財務演算法期末報告

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Objection:

Price an arithmetic average call with the following payoff using the binomial tree model.

$$\text{Payoff}_\tau = \max\left(S_{\text{ave},\tau} - K, 0\right),$$

where $S_{ave,\tau}$ is the arithmetic average of stock prices calculated from the issue date until the current time point τ

Code Files:

1. *Arithmetic_Average_Call_Option_Pricing_by_BT_and_MT.py*
2. *Arithmetic_Average_Call_by_BT_with_Linearly_vs_Logarithmically_Equally_Spaced_Placement_Method.py*
3. *Arithmetic_Average_Call_by_BT_with_Different_FindingWays.py*

Methods:

1. Implement the binomial tree model to price both European and American arithmetic average calls.
   (*Arithmetic_Average_Call_Option_Pricing_by_BT_and_MT.py*)
   The algorithm of Hull and White (1993):
   (1) For any node(i, j), the maximum arithmetic average price is contributed by a price path starting with $i - j$ consecutive up movements followed by j consecutive down movements, and the minimum arithmetic average price can be calculated from a price path starting with j consecutive down movements followed by $i - j$ consecutive up movements.

```
49      def record_possible_Save(self):
50          self.tree[0, 0].possibleSaveList.append(self.S_avet)
51          for ithCol in range(1, self.n + 1):
52              for jthRow in range(ithCol + 1):
53                  currentNode = self.tree[jthRow, ithCol]
54
55                  A_max = (self.S_avet * self.previous_n
56                          + self.St * self.u * ((1 - self.u ** (ithCol - jthRow)) / (1 - self.u))
57                          + self.St * (self.u ** (ithCol - jthRow)) * self.d * ((1 - self.d ** jthRow) / (1 - self.d)))
58                  A_max /= (self.previous_n + ithCol)
59
60                  A_min = (self.S_avet * self.previous_n
61                          + self.St * self.d * ((1 - self.d ** jthRow) / (1 - self.d))
62                          + self.St * (self.d ** jthRow) * self.u * ((1 - self.u ** (ithCol - jthRow)) / (1 - self.u)))
63                  A_min /= (self.previous_n + ithCol)
64
```

   (2) For each node, representative average prices are placed (logarithmically) equally spaced from the maximum to the minimum arithmetic average prices for each node via the following formula.

```
            if((jthRow == 0) or (jthRow == ithCol)):
                currentNode.possibleSaveList.append(A_max)
            else:
                if(self.Savg_spaced_way == 'Linear'):
                    for k in range(0, self.M + 1):
                        A_i_j_k = ((self.M - k) / self.M) * A_max + (k / self.M) * A_min
                        currentNode.possibleSaveList.append(A_i_j_k)
                elif(self.Savg_spaced_way == 'Logarithmically'):
                    for k in range(0, self.M + 1):
                        A_i_j_k = np.exp(((self.M - k) / self.M) * np.log(A_max) + (k / self.M) * np.log(A_min))
                        currentNode.possibleSaveList.append(A_i_j_k)
```

(3) For each terminal node (n, j), decide the payoff for each representative average price $A(n, j, k)$.

```
    def calc_terminal_node_Payoff(self):
        lastColIndex = self.n
        for jthRow in range(lastColIndex + 1):
            # Map each element in possibleSmaxList to possibleCallList
            currentNode = self.tree[jthRow, lastColIndex]
            for i in range(len(currentNode.possibleSaveList)):
                currentNode.possibleCallList.append(max(currentNode.possibleSaveList[i] - self.K, 0))
```

