Matrix Multiplications as Linear Combinations of Vectors

- **Recall**: we defined matrix-matrix multiplications as the inner product between the row on the left and the column on the right:
  - e.g., $\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 1 \\ 9 & -3 \end{bmatrix}$
- Inspired by the vector equation, we can view $A\mathbf{x}$ as a linear combination of columns of the left matrix:
  - $\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A\mathbf{x} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3$

Matrix Multiplications as Column Combinations

- Linear combinations of columns
  - Left matrix: bases, right matrix: coefficients

One column on the right

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}1+\begin{bmatrix}1\\0\\-1\end{bmatrix}2+\begin{bmatrix}0\\1\\1\end{bmatrix}3$$

Multi-columns on the right

$$\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\2 & 0\\3 & 1\end{bmatrix}=\begin{bmatrix}x_1 & y_1\\x_2 & y_2\\x_3 & y_3\end{bmatrix}=[\mathbf{x}\ \mathbf{y}]$$

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}1+\begin{bmatrix}1\\0\\-1\end{bmatrix}2+\begin{bmatrix}0\\1\\1\end{bmatrix}3$$

$$\mathbf{y}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}(-1)+\begin{bmatrix}1\\0\\-1\end{bmatrix}0+\begin{bmatrix}0\\1\\1\end{bmatrix}1$$

Matrix Multiplications as Row Combinations

- Linear combinations of rows of the right matrix
  - Right matrix: bases, left matrix: coefficients

One row on the left

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{matrix} 1 \times [1 & 1 & 0] \\ +2 \times [1 & 0 & 1] \\ +3 \times [1 & -1 & 1] \end{matrix}$$

Multiple rows on the left

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{\mathrm{T}} \\ \mathbf{y}^{\mathrm{T}} \end{bmatrix}$$

$$\mathbf{x}^{\mathrm{T}} = [x_1 \quad x_2 \quad x_3] = 1[1 \quad 1 \quad 0] + 2[1 \quad 0 \quad 1] + \quad 3[1 \quad -1 \quad 1]$$

$$\mathbf{y}^{\mathrm{T}} = [y_1 \quad y_2 \quad y_3] = 1[1 \quad 1 \quad 0] + 0[1 \quad 0 \quad 1] + (-1)[1 \quad -1 \quad 1]$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- (Rank-1) outer product

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$$

- Sum of (Rank-1) outer products

$$\begin{bmatrix}1 & 1\\1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}+\begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}4 & 5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}+\begin{bmatrix}4 & 5 & 6\\-4 & -5 & -6\\4 & 5 & 6\end{bmatrix}$$

Matrix Multiplications as Sum of (Rank-1) Outer Products

- Sum of (Rank-1) outer products is widely used in machine learning
  - Covariance matrix in multivariate Gaussian
  - Gram matrix in style transfer