Vector Norm

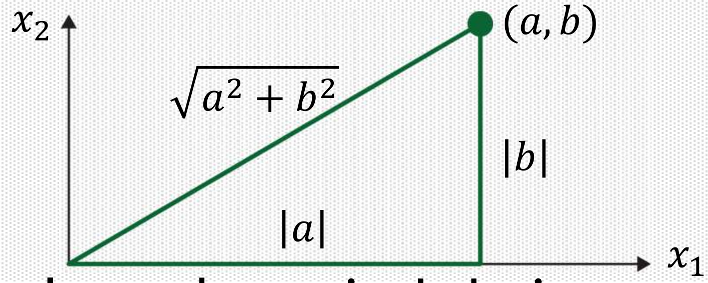
- For $\mathbf{v} \in \mathbb{R}^n$, with entries $v_1, \ldots, v_n$, the square root of $\mathbf{v} \cdot \mathbf{v}$ is defined because $\mathbf{v} \cdot \mathbf{v}$ is nonnegative.

- **Definition**: The length (or norm) of $\mathbf{v}$ is the non-negative scalar $\|\mathbf{v}\|$ defined as the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \text{ and } \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$$

Geometric Meaning of Vector Norm

- Suppose $\mathbf{v} \in \mathbb{R}^2$, say, $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.
- $\|\mathbf{v}\|$ is the length of the line segment from the origin to $\mathbf{v}$.
- This follows from Pythagorean Theorem applied to a triangle such as the one shown in the following figure:



Interpretation of ||v|| as length

- For any scalar c, the length $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$ That is,

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

Unit Vector

- A vector whose length is 1 is called a unit vector.

- Normalizing a vector: Given a nonzero vector $\mathbf{v}$, if we divide it by its length, we obtain a unit vector $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$.

- $\mathbf{u}$ is in the same direction as $\mathbf{v}$, but its length is 1.

Distance between Vectors in $\mathbb{R}^n$

- **Definition:** For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between $\mathbf{u}$ and $\mathbf{v}$**, written as dist $(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$ That is,

$$\text{dist}\,(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- **Example**: Compute the distance between the vector

$$\mathbf{u} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

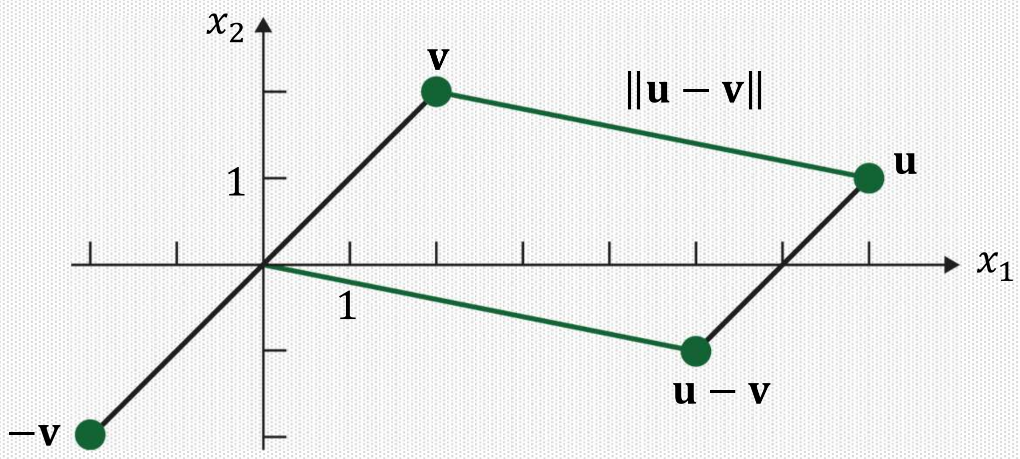
- **Solution**: Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

Distance between Vectors in $\mathbb{R}^n$

- The distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u} - \mathbf{v}$ to 0.



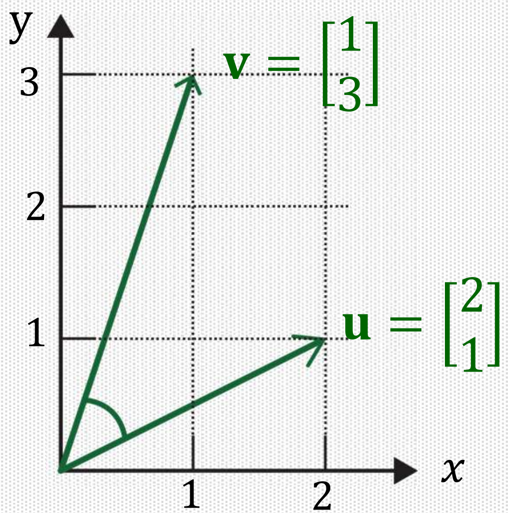
The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$

Inner Product and Angle Between Vectors

- Inner product between $\mathbf{u}$ and $\mathbf{v}$ can be rewritten using their norms and angle:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$

- **Example**:



$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5$$

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5} \quad \|\mathbf{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\mathbf{u} \cdot \mathbf{v} = 5 = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta = \sqrt{5} \cdot \sqrt{10} \cos\theta$$

$$\Rightarrow \cos\theta = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = 45^\circ$$

Orthogonal Vectors

- **Definition:** $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v}=0$

  That is,
  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = 0.$
  ⇒ $\cos\theta = 0$ for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$
  ⇒ $\theta = 90°$ ($\mathbf{u} \perp \mathbf{v}$).
  ⇒ $\mathbf{u}$ and $\mathbf{v}$ are perpendicular each other.