(4) Backward induction

```
    def backward_induction(self, type='European', searchWay='Sequential search'):
        p = self.p

        for ithCol in range(self.n - 1, -1, -1):
            for jthRow in range(ithCol + 1):
                # Process each node
                currentNode = self.tree[jthRow, ithCol]
                upperChildNode = self.tree[jthRow, ithCol + 1]
                lowerChildNode = self.tree[jthRow + 1, ithCol + 1]
                call_upperChildNode = -1
                call_lowerChildNode = -1

                # Match every element
                for k in range(len(currentNode.possibleSaveList)):
                    # The upper child
                    Au = (((ithCol + self.previous_n) * currentNode.possibleSaveList[k] + upperChildNode.sPrice)
                        / (self.previous_n + ithCol + 1))

                    j = 0
                    if(searchWay == 'Sequential search'):
                        while(upperChildNode.possibleSaveList[j] - Au > 10**-8):
                            j += 1
                    elif(searchWay == 'Binary search'):
                        j = self.binary_search(upperChildNode.possibleSaveList, Au, 0, len(upperChildNode.possibleSaveList) - 1)
                    elif(searchWay == 'Linear interpolation search'):
                        j = self.linear_interpolation_search(upperChildNode.possibleSaveList, Au, 0, len(upperChildNode.possibleSaveList) - 1)

                    if(abs(upperChildNode.possibleSaveList[j] - Au) <= 10**-8):
                        call_upperChildNode = upperChildNode.possibleCallList[j]
                    elif(upperChildNode.possibleSaveList[j] < Au):
                        # Inner interploation
                        Wu = (upperChildNode.possibleSaveList[j - 1] - Au) / (upperChildNode.possibleSaveList[j - 1] - upperChildNode.possibleSaveList[j])
                        call_upperChildNode = Wu * (upperChildNode.possibleCallList[j]) + (1 - Wu) * (upperChildNode.possibleCallList[j - 1])
```

```
                    # The lower child
                    Ad = (((ithCol + self.previous_n) * currentNode.possibleSaveList[k] + lowerChildNode.sPrice)
                        / (self.previous_n + ithCol + 1))

                    j = 0
                    if(searchWay == 'Sequential search'):
                        while(lowerChildNode.possibleSaveList[j] - Ad > 10**-8):
                            j += 1
                    elif(searchWay == 'Binary search'):
                        j = self.binary_search(lowerChildNode.possibleSaveList, Ad, 0, len(lowerChildNode.possibleSaveList) - 1)
                    elif(searchWay == 'Linear interpolation search'):
                        j = self.linear_interpolation_search(lowerChildNode.possibleSaveList, Ad, 0, len(lowerChildNode.possibleSaveList) - 1)

                    if(abs(lowerChildNode.possibleSaveList[j] - Ad) <= 10**-8):
                        call_lowerChildNode = lowerChildNode.possibleCallList[j]
                    elif(lowerChildNode.possibleSaveList[j] < Ad):
                        # Inner interploation
                        Wd = (lowerChildNode.possibleSaveList[j - 1] - Ad) / (lowerChildNode.possibleSaveList[j - 1] - lowerChildNode.possibleSaveList[j])
                        call_lowerChildNode = Wd *  (lowerChildNode.possibleCallList[j])+ (1 - Wd) * (lowerChildNode.possibleCallList[j - 1])

                    currentNodeCall = np.exp((-1) * self.r * (self.deltaT / self.n)) * (p * call_upperChildNode + (1 - p) * call_lowerChildNode)

                    if(type == 'American'):
                        # Early exercise or not
                        if(currentNodeCall < (currentNode.possibleSaveList[k] - self.K)):
                            currentNodeCall = (currentNode.possibleSaveList[k] - self.K)
                    currentNode.possibleCallList.append(currentNodeCall)
```

2. Implement the Monte Carlo simulation to price European arithmetic average calls. (*Arithmetic_Average_Call_Option_Pricing_by_BT_and_MT.py*)

(1) Inputs: St, K, r, q, σ, t, $T-t$, M, n, $S_{ave,t}$, number of simulations, number of repetitions.

