Eigendecomposition

- If A is diagonalizable, we can write $D = V^{-1}AV$.

- We can also write $A = VDV^{-1}$.
  which we call eigendecomposition of A.

- A being diagonalizable is equivalent to
  A having eigendecomposition.

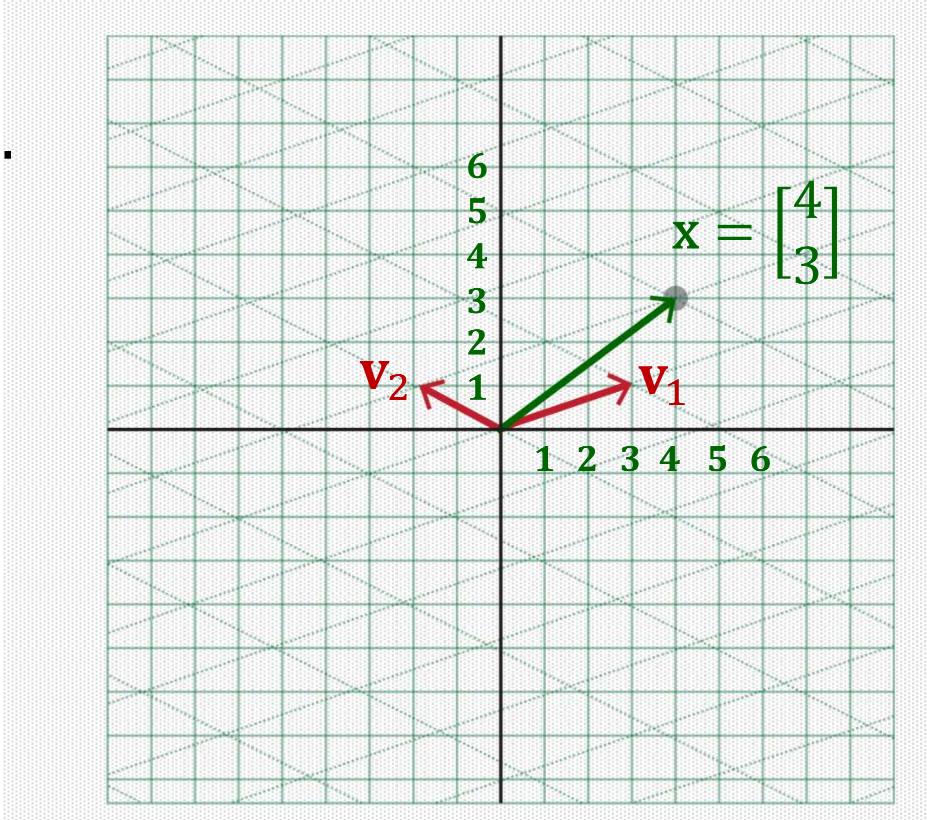
Linear Transformation via Eigendecomposition

- Suppose A is diagonalizable, thus having eigendecompositio
  n
  $$A = VDV^{-1}$$

- Consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V\left(D(V^{-1}\mathbf{x})\right)$.

Change of Basis

- Suppose $A\mathbf{v}_1 = -1\mathbf{v}_1$ and $A\mathbf{v}_2 = 2\mathbf{v}_2$.
- $T(\mathbf{x}) = A\mathbf{x} = VDV^{-1}\mathbf{x} = V\left(D(V^{-1}\mathbf{x})\right)$
- Let $\mathbf{y} = V^{-1}\mathbf{x}$. Then,
$$V\mathbf{y} = \mathbf{x}$$
- $\mathbf{y}$ is a new coordinate of $\mathbf{x}$ with respect to a new basis of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.



