1

$$f(x_1,x_2)\in C^\infty(\mathbb{R}^2), pf: e^{t\partial_{x_1}^n\partial_{x_2}^m}f(x_1,x_2)e^{-t\partial_{x_1}^n\partial_{x_2}^m}=f(x_1+nt\partial_{x_1}^{n-1}\partial_{x_2}^m, x_2+mt\partial_{x_1}^n\partial_{x_2}^{m-1})$$

$$\text{sol: } f(x_1,x_2)\in C^\infty(\mathbb{R}^2)\Rightarrow f(x_1,x_2)=\sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}x_1^\alpha x_2^\beta$$

$$\begin{aligned}
e^{t\partial_{x_1}^n\partial_{x_2}^m}f(x_1,x_2)e^{-t\partial_{x_1}^n\partial_{x_2}^m} &= e^{t\partial_{x_1}^n\partial_{x_2}^m}\left(\sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}x_1^\alpha x_2^\beta\right)e^{-t\partial_{x_1}^n\partial_{x_2}^m}\\
&= \sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}\left[e^{t\partial_{x_1}^n\partial_{x_2}^m}x_1^\alpha x_2^\beta e^{-t\partial_{x_1}^n\partial_{x_2}^m}\right]\\
&= \sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}\left[e^{t\partial_{x_1}^n\partial_{x_2}^m}x_1^\alpha e^{-t\partial_{x_1}^n\partial_{x_2}^m}\right]\left[e^{t\partial_{x_1}^n\partial_{x_2}^m}x_2^\beta e^{-t\partial_{x_1}^n\partial_{x_2}^m}\right]\\
&= \sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}\left[e^{t\partial_{x_1}^n\partial_{x_2}^m}x_1 e^{-t\partial_{x_1}^n\partial_{x_2}^m}\right]^\alpha\left[e^{t\partial_{x_1}^n\partial_{x_2}^m}x_2 e^{-t\partial_{x_1}^n\partial_{x_2}^m}\right]^\beta\\
&= \sum_{\alpha,\beta\in\mathbb{N}} f_{\alpha\beta}\left[x_1+nt\partial_{x_1}^{n-1}\partial_{x_2}^m\right]^\alpha\left[x_2+mt\partial_{x_1}^n\partial_{x_2}^{m-1}\right]^\beta\\
&= f(x_1+nt\partial_{x_1}^{n-1}\partial_{x_2}^m, x_2+mt\partial_{x_1}^n\partial_{x_2}^{m-1})
\end{aligned}$$

$$\begin{aligned}
&\exp\left[t(\partial_{x_1},\partial_{x_2})\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\gamma\\ \sigma_1\sigma_2\gamma & \sigma_2^2\end{pmatrix}\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\end{pmatrix}\right]\phi(x_1,x_2)\\
&= \frac{1}{4\pi\sigma_1\sigma_2 t\sqrt{1-\gamma^2}}\iint_{\mathbb{R}^2}\phi(\xi_1,\xi_2)\exp\left\{-\frac{1}{4t}(x_1-\xi_1,x_2-\xi_2)\begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\gamma\\ \sigma_1\sigma_2\gamma & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}x_1-\xi_1\\ x_2-\xi_2\end{pmatrix}\right\}d\xi_1 d\xi_2 \quad |\gamma|<1
\end{aligned}$$

$$\exp\left[t(\partial_{x_1},\partial_{x_2})\mathrm{diag}(\sigma_1^2,\sigma_2^2)\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\end{pmatrix}\right]\phi(x_1,x_2)=\frac{1}{4\pi t\sigma_1\sigma_2}\iint_{\mathbb{R}^2}\phi(\xi_1,\xi_2)\exp\left\{-\frac{1}{4t}(x_1-\xi_1,x_2-\xi_2)\mathrm{diag}(\sigma_1^2,\sigma_2^2)^{-1}\begin{pmatrix}x_1-\xi_1\\ x_2-\xi_2\end{pmatrix}\right\}d\xi_1 d\xi_2$$

