e,f}\right) + \frac{1}{2}\beta^2\sum_{e<f} Q_{e,f}^2 - \frac{\beta^2 n}{4}\right) \\ &= \frac{\partial}{\partial Q_{c,d}}\left(\beta^2 Q_{a,b} - \frac{\sum_{\{S\}}\left(\beta^2 S_a S_b \exp\left(\beta^2 h\sum_{e=1}^{n} S_e + \beta^2\sum_{e<f} S_e S_f Q_{e,f}\right)\right)}{\sum_{\{S\}}\left(\exp\left(\beta^2 h\sum_{e=1}^{n} S_e + \beta^2\sum_{e<f} S_e S_f Q_{e,f}\right)\right)}\right) \\ &= \beta^2\frac{\partial Q_{a,b}}{\partial Q_{c,d}} - \beta^4\langle S_a S_b S_c S_d\rangle + \beta^4\langle S_a S_b\rangle\langle S_c S_d\rangle \end{aligned} \tag{79}$$

Here we denote $\langle S_a S_b S_c S_d\rangle$ by $r$. Like equation(77), we can get:

$$r = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{d}z\exp\left(-\frac{z^2}{2}\right)\tanh^4\left(\beta h + \beta\sqrt{q}z\right) \tag{80}$$

Since the value of $S_i$ is $\pm 1$, we have:

$$M_{(a,b)(c,d)} = \begin{cases} -\beta^4 + \beta^2 + \beta^4 q^2 = M_1,\ (a,b) = (c,d) \\ \beta^4 q^2 - \beta^4 q = M_2,\ a = b\ or\ c = d \\ \beta^4 q^2 - \beta^4 r = M_3,\ a \neq c\ and\ b \neq d \end{cases} \tag{81}$$

The eigenvalue $m$ of $M$ meet the following condition:

$$Mv = mv \tag{82}$$

where $v$ is the eigenvector, i.e.

$$\sum_{(c,d)} M_{(a,b)(c,d)} v_{(c,d)} = m v_{(a,b)} \tag{83}$$

where $\sum_{(c,d)}$ denotes the sum over $(1,2), (1,3), ..., (n-1,n)$.

By analyzing equation (83) we can get three eigenspaces of $M$ (and they are all):

Longitudinal:

$$v_{(a,b)} = v \tag{84}$$

$$m_L = M_1 - 4M_2 + 3M_3 \tag{85}$$

Anomalous:

$$v_{(a,b)} = \frac{1}{2}(v_a + v_b),\ where\sum_a v_a = 0 \tag{86}$$

$$m_A = M_1 - 4M_2 + 3M_3 \tag{87}$$

Repicon:

$$v_{(a,b)},\ where\sum_a v_{(a,b)} = 0 \tag{88}$$

$$m_R = M_1 - 2M_2 + M_3 \tag{89}$$

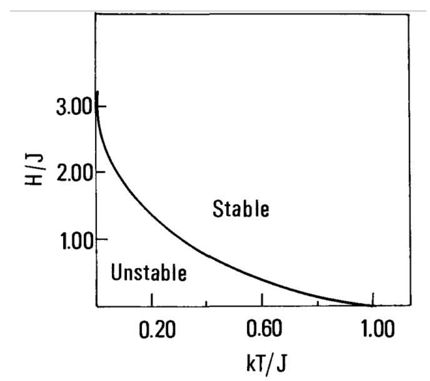


FIG. 6. de Almeida - Thouless line

If the RS solution was the real minimum, $m_L$, $m_A$ and $m_R$ should all be positive. De Almeida and Thouless pointed out, that in low temperature and low external magnetic field, they are all not still positive.[3] The boundary of the stable (real minimum) and the unstable (not minimum) solution is called the de Almeida - Thouless line (see FIG 6).

**Breaking replica symmetry**

In the TAP method part we learn the difference between the fixed solution and the pure solution. We can express this difference with spin $s_i$.

$$\langle s_1 s_2 \cdots s_k\rangle = \sum_\alpha \omega_\alpha \langle s_1 s_2 \cdots s_k\rangle_\alpha \tag{90}$$

$$\langle s_1 s_2 \cdots s_k\rangle_\alpha = m_1^\alpha m_2^\alpha \cdots m_k^\alpha \tag{91}$$

Equation(91) is written because in a pure state, connected correlation function vanishes when $N \to \infty$(for more detailed calculation see paper [6]). As $\omega_\alpha$ is unknown weights, there is no reason for the mixed state to equip the same property.