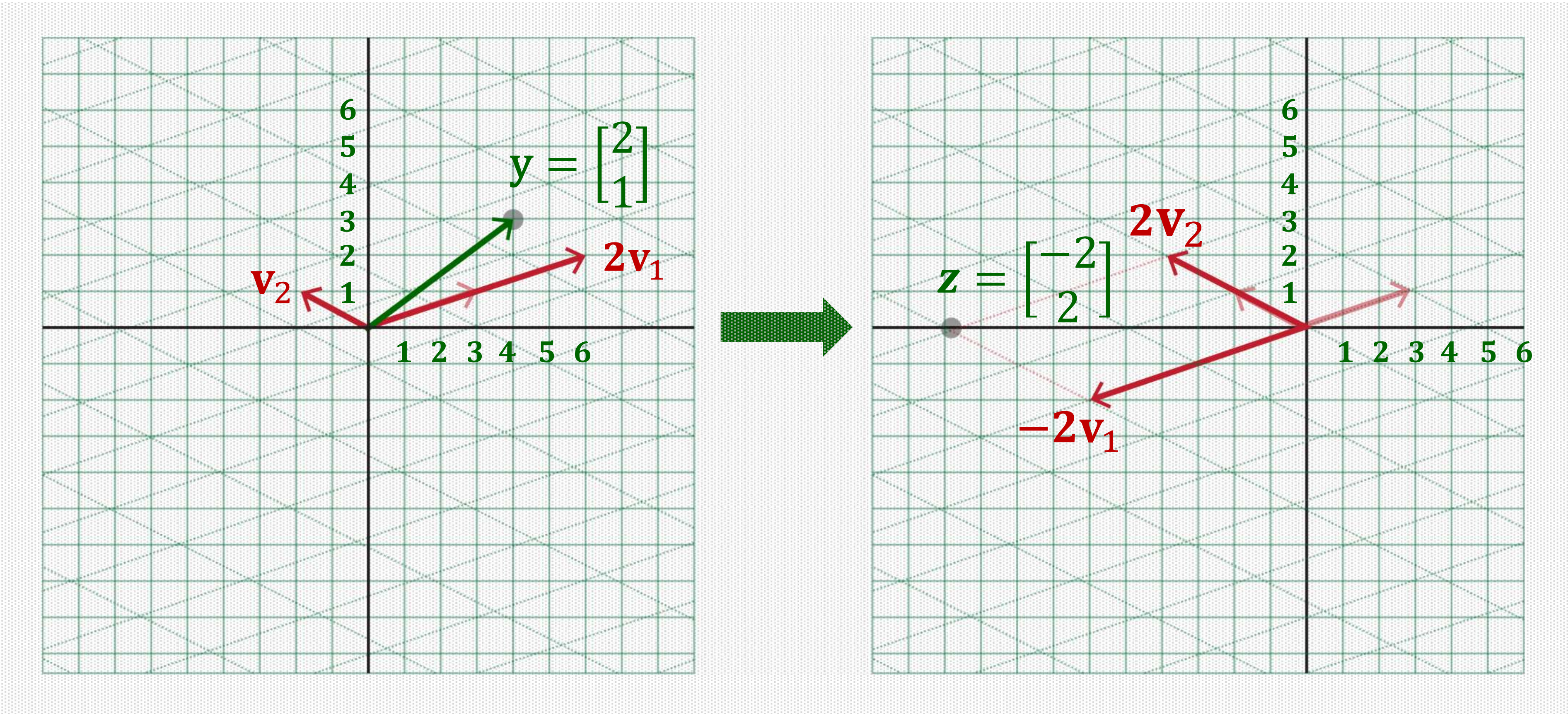
$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = V\mathbf{y} = [\mathbf{v}_1 \quad \mathbf{v}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2\mathbf{v}_1 + 1\mathbf{v}_2 \Longrightarrow \mathbf{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Element-wise Scaling

- $T(\mathbf{x}) = V\big(D(P^{-1}\mathbf{x})\big) = V(D\mathbf{y})$
- Let $z = D\mathbf{y}$. This computation is a simple element-wise scaling of $\mathbf{y}$.
- **Example**: Suppose $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Then

$$z = D\mathbf{y} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-1) \times 2 \\ 2 \times 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Dimension-wise Scaling



Back to Original Basis

- $T(\mathbf{x}) = V(D\mathbf{y}) = Vz$
- z is still a coordinate based on the new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- Vz converts z to another coordinates based on the original standard basis.
- That is, Vz is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ using the coefficient vector z.
- That is,

$$Vz = [\mathbf{v}_1 \quad \mathbf{v}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{v}_1 z_1 + \mathbf{v}_2 z_2$$

