# Null Space

- **Definition**: The **null space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all solutions of a homogeneous linear system, $A\mathbf{x} = \mathbf{0}$. We denote the null space of $A$ as Nul $A$.

- For $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$, $\mathbf{x}$ should satisfy the following:

$$\mathbf{a}_1^T\mathbf{x} = 0, \mathbf{a}_2^T\mathbf{x} = 0, \dots, \mathbf{a}_m^T\mathbf{x} = 0$$

- That is, $\mathbf{x}$ should be orthogonal to every row vector in $A$.

# Null Space is a Subspace

- **Theorem**: The null space of a matrix $A \in \mathbb{R}^{m \times n}$ is a subspace of $\mathbb{R}^n$. In other words, the set of all the solutions of a system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

- **Note**: An eigenspace thus have a set of basis vectors with a particular dimension.

# Orthogonal Complement

- If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace $W$ of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** $W$.
- The set of all vectors $\mathbf{z}$ that are orthogonal to $W$ is called the **orthogonal complement** of $W$ and is denoted by $W^{\perp}$ (and read as "$W$ perpendicular" or simply "$W$ perp").
- A vector $\mathbf{x} \in \mathbb{R}^n$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans $W$.
- $W^{\perp}$ is a subspace of $\mathbb{R}^n$.
- Nul $A = (\mathrm{Row}\, A)^{\perp}$.
- Likewise, Nul $A^T = (\mathrm{Col}\, A)^{\perp}$.

# Fundamental Subspaces Given by $A$

- Nul $A = (\text{Row } A)^{\perp}$.
- Nul $A^T = (\text{Col } A)^{\perp}$.

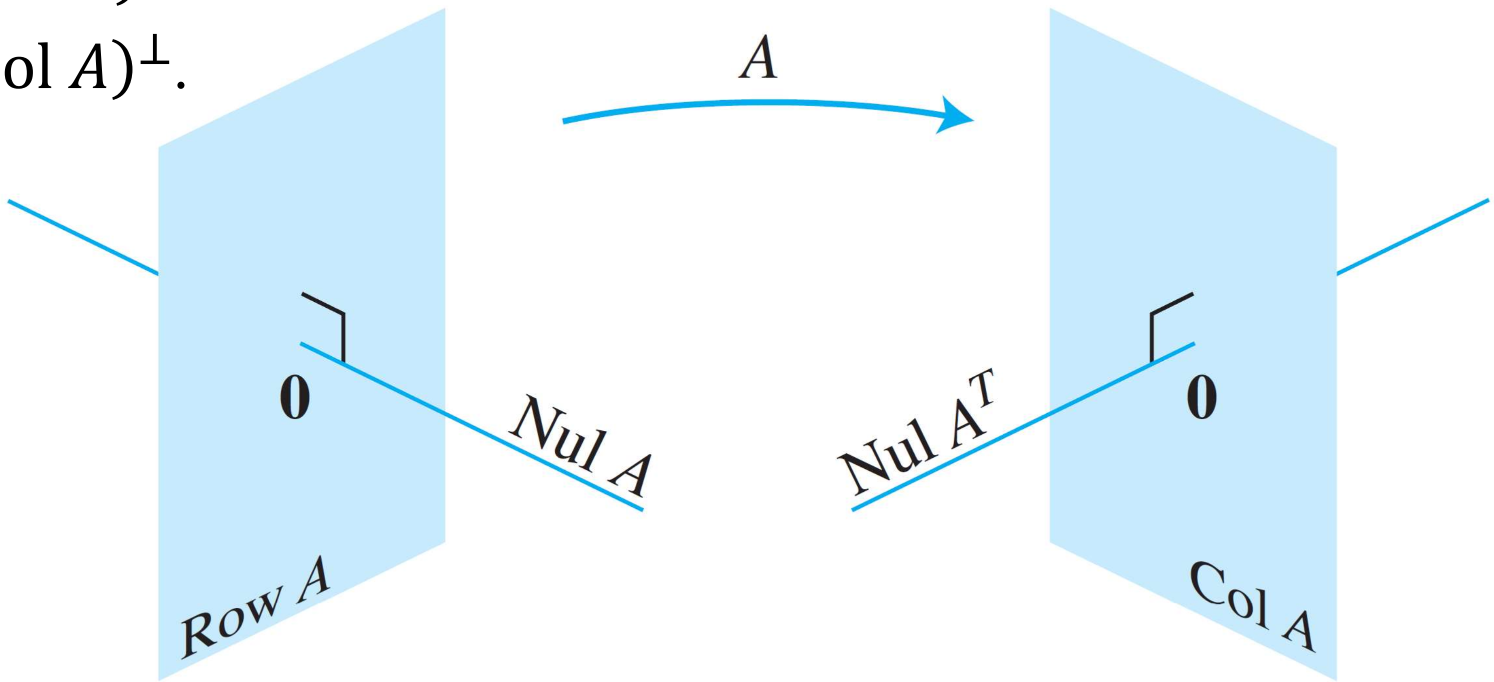
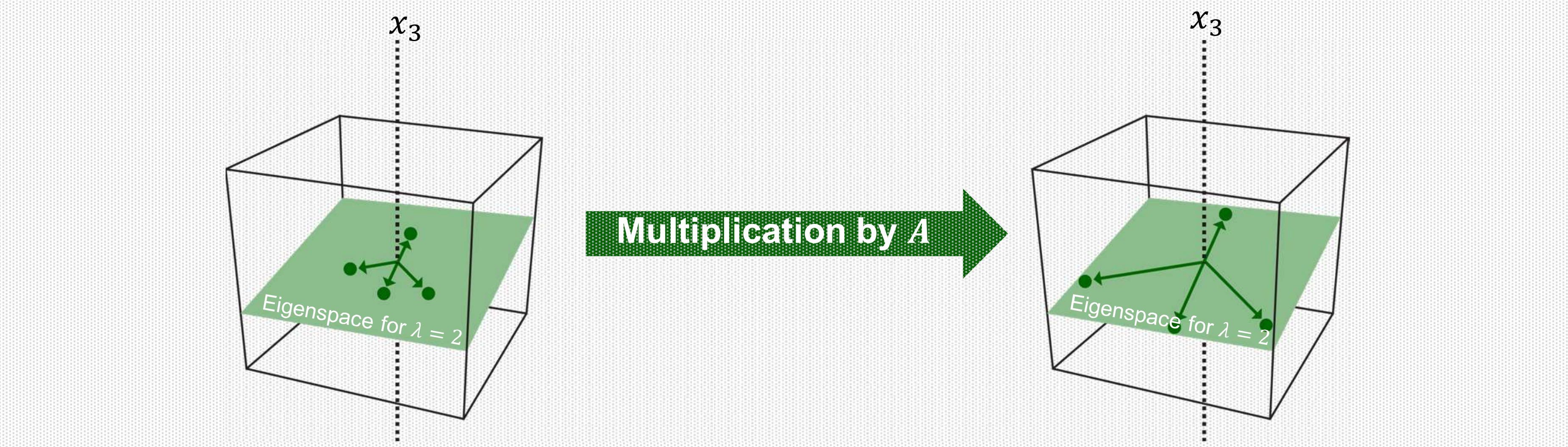



**FIGURE 8** The fundamental subspaces determined by an $m \times n$ matrix $A$.

# Eigenspace

- Note that the dimension of the eigenspace (corresponding to a particular $\lambda$) can be larger than one. In this case, any vector in the eigenspace satisfies

$$T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$$




$A$ acts as a dilation on the eigenspace