Diagonalization

- We want to change a given square matrix $A \in \mathbb{R}^{n \times n}$ into a diagonal matrix via the following form:
$$D = V^{-1}AV$$
where $P \in \mathbb{R}^{n \times n}$ is an invertible matrix and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix. This is called a diagonalization of A.

- It is not always possible to diagonalize A. For A to be diagonalizable, an invertible V should exist such that $V^{-1}AV$ becomes a diagonal matrix.

Finding V and D

- How can we find an invertible P and the resulting diagonal matrix $D = V^{-1}AV$?
- $D = V^{-1}AV \Longrightarrow VD = AV$
- Let us represent the following:
- $V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ where $\mathbf{v}_i$'s are column vectors of V
- $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

Finding V and D

- $AV = A[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] = [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n]$
- $VD = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

  $= [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n]$
- $VD = AV \Leftrightarrow [A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n] = [\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n]$

Finding V and D

- Equating columns, we obtain

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \dots,\ A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

- Thus, $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ should be eigenvectors and λ_1, λ_2, …, λ_n should be eigenvalues.
- Then, for $VD = AV \Longrightarrow D = V^{-1}AV$ to be true, V should invertible.
- In this case, the resulting diagonal matrix D has eigenvalues as diagonal entries.

Diagonalizable Matrix

- For V to be invertible,
  V should be a square matrix in $\mathbb{R}^{n \times n}$, and
  Vshould have n linearly independent columns.

- Recall columns of V are eigenvectors.
  Hence, A should have n linearly independent eigenvectors.

- It is not always the case, but if it is, A is diagonalizable.