$$
\begin{aligned}
e^{a^2t\partial_x^2}e^{-kx^2} &= \frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}} e^{-ks^2}e^{\frac{-(x-s)^2}{4a^2t}}ds \\
&= \frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}} e^{\frac{-1}{4a^2t}\left[(1+4a^2tk)\left(s-\frac{x}{1+4a^2tk}\right)^2+\frac{4a^2tkx^2}{1+4a^2tk}\right]}ds \\
&= e^{\frac{-kx^2}{1+4a^2tk}}\frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}} e^{\frac{-1}{4a^2t}(1+4a^2tk)\left(s-\frac{x}{1+4a^2tk}\right)^2}ds \\
&= e^{\frac{-kx^2}{1+4a^2tk}}\frac{1}{2a\sqrt{\pi t}}\frac{1}{\sqrt{\frac{1+4a^2tk}{4a^2t}}}\sqrt{\pi} \\
&= \frac{1}{\sqrt{1+4a^2tk}}e^{\frac{-kx^2}{1+4a^2tk}}
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(\beta\frac{\partial^2}{\partial x\partial y}\right)\exp(-ax^2-by^2) \\
&\qquad= \exp\left[-a(x+\beta\partial_y)^2\right]\exp\left[-b(y+\beta\partial_x)^2\right]\cdot 1
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left[-a(x+\beta\partial_y)^2\right]\exp[-by^2] \\
&= e^{-ax^2}e^{-2ax\beta\partial_y}e^{-a\beta^2\partial_y^2}\exp[-by^2] \\
&= e^{-ax^2}e^{-2ax\beta\partial_y}\frac{1}{\sqrt{1-4ab\beta^2}}e^{\frac{-by^2}{1-4ab\beta^2}} \\
&= e^{-ax^2}\frac{1}{\sqrt{1-4ab\beta^2}}e^{\frac{-b(y-2a\beta x)^2}{1-4ab\beta^2}} \\
&= \frac{1}{\sqrt{1-4ab\beta^2}}\exp\left[\frac{-1}{1-4ab\beta^2}(by^2-4abxy\beta+4a^2b\beta^2x^2+ax^2-4a^2b\beta^2x^2)\right] \\
&= \frac{1}{\sqrt{1-4ab\beta^2}}\exp\left[\frac{-1}{1-4ab\beta^2}(by^2-4abxy\beta+ax^2)\right]
\end{aligned}
$$

Let $\begin{cases} \beta = 2\sigma_1\sigma_2\gamma t \\ a = \frac{1}{4t\sigma_1^2} \\ b = \frac{1}{4t\sigma_2^2} \end{cases}$ Then $\exp\left(2\sigma_1\sigma_2\gamma t \dfrac{\partial^2}{\partial x \partial y}\right) \exp\left(-\dfrac{1}{4t\sigma_1^2}x^2 - \dfrac{1}{4t\sigma_2^2}y^2\right)$

$$= \frac{1}{\sqrt{1 - 4\frac{1}{4t\sigma_1^2}\frac{1}{4t\sigma_2^2}(2\sigma_1\sigma_2\gamma t)^2}} \exp\left[\frac{-1}{1 - 4\frac{1}{4t\sigma_1^2}\frac{1}{4t\sigma_2^2}(2\sigma_1\sigma_2\gamma t)^2}\left(\frac{1}{4t\sigma_2^2}y^2 - 4\frac{1}{4t\sigma_1^2}\frac{1}{4t\sigma_2^2}xy2\sigma_1\sigma_2\gamma t + \frac{1}{4t\sigma_1^2}x^2\right)\right]$$

$$= \frac{1}{\sqrt{1-\gamma^2}} \exp\left[\frac{-1}{4t(1-\gamma^2)}\left(\frac{y^2}{\sigma_2^2} - \frac{2\gamma xy}{\sigma_1\sigma_2} + \frac{x^2}{\sigma_1^2}\right)\right]$$