Recall: Linear System

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\Rightarrow \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$

- Or, a vector equation is written as

$$\Rightarrow \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

Uniqueness of Solution for $A\mathbf{x} = \mathbf{b}$

- The solution exists only when $\mathbf{b} \in \text{Span}\ \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- If the solution exists for $A\mathbf{x} = \mathbf{b}$, when is it unique?
- It is unique when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly independent.
- Infinitely many solutions exist when $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are linearly dependent.

Linear Independence

(Practical) Definition:

- Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, check if $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$ for $j = 1, \ldots, p$, e.g.,

$$\mathbf{v}_j \in \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\} \text{ for some } j = 1, \ldots, p?$$

- If at least one such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly dependent.
- If no such $\mathbf{v}_j$ is found, then $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

(Formal) Definition:

- Consider $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots + x_p\mathbf{v}_p = \mathbf{0}$.
- Obviously, one solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

  which we call a trivial solution.

- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly independent if this is the only solution.
- $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent if this system also has other nontrivial solutions, e.g., at least one x_i being nonzero.

Two Definitions are Equivalent

- If $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are linearly dependent, consider a nontrivial solution.
- In the solution, let's denote j as the last index such that $x_j \neq 0$.
- Then, one can write $x_j \mathbf{v}_j = -x_1 \mathbf{v}_1 - \cdots - x_{j-1} \mathbf{v}_{j-1}$,
  and safely divide it by x_j, resulting in

$$\mathbf{v}_j = -\frac{x_1}{x_j}\mathbf{v}_1 - \cdots - \frac{x_{j-1}}{x_j}\mathbf{v}_{j-1} \in \text{Span}\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\right\}$$

  which means $\mathbf{v}_j$ can be represented as a linear combination of the previous vectors.