The pure state and the mixed state is important because of the following reason: an nonequilibrium state evolves toward a pure state and the time needed is exponentially large with the size of the system. For spin glass system, the spin site with freezing property is not static for an infinity time, but maintain its spin for much longer time than the observation time. Naturally, we want to derive an indicator to measure the "distance" between two pure states in the phase space. Here we define $d_{\alpha\beta}$

$$d_{\alpha\beta}^2 = 1/N\sum_i (m_i^\alpha - m_i^\beta)^2 \tag{92}$$

Also we can define the overlap between two solutions $q_{\alpha\beta}$

$$q_{\alpha\beta} = 1/N\sum_i m_i^\alpha m_i^\beta \tag{93}$$

and $q_{\alpha\alpha} = q_{EA}$($q_{EA}$ is denoted in equation(46)). An acceptable argument is that the distribution of the overlap is helpful for us to describe the state distribution. We introduce $P(q)$ as

$$P(q) = \overline{P_J(q)} = \overline{\sum_{\alpha,\beta}\omega_\alpha^J\omega_\beta^J\delta(q_{\alpha\beta}-q)} \tag{94}$$

Here the bar denotes the average for all the coupling $J$. According to the Boltzmann statistical distribution law the $\omega_\alpha$is written as

$$\omega_\alpha \propto \exp(-\beta F_\alpha)$$

here $\beta = \frac{1}{k_B T}$ and $F_\alpha$ is the free energy of the $\alpha$ solution.

As we've talked about above, $P_J(q)$ is sensitive to the change of $J$, and it fluctuates as N approaches infinity, so the following expression is obvious

$$\overline{P_J(q_1)P_J(q_2)} \neq \overline{P_J(q_1)}\times\overline{P_J(q_2)} \tag{95}$$

This relationship is the main point that is ignored in the replica method. Next, we will compute the $P(q)$ in the replica method to show the difference.

Let's consider the quantities as follow

$$\begin{aligned} q_J^{(1)} \equiv& 1/N\sum_{i=1}^N \langle s_i\rangle\langle s_i\rangle \\ =& 1/N\sum_{i=1}^N\left\{\sum_\alpha \omega_\alpha\langle s_i\rangle_\alpha\right\}\left\{\sum_\beta\omega_\beta\langle s_i\rangle_\beta\right\} \\ =&\sum_{\alpha,\beta}\omega_\alpha\omega_\beta q_{\alpha,\beta} \\ =&\int dqP_J(q)q \end{aligned} \tag{96}$$

Similarly we can compute the $q^{(k)}$ as follow

$$\begin{aligned} q_J^{(k)} \equiv& 1/N^k\sum_{i=1}^N\langle s_{i_1}s_{i_2}\cdots s_{i_k}\rangle\langle s_{i_1}s_{i_2}\cdots s_{i_k}\rangle \\ =&\int dqP_J(q)q^k \end{aligned} \tag{97}$$

Equation(97) holds because of the vanishing connected correlation function when $N\to\infty$. Such equation indicates the following expression

$$q^{(k)} \equiv \overline{\langle s_{i_1}s_{i_2}\cdots s_{i_k}\rangle\langle s_{i_1}s_{i_2}\cdots s_{i_k}\rangle} = \int dqP(q)q^k \tag{98}$$

where the bar denotes the average over all coupling $J$ and $i_1, i_2$ can be randomly chosen.

If we apply the replica method to compute the $q^{(k)}$, we will obtain

$$\begin{aligned} q^{(k)} =& \lim_{n\to 0}\left\{\langle s_i^a s_i^b\rangle\right\}^k = \lim_{n\to 0}[Q_{a,b}]^k \\ Q_{a,b} \equiv& \langle s_i^a s_i^b\rangle \end{aligned} \tag{99}$$

Here $Q_{a,b}$ is defined the same way we used in equation(29).If the replica symmetry is not broken, equation(99) is not ambiguous because $a$ and $b$ replicates are equivalent, here we should carefully write $q^{(k)}$ as following(since we are not sure if each replicate is equivalent)

$$q^{(k)} = \lim_{n\to 0}\frac{2}{n^2-n}\sum_{a,b}[Q_{a,b}]^k \tag{100}$$

If one solution of $Q_{a,b}$ is found which is a stable solution of the saddle-point equations, its evident that the matrix obtained by exchanging two columns or two rows of the previous matrix is also a solution, and the result is the average of all these solutions corresponding to the same saddle point. That's the meaning of equation(100).