(2) Outputs: Option values for both methods and 95% confidence interval for Monte Carlo simulation.

```python
def monteCarlo_calc():
    '''
    Calculate call and put pirce by Monte Carlo simulation.
        Parameters:
            paramsList: list
        Outputs:
            c: float -> call price
            p: float -> put price
    '''
    print('Monte Carlo simulation')

    # St, K, r, q, σ, t, T-t, n, S_avet, simCnt, repCnt
    paramsList = [float(i) for i in input('# St, K, r, q, σ, t, T-t, n, S_avet, simCnt, repCnt = ').split()]
    St, K, r, q, sigma, t, deltaT, n, S_avet, simCnt, repCnt = paramsList
    n, simCnt, repCnt = int(n), int(simCnt), int(repCnt)
    previous_n_to_timet = n * (deltaT + t) / deltaT - n + 1
    total_n = previous_n_to_timet + n

    def drawSample(St, K, r, q, sigma, t, deltaT, n, S_avet, simCnt):
        mean = np.log(St) + ((r - q - sigma ** 2 / 2)*(deltaT / n))
        std = sigma*(((deltaT) / n) ** 0.5)
        lnSt_list = np.random.normal(loc=mean, scale=std, size=int(simCnt))

        St_sum_list = np.add(np.multiply(S_avet, previous_n_to_timet), np.exp(lnSt_list))
        for i in range(n-1):
            mean = lnSt_list + ((r - q - sigma ** 2 / 2) * (deltaT / n))
            lnSt_list = np.random.normal(loc=mean, scale=std)
            St_sum_list = np.add(np.exp(lnSt_list), St_sum_list)

        Save_list = St_sum_list / total_n
        payoff_list = np.maximum(Save_list - K, 0) * np.exp(-r * deltaT)
        return np.mean(payoff_list)

    resultList = []
    for i in range(int(repCnt)):
        p = drawSample(St, K, r, q, sigma, t, deltaT, n, S_avet, simCnt)
        resultList.append(p)

    print(f'  Average call price upper bound: {np.round(np.mean(resultList) + 2*np.std(resultList), 4)}')
    print(f'  Average call priceclower bound: {np.round(np.mean(resultList) - 2*np.std(resultList), 4)}', '\n')
```

Results:

1. *Arithmetic_Average_Call_Option_Pricing_by_BT_and_MT.py* 裡面有 binomial tree method & Monte Carlo simulation 的程式碼。
   按執行後，依序輸入 50 50 0.1 0.05 0.8 0 0.25 100 100 50
   會得到$t = 0$時的 European & American average call price，如下圖：

```
# St, K, r, q, σ, t, T-t, M, n, S_avet = 50 50 0.1 0.05 0.8 0 0.25 100 100 50
Binomial Tree method
European Average Call price:  4.7354
American Average Call price:  5.4146
```

2. 接著再輸入 50 50 0.1 0.05 0.8 0.25 0.25 100 100 50，
   會得到$t = 0.25$時的 European & American average call price，如下圖：

```
# St, K, r, q, σ, t, T-t, M, n, S_avet = 50 50 0.1 0.05 0.8 0.25 0.25 100 100 50
Binomial Tree method
European Average Call price:  2.3795
American Average Call price:  2.5079
```

3. 接著再輸入 50 50 0.1 0.05 0.8 0 0.25 100 50 10000 20，
   會得到 Monte Carlo simulation 的上下界，如下圖：

```
Monte Carlo simulation
# St, K, r, q, σ, t, T-t, n, S_avet, simCnt, repCnt = 50 50 0.1 0.05 0.8 0 0.25 100 50 10000 20
  Average call price upper bound: 4.7968
  Average call priceclower bound: 4.5167
```

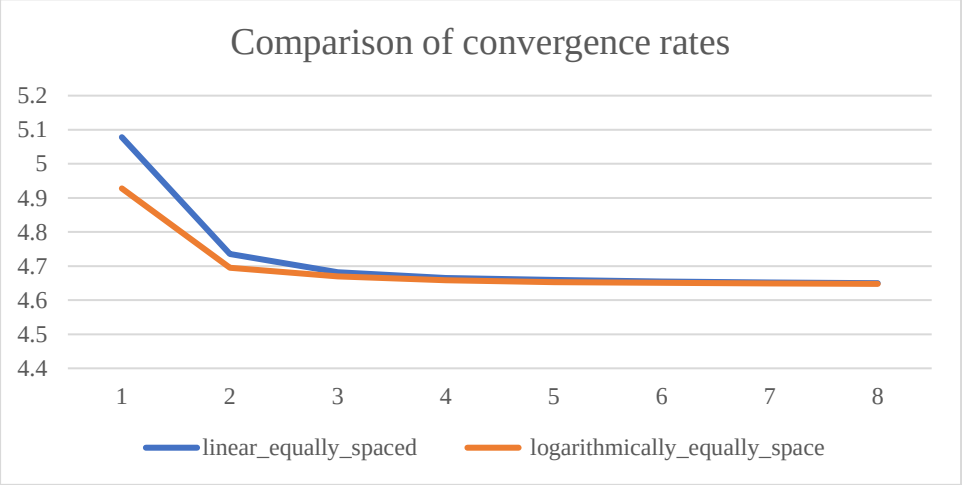
Time Complexity Analysis:

