Span and Subspace

- **Definition**: A **subspace** H is defined as a subset of $\mathbb{R}^n$ closed under linear combination:
  - For any two vectors, $\mathbf{u}_1$,$\mathbf{u}_2 \in H$, and any two scalars c and d, $c\mathbf{u}_1 + d\mathbf{u}_2 \in H$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is always a subspace. Why?
  - $\mathbf{u}_1 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$, $\mathbf{u}_2 = b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p$
  - $$c\mathbf{u}_1 + d\mathbf{u}_2 = c\left(a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p\right) + d\left(b_1\mathbf{v}_1 + \cdots + b_p\mathbf{v}_p\right)$$
    $$= (ca_1 + db_1)\mathbf{v}_1 + \cdots + \left(ca_p + db_p\right)\mathbf{v}_p$$
- In fact, a subspace is always represented as Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

Basis of a Subspace

- **Definition**: A **basis** of a subspace H is a set of vectors that satisfies both of the following:
  - Fully spans the given subspace H
  - Linearly independent (i.e., no redundancy)

- In the previous example, where $H = \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H, but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

Non-Uniqueness of Basis

- Consider a subspace H (green plane).
- Is a basis unique?
- That is, is there any other set of linearly independent vectors that span the same subspace H?