The Singular Value Decomposition (1 of 11)

- **Theorem 10: The Singular Value Decomposition**
  Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ as in (3) for which the diagonal entries in D are the first r singular values of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = U \Sigma V^T$$

The Singular Value Decomposition (2 of 11)

- Any factorization $A = U\Sigma V^T$, with *U* and *V* orthogonal, Σ as in (3), and positive diagonal entries in *D*, is called a **singular value decomposition** (or **SVD**) of *A*.
- The columns of *U* in such a decomposition are called **left singular vectors** of *A*, and the columns of *V* are called **right singular vectors** of *A*.
- **Proof** Let λ_i and $\mathbf{v}_i$ be as in Theorem 9, so that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for Col *A*.

The Singular Value Decomposition (3 of 11)

- Normalize each $A\mathbf{v}_{\text{i}}$ to obtain an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_{\text{r}}\}$, where

$$\mathbf{u}_i = \frac{1}{\|A\mathbf{v}_i\|} A\mathbf{v}_i = \frac{1}{\sigma_1} A\mathbf{v}_i$$

- And

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i \qquad (1 \le i \le r) \qquad (4)$$

- Now extend $\{\mathbf{u}_1, \ldots, \mathbf{u}_{\text{r}}\}$ to an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of $\mathbb{R}^m$, and let

$$U = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \ldots \quad \mathbf{u}_m] \text{ and } V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \ldots \quad \mathbf{v}_n]$$

- By construction, *U* and *V* are orthogonal matrices.

The Singular Value Decomposition (4 of 11)

- Also, from (4),

$$AV = \begin{bmatrix} A\mathbf{v}_1 & \dots & A\mathbf{v}_r & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \dots & \sigma_r\mathbf{u}_r & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$$

- Let D be the diagonal matrix with diagonal entries $\sigma_1, \dots, \sigma_r$, and let Σ be as in (3) above. Then

$$U\Sigma = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m\,] \left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= [\,\sigma_1\mathbf{u}_1 \quad \cdots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\,]$$

$$= AV$$

- Since V is an orthogonal matrix, $U\Sigma V^T = AVV^T = A$.