If we compare equation(97) and equation(100) carefully, we can find an interesting result:

$$P(q) = \lim_{n\to 0}\frac{2}{n^2-n}\sum_{a,b}\delta(Q_{a,b}-q) \tag{101}$$

$P(q)$ and $Q_{a,b}$ have different sources. The left side of the equation is the possibility distribution of $q$ considering all couplings $J$, and $q$ is the coincidence degree of two solutions. Intuitively $P(q)$ includes information on relaxation time. For example, above the critical temperature $P(q)$ is a Delta function in a ferromagnet, indicating $m$ always keeps a single solution. The right side of the equation comes from the replica method which is talked about in the SK model. Through the analysis for $q^{(k)}$, the broken symmetry is explained by the equilibrium states.

Up till now, we've acquired an explanation for the broken symmetry by talking about the pure equilibrium state and the expected equilibrium state. We finally explain the phase transition for the following reason: under certain critical temperatures, evolution between pure states needs much longer time than mesoscopic time, resulting in macroscopic fluctuation. Using the replica method, we find the order parameter matrix for this phase transition $Q_{a,b}$. Then we find the system is unstable if all the replicates are equivalent. To give a physical explanation for the inequality of replicates, we introduce the TAP method and successfully build the bridge between the equilibrium states and the broken replica symmetry.

In the next part, we will provide the solution with breaking replica symmetry.

### Order parameter matrix and fractal

In the standard treatment of the S-K model, $Q_{\alpha,\beta} = const = q$, where q is the Edward-Anderson order parameter mentioned above.

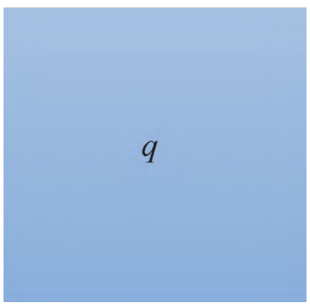


FIG. 7. Q (RS)

This treatment with the computer simulations gives a negative entropy at zero temperature, i.e., $S(0) = -0.17$. However, we all know that the entropy of the model must be zero or positive by definition. Given the existence of different states, replicates should differ from each other, and $q$ in matrix $Q_{\alpha,\beta}$ has no reason to be a constant, so we have the opportunity to make $Q_{\alpha,\beta}$ a variational parameter.

Parisi first parametrized the matrix $Q_{\alpha,\beta}$ as a function of three variables, which is also called "first-order replica symmetry broken (1-RSB)".

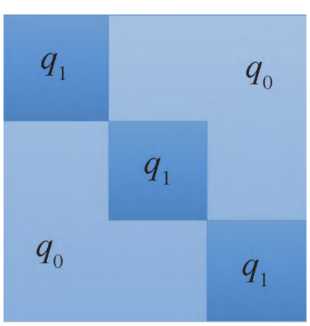


FIG. 8. Q (1-RSB)

After computer simulations, the result he obtained for the S-K model is excellent. The zero-temperature entropy came to $S(0) = -0.01$. Naturally, along this line of thought, he further examined the second-order (2-RSB, FIG9), the third-order (3-RSB), and so on.

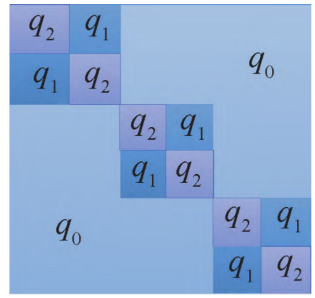


FIG. 9. Q (2-RSB)

Parisi's calculation shows that as the number of broken orders increases, $S(0)$ gradually tends to zero (TableII).

| The number of broken orders | S(0) |
|---|---|
| 0 | -0.16 |
| 1 | -0.01 |
| 2 | -0.004 |
| $\infty$ | 0 |

TABLE II. S(0) obtained by different broken replicas' order

These attempts led Parisi to speculate that only an infinite-order replica symmetry breaking (called a full-order replica symmetry breaking) would give a truly rigorous solution to the spin glass phase. The theory of full-replica symmetry breaking indeed obtains the entropy S(0) = 0 at zero temperature, and the analysis of the free energy function shows that the full-replica symmetry breaking solution is stable at low temperature. In addition, the results of the full replica symmetry breaking theory are also in good agreement with the simulation results of the SK model.