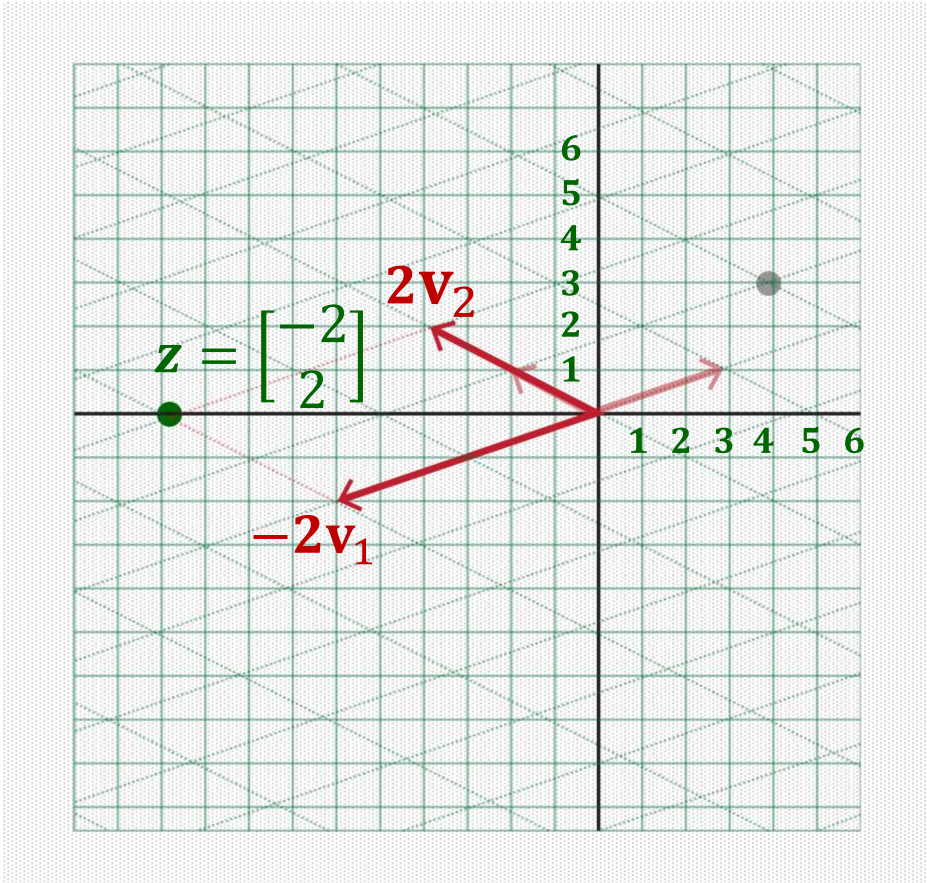
Back to Original Basis

- $T(\mathbf{x}) = V{\color{red}z} = [\mathbf{v}_1 \quad \mathbf{v}_2]\begin{bmatrix}-2\\2\end{bmatrix}$

$$= -2\mathbf{v}_1 + 2\mathbf{v}_2$$

$$= -2\begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}-2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-10\\0\end{bmatrix}$$



Overview of Transformation using Eigendecomposition

$$
\begin{tikzcd}
x \arrow[r, "A"] \arrow[d, "V^{-1}"'] & T(x) = Ax = V\left(D(V^{-1}x)\right) \\
y \arrow[r, "D"] & z \arrow[u, "V"']
\end{tikzcd}
$$

Linear Transformation via A^k

- Now, consider recursive transformation $A \times A \times \cdots \times A\mathbf{x} = A^k \mathbf{x}$.
- If A is diagonalizable, A has eigendecomposition

$$A = VDV^{-1}$$

- $A^k = (VDV^{-1})(VDV^{-1}) \cdots (VDV^{-1}) = VD^kV^{-1}$
- D^k is simply computed as

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$

Linear Transformation via A^k

- $A^k\mathbf{x} = VD^kV^{-1}\mathbf{x}$ can be computed in the similar manner to the previous example.
- It is much faster to compute $V\left(D^k(V^{-1}\mathbf{x})\right)$ than to compute $A^k\mathbf{x}$.