1. Compare the convergence rates of the linearly and logarithmically equally-spaced placement methods (*Arithmetic_Average_Call_by_BT_with_Linearly_vs_Logarithmically_Equally_Spaced_Placement_Method.py*)

   執行後得到數據如下圖：

6	Option Value\ M	50	100	150	200	250	300	350	400
7	Linearly equally-spaced	5.0781	4.7354	4.6821	4.6652	4.659	4.6543	4.6514	4.6498
8	logarithmically equally-spaced	4.9278	4.6949	4.6699	4.6586	4.6531	4.6505	4.6488	4.6478

   將結果繪出如圖：



   得到 Logarithmically equally space method 會收斂比較快的結論。

2. Compare the computational time of the following three methods to locate the positions of A_u and A_d.

   (*Arithmetic_Average_Call_by_BT_with_Different_FindingWays.py*, 需要引入 *Arithmetic_Average_Call_Option_Pricing_by_BT_and_MT.py*，因為 binary search & linear interpolation method 都在這份.py 檔中，如下圖)

```
85
86      def binary_search(self, array, target, low, high):
87          '''
88          Using binary search to find the index that target will larger or equal to array[index]
89          and smaller than array[index - 1].
90          '''
91          if(high >= low):
92              mid = low + (high - low) // 2
93
94              if((abs(array[mid] - target) < 10 ** -8) or ((array[mid] < target) and (array[mid - 1] > target))):
95                  return mid
96
97              elif(array[mid] < target):
98                  return self.binary_search(array, target, low, mid - 1)
99              elif(array[mid] > target):
100                 return self.binary_search(array, target, mid + 1, high)
101
102     def linear_interpolation_search(self, array, target, low, high):
103         '''
104         Using linear interpolation search to find the index that target will larger or equal to array[index]
105         and smaller than array[index - 1].
106         '''
107         if high >= low:
108             if high == low:
109                 return low
110
111             index = int(((array[low] - target) * high + (target - array[high]) * low) / (array[low] - array[high]))
112
113             counts = 0
114             while(~((abs(array[index] - target) < 10 ** -8) or ((array[index] < target) and (array[index - 1] > target)))):
115                 if((abs(array[index] - target) < 10 ** -8) or ((array[index] < target) and (array[index - 1] > target))):
116                     return index
117                 else:
118                     if(counts < 0):
119                         counts -= 1
120                     else:
121                         counts += 1
122                     counts = -counts
123                     index = index + counts
124             return index
```

在 Arithmetic_Average_Call_by_BT_with_Different_FindingWays.py 的執行檔中按下執行後，輸入 50 50 0.1 0.05 0.8 0 0.25 500 100 50，會得到各個方法在 $M = 500$時的計算時間，如圖:

```
# St, K, r, q, σ, t, T-t, M, n, S_avet = 50 50 0.1 0.05 0.8 0 0.25 500 100 50
Binomial Tree method
--- Sequential search: 272.821049451828 seconds ---

Binomial Tree method
--- Binary search: 239.4256603717804 seconds ---

Binomial Tree method
--- Linear interpolation search: 204.05728435516357 seconds ---
```

得到 Linear interpolation search 快過 Binary search 再快過 Sequential search 的結論。

Reference:

http://homepage.ntu.edu.tw/~jryanwang/