From an intuitive point of view, the phase space of spin-glass is divided into infinite levels, each of which is composed of multiple subspaces. The replicas within the same subspace are symmetrical, and the subspaces also have rearrangement invariance. This splitting can continue in the form of a fractal (FIG10) until it reaches the smallest subspace.

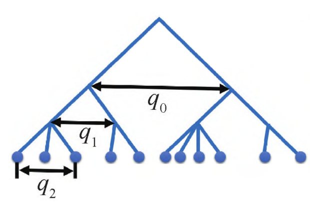


FIG. 10. Fractal

## DISCUSSIONS

Since nature is a huge complex system, the study of complex systems in statistics can also be widely used in a large number of other fields. As the simplest model of a complex disordered system, the spin-glass model can also inspire future generations in a large number of fields, which is one of the reasons why Parasi can win the Nobel Prize. After getting a rough idea of Parisi's work, we also learned about applications in some other areas.

### Application of spin glass model in neural networks

In this section, we will introduce a famous application of the spin-glass theory, the Hopfield network. In 1982, J.J.Hopfield proposed the algorithm base on the biological characteristics of neurons: Conduction with time interval, asynchronous processing, and emergent collective behaviors[7].

In many neural networks, the information process is considered a sequential structure. For example, the input state $V_{in}$(a vector) is put in a matrix to obtain an output state, i.e.

$$V_{out} = WV_{in} + b$$

where, W is a matrix and b is an offset vector. With this opinion formation of memory is to adjust a set of matrix $W_i$ and offset vector $b_i$.

However, the Hopfield network suggests a different way. In this model, each neuron has two states $V_i = 0(not activated), V_i = 1(activated)$, and N neurons form a piece of information, which is familiar with the magnetization $m$ in the spin-glass system. We've known that in the spin-glass system, the coupling construction $J$ determines the distribution of solutions $m^\alpha$, and any non-equilibrium state m would fall into a certain pure state with the function of $J$. The same idea can be transferred to neural networks. If we want to remember a group of state $V(includes V_1, V_2, V_3, \cdots)$, we should adjust the coupling $J$. To hit a certain local minimum point $m^\alpha$, the networks perform the following operations on each neuron:

$$\begin{aligned} V_i \to 1, \quad if \sum_j J_{ij} V_j > U_i \\ V_i \to 0, \quad if \sum_j J_{ij} V_j < U_i \end{aligned} \tag{102}$$

$U_i$ is the activation threshold and is usually 0.

Hopfield network gives the construction of a special $J$ for a set of information $V^s, (s = 1, 2, \cdots, k)$.

$$J_{i,j} = \sum_s (2V_i^s - 1)(2V_j^s - 1) \tag{103}$$

but with $J_{i,i} = 0$. If we use equation(102) to judge the signal of $V_i^{s'}$

$$\sum_j J_{i,j} V_j^{s'} = \sum_s (2V_i^s - 1)\left[\sum_j V_j^{s'}(2V_j^s - 1)\right] \tag{104}$$

The bracket in equation(104) approaches zero unless $s = s'$, so

$$\sum_j J_{i,j} V_j^{s'} \approx (2V_i^{s'} - 1)N/2 \tag{105}$$

It's evident that $V^s$ is the equilibrium state of the system when N is a large number. Another main point for this system is: would the original state $V$ approach toward the equilibrium state spontaneously, and how to define an "adjacent" state? We can answer the question with the spin glass theory.

In the Hopfield network, we can define

$$E = -\sum\sum_{i \neq j} T_{i,j} V_i V_j \tag{106}$$

the $\Delta E$ and $\Delta V_i$ is given by:

$$\Delta E = -\Delta V_i \sum\sum_{i \neq j} V_i V_j \tag{107}$$

We will find that the system is isomorphic with the spin-glass model. Therefore, the network has a set of stable states as the recorded information, and naturally $q_{s,s'} = \sum_i V_i^s V_i^{s'}$ can be an appropriate indicator for measuring two adjacent states.

### Application of spin glass in gravity

In this section, we will introduce another interesting application of the spin-glass theory, the glassy gravity.

Our current understanding of the universe is an unhappy marriage of quantum field theory and general relativity. A typical example is that our expected value of the Cosmological Constant is off by some 120 orders of magnitude from its observed value. One possible explanation is that some quantities are dynamical in principle, but fixed over the observable universe or timescales, and we can observe just one of a multitude of possibilities. This situation is equivalent to that in the study of glass, for which an extensive number of degrees of freedom are fixed on experimental timescales.

