Linear System: Set of Equations

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables - say, $x_1, \cdots, x_n$.

Linear System Example

- Suppose we collected persons' weight, height, and life-span (e.g., how long s/he lived)

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

- We want to set up the following linear system:

$$60x_1+5.5x_2+1\cdot x_3=66$$
$$65x_1+5.0x_2+0\cdot x_3=74$$
$$55x_1+6.0x_2+1\cdot x_3=78$$

- Once we solve for x_1, x_2, and x_3, given a new person with his/her weight, height, and is_smoking, we can predict his/her life-span.

Linear System Example

- The essential information of a linear system can be written compactly using a **matrix**.
- In the following set of equations,

$$60x_1 + 5.5x_2 + 1 \cdot x_3 = 66$$
$$65x_1 + 5.0x_2 + 0 \cdot x_3 = 74$$
$$55x_1 + 6.0x_2 + 1 \cdot x_3 = 78$$

- Let's collect all the coefficients on the left and form a matrix

$$A = \begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}$$

- Also, let's form two vectors: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$

From Multiple Equations to Single Matrix Equation

- Multiple equations can be converted into a **single** matrix equations

$$
\begin{aligned}
60x_1 + 5.5x_2 + 1 \cdot x_3 &= 66 \\
65x_1 + 5.0x_2 + 0 \cdot x_3 &= 74 \\
55x_1 + 6.0x_2 + 1 \cdot x_3 &= 78
\end{aligned}
\quad \Rightarrow \quad
\underbrace{\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}}_{\mathbf{b}}
\quad \Leftarrow \quad
\begin{aligned}
\mathbf{a}_1^T \mathbf{x} &= 66 \\
\mathbf{a}_2^T \mathbf{x} &= 74 \\
\mathbf{a}_3^T \mathbf{x} &= 78
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b}$$

- How can we solve for $\mathbf{x}$?

Identity Matrix

- **Definition**: An identity matrix is a square matrix whose diagonal entries are all 1's, and all the other entries are zeros. Often, we denote it as $I_n \in \mathbb{R}^{n \times n}$.
  - e.g., $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- An identity matrix I_n preserves any vector $\mathbf{x} \in \mathbb{R}^n$ after multiplying $\mathbf{x}$ by I_n :

$$\forall \mathbf{x} \in \mathbb{R}^n, \qquad I_n\mathbf{x} = \mathbf{x}$$