1 Introduction

Let's consider a dynamic Item Response Theory (IRT) model for N students with, $\theta_{i,t}$, proficiency of the i-th student at time t, following a Gaussian random walk with a fixed variance parameter $\sigma^2_{i,\theta}$. The parameter $d_{j,t}$, difficulty of question j-th question at time t is unknown. For simplicity, let's assume that each student receives the same set of problems at each t. The number of each problems solved by students may vary over time $j \in \{1, \dots, n_t\}$. Let $y_{i,t} = \{y_{i,1,t}, \dots, y_{i,n_t,t}\}$. We have the following Bayesian hierarchical representation of the model:

$$
\begin{aligned}
&y_{i,t,j} \sim Binomial(n_t, \theta_{i,t} - d_{j,t}) \\
&[\theta_{i,0}] \sim N(0, \sigma^2_{i,0}) \\
&[\theta_{i,t} - \theta_{i,t-1}|, \sigma^2_{i,\theta}] \sim N(0, \tau^2 \lambda^2_{i,t}) \\
&\tau \sim C^+(0,1) \\
&\lambda_{i,t} \sim C^+(0,1) \\
&[d] \sim N(0, I)
\end{aligned}
$$

In this set-up, we have the following likelihood:

$$
f(y|\theta, d) \propto \prod_{t=1}^{T} \prod_{i=1}^{N} \prod_{j=1}^{n_t} \left(\frac{exp\{\theta_{i,t} - d_{j,t}\}^{y_{i,j,t}}}{1 + exp\{\theta_{i,t} - d_{j,t}\}} \right).
$$

By Polya Gamma data augmentation, we have

$$
\begin{aligned}
f(y_{i,j,t}|\theta_{i,t}, d_{j,t}) &= \frac{exp\{\theta_{i,t} - d_{j,t}\}^{y_{i,j,t}}}{1 + exp\{\theta_{i,t} - d_{j,t}\}} \\
&\propto exp\{\kappa_{i,j,t}(\theta_{i,t} - d_{j,t})\} \int_0^\infty exp\left\{ \frac{-w_{i,j,t}(\theta_{i,t} - d_{j,t})^2}{2} \right\} dw \\
&\propto exp\{\kappa_{i,j,t}(\theta_{i,t} - d_{j,t})\} \int_0^\infty exp(-w_{i,j,t}(\theta_{i,t} - d_{j,t})^2/2) f(w_{i,j,t}|\theta_{i,t} - d_{j,t}) dw_{i,j,t}
\end{aligned}
$$

Introducing the variable $\omega_{i,j,t} \sim PG(1,0)$, we have the following likelihood:

$$
f(y_{i,t,j}|\theta_{i,t}, d_{j,t}, \omega_{i,j,t}) \propto exp\left\{ -\frac{w_{i,j,t}}{2} \left(\frac{k_{i,j,t}}{w_{i,j,t}} - (\theta_{i,t} - d_{j,t}) \right)^2 \right\},
$$

which is indeed Gaussian Likelihood.

2 $\theta_{i,t}$

First, Let's now assume that $d_{j,t}$ is given,

$$f(y|\theta, d) \propto \prod_{t=1}^{T}\prod_{i=1}^{N}\prod_{j=1}^{n_t}\left(\frac{exp\{\theta_{i,t} - d_{j,t}\}^{y_{i,j,t}}}{1 + exp\{\theta_{i,t} - d_{j,t}\}}\right).$$

$$f(y|\theta, d) \propto \prod_{t=1}^{T}\prod_{i=1}^{N}\frac{exp\{\theta_{i,t} - d_{j,t}\}^{\sum_{j=1}^{n_t} y_{i,j,t}}}{(1 + exp\{\theta_{i,t} - d_{j,t}\})^{n_t}}$$

Let $y^*_{i,t} = \sum_{j=1}^{n_t} y_{i,j,t}$ and $\kappa_{i,t} = y^*_{i,t} - \frac{n_t}{2}$ Then, by the parameter expansion described above, the likelihood

$$[\kappa_{i,t}|w_{i,t}, \theta_{i,t}, d_{j,t}] \sim N(w_{i,t}(\theta_{i,t} - d_{j,t}), w_{i,t}) \qquad [w_{i,t}|n_t] \sim PG(n_t, 0),$$

Gaussian prior on $\theta_{i,t}$ results in Gaussian posterior.

3 $d_{j,t}$

Similarly, define $y^*_{j,t} = \sum_{i=1}^{N} y_{i,j,t}$ and $\kappa_{j,t} = y^*_{j,t} - \frac{N}{2}$. Then the likelihood can be written as,

$$\kappa_{j,t} \sim N(w_{j,t}(\theta_{i,t} - d_{j,t}), w_{j,t}) \qquad [w_{j,t}|N] \sim PG(N, 0)$$

These two are equivalent:

Horseshoe prior:

$$
\begin{aligned}
&[\theta_{i,t} - \theta_{i,t-1}|, \sigma^2_{i,\theta}] \sim N(0, \tau^2 \lambda^2_{i,t}) \\
&\tau \sim C^+(0,1) \\
&\lambda_{i,t} \sim C^+(0,1)
\end{aligned}
$$

Parameter expansion on

$$
\begin{aligned}
&[\theta_{i,t} - \theta_{i,t-1}|, \sigma^2_{i,\theta}] \sim N(0, \tau^2 \lambda^2_{i,t}) && \\
&\tau^2|\tau_x \sim IG(1/2, 1/\tau_x) && \tau_x \sim IG(1/2, 1) \\
&\lambda^2_{i,t}|\lambda^*_{i,t} \sim IG(1/2, 1/\lambda^*_{i,t}) && \lambda^*_{i,t} \sim IG(1/2, 1)
\end{aligned}
$$