A.De Giuli and E.Zee[4] hypothesized that Euclidean gravity, considered as a statistical field theory, is glassy. Thus there will be many local minima we call metastable states of the action we can not see by the formal partition function.

However, the unstable vacua are missed in the Euclidean computation because of the Euclidean regularity conditions. And this paper suggests that the unstable vacua are present as metastable states, just like in the partition function for a glass.

Consider a statistical field theory in a field $\phi(x)$, to which is added an external field $\psi(x)$, coupling through a term $U = h(\psi(x) - \phi(x))^2$. The free energy can be expressed as:

$$F_\phi[\psi, h, \beta] = \frac{-1}{\beta} \log \int D\phi \exp(-\beta H[\phi] - \frac{1}{2}U)$$

where H is the Hamiltonian of the field.

When the field is aligned with the constant field, we can scan phase space to find metastable states by varying $\psi$. If these states remain when $h \to 0^+$, then ergodicity is spontaneously broken.

In particular, consider the free energy of the field $\psi$ at inverse temperature $\beta m$:

$$F_\psi(m, \beta) = \lim_{h \to 0} \frac{-1}{\beta m} \log \int D\phi \exp(-m\beta F_\phi[\psi, h, \beta])$$

Obviously, calculating $F_\psi$ for arbitrary $m$ is very difficult for interaction theory in general. However, when $m$ is an integer, $\exp(m\beta F_\phi)$ can be expressed in m replicas. Then a term $\frac{h}{2m} \int dx(\phi_a(x) - \phi_b(x))^2$ can be introduced, where $a, b = 1, \cdots, m$ are replica indices. This item creates a coupling between replicas.

This concept is very similar to the free energy of a spin-glass system, so we can handle glassy gravity using the method of processing spin-glass used above, and finally analytically continued to $m = 1$. In this analytical continuation, a wealth of non-perturbation phenomena can occur.

In statistical field theory, replicas are distinct systems that evolve with independent thermal perturbations, but are coupled together by attractive interactions and are eventually removed to detect spontaneous destruction traversal. In quantum field theory, replicas have independent quantum fluctuations.

If Euclidean quantum gravity is glassy, then the partition function $Z$ vastly overcounts states, since on observable timescales the universe is stuck in one sector of phase space. If we are stuck in an atypical vacuum, then $Z$ may not resemble at all the contribution $Z_{local}$ that we infer from local measurements. Thus, this may be a possible solution to the Cosmological Constant problem.

## Application of spin glass in stock market

In this section, we broaden our horizon and find an interesting application in economics.

We know that in a complex disorder system, sometimes the microscopic details of interactions are not necessary to explain the macroscopic structures. This suggests that the most relevant properties ruling macroscopic behaviours of such complex systems are not the nature of microscopic entities but are order of interactions, their range and the topology[1].

Since financial and neural networks have topological similarities (modular, hierarchical, small-world organization highlighted by an asset tree-based approach), and we knew that neural networks can be described by the spin-glass theory, we can describe the stock market by the same method we applied in the spin-glass theory, even though spins and stocks are completely different elementary entities.

Thomas provided empirical evidence that the financial network can be accurately described by a statistical model, which can be thought of as the Ising model on a complex graph with scaled interaction. We consider stocks as economic entities influencing each other, taking no account of the interaction process itself. Then we binarize the prizes to interpret the daily movement as a bullish (or bearish) orientation like what we did in the spin-glass system (binarize the spin). So it is possible that we can apply spin-glass exact mean-field models to the study of the stock market.

We consider a set of N market indices or N stocks with binary states $s_i = \pm 1 (i = 1, 2, \cdots, N)$.The binary variables will be equal to 1 if the associated closing price is larger than (or equal to) the opening one and equal to $-1$ if not.In this model, we can also define:

$$H(s) = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} A_{ij} s_i s_j - \sum_{i=1}^{N} h_i s_i \tag{108}$$

Pairwise interactions between economic entities are modelled by interaction strength $J_{i,j}$ here, which is equivalent to that in the spin glass system.

And natrually we can define the probability distribution through this equation:

$$p_2(\mathbf{s}) = Z^{-1} \exp(\frac{1}{2} \sum_{i,j}^{N} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i) \equiv \frac{e^{-H(s)}}{Z} \tag{109}$$

Thus we can find that this model is isomorphic with the spin glass model. So they have many of the same properties, especially mathematically.

