Transformations between Vectors

- $T: \mathbf{x} \in \mathbb{R}^n \mapsto \mathbf{y} \in \mathbb{R}^m$: Mapping n-dim vector to m-dim vector
- Example:

$$T: \mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{y} \in \mathbb{R}^2 \quad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \quad \mapsto \quad \mathbf{y} = T(\mathbf{x}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \in \mathbb{R}^2$$

Matrix of Linear Transformation

- Example: Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \text{ With no additional information,}$$

find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{x}) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix of Linear Transformation

- In general, let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is always written as a matrix-vector multiplication, i.e.,

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

- In fact, the j-th column of $A \in \mathbb{R}^{m \times n}$ is equal to the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j-th column of the identity matrix in $\mathbb{R}^{n \times n}$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)]$$

- Here, the matrix A is called the **standard matrix** of the linear transformation T

Matrix of Linear Transformation

- Example: Find the standard matrix A of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}, T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}4\\3\end{bmatrix}, \text{ and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}.$$

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow T(\mathbf{x}) = T\left(x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = x_1 T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + x_2 T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + x_3 T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}4\\3\end{bmatrix} + x_2\begin{bmatrix}5\\6\end{bmatrix} = \begin{bmatrix}1 & 4 & 5\\2 & 3 & 6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = A\mathbf{x}$$