However, the N in stock market is often limited (about$10^2$ $10^3$), which is far less than the number of spins in the spin-glass model. It means that the fluctuation in this model may be significant even if the relevant variables are correctly scaled. Further study is still needed to overcome this problem.

## CONCLUSIONS

In this paper, we use mathematical derivation and physical analysis to explain the reason for the spin-glass theory and provide its application in other fields. We begin with the phase transition theory, introducing an important concept "order parameter". We try to find out the "hidden" order parameter in the spin-glass system.

Following the model derivation using the replica method, we successfully derive the expression of the system's partition function with a matrix variable $Q$. By the saddle-point method and natural hypothesis that all replicates are equivalent, we can figure out the physical property of the spin-glass system. However, this method leads to unstable results with negative zero temperature entropy.

TAP method provides a new view that there exist many pure states in the spin-glass system, which indicates the intuitive physical meaning of the replicates introduced in the replica method. Through comparing the TAP method and the replica method, the replicates are related to the many states, thus symmetry between replicates breaks since in the TAP method the number of the states' solutions is proportional to the number of spin sites. The breaking replica symmetry suggests a special construction of the parameter matrix $Q$. Finally, the result can well meet the results of numerical simulation.

Still we are not clear whether the matrix of order 0 appearing in derivation has any physical meaning, or if we can find another method without the introduction of replicates that avoids using the analytic continuation of the number of replicates tending to zero. The exploration of these questions is an interesting topic. Besides, the model is only discussed in 1-D case. It's hard to figure out the 2-D and 3-D cases, which is the actual physical scene. In a word, spin glass theory is still a puzzle for people. Although the spin glass theory has not been thoroughly understood, many interesting ideas have been created and successfully applied to other fields. This model has good applicability to other complex systems, like the neural network, the quantum gravity and the stock market mentioned in the "Discussion" part. We expect more intriguing ideas in the further study to spin glass.

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[1] Thomas Bury. A statistical physics perspective on criticality in financial markets. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(11):P11004, nov 2013.

[2] V. Cannella and J. A. Mydosh. Magnetic ordering in gold-iron alloys. *Phys. Rev. B*, 6:4220–4237, Dec 1972.

[3] J. R. L. De Almeida and D. J. Thouless. Stability of the sherrington-kirkpatrick solution of a spin glass model. *Journal of physics. A, Mathematical and general*, 11(5):983–990, 1978.

[4] A. De Giuli, E. Zee. Glassy gravity. *Europhysics Letters*, 133, 2021.

[5] S. F. Edwards and P. W. Anderson. Theory of spin glasses. *Journal of Physics F: Metal Physics*, 5(5):965–974, 1975.

[6] L. Gonçalves and Jairo de Almeida. One dimensional decorated lattice: The v –> limit. *Journal of Magnetism and Magnetic Materials - J MAGN MAGN MATER*, 31:1218–1220, 02 1983.

[7] J J Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the National Academy of Sciences*, 79(8):2554–2558, 1982.

[8] Ernst Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik*, 31(1):253–258, 1925.

[9] Lev Davidovich Landau. On the theory of phase transitions. I. *Phys. Z. Sowjet.*, 11:26, 1937.

[10] W Lenz. Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern. *Z. Phys.*, 21:613–615, 1920.

[11] Lars Onsager. Crystal statistics. i. a two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117–149, Feb 1944.

[12] G. Parisi. Infinite number of order parameters for spin-glasses. *Phys. Rev. Lett.*, 43:1754–1756, Dec 1979.

[13] G Parisi. The order parameter for spin glasses: a function on the interval 0-1. *Journal of Physics A: Mathematical and General*, 13(3):1101–1112, mar 1980.

[14] T Plefka. Convergence condition of the TAP equation for the infinite-ranged ising spin glass model. *Journal of Physics A: Mathematical and General*, 15(6):1971–1978, jun 1982.

[15] David Sherrington and Scott Kirkpatrick. Solvable model of a spin-glass. *Phys. Rev. Lett.*, 35:1792–1796, Dec 1975.

[16] D. J. Thouless, P. W. Anderson, and R. G. Palmer. Solution of 'solvable model of a spin glass'. *The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics*, 35(3):593–601, 1977.

[17] Pierre Weiss. L'hypothèse du champ moléculaire et la propriété ferromagnétique. *Journal de Physique Théorique et Appliquée*, 6(1):661690, 1907.

[18] C. N. Yang. The spontaneous magnetization of a two-dimensional ising model. *Phys. Rev.*, 85:808–816, Mar 1952.