# The Singular Values of an *m* by *n* Matrix (1 of 3)

- **Theorem 9** Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, arranged so that the corresponding eigenvalues of $A^T A$ satisfy $\lambda_1 \geq \cdots \geq \lambda_n$, and suppose *A* has *r* nonzero singular values. Then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an orthogonal basis for Col *A*, and rank $A = r$.

- **Proof** Because $\mathbf{v}_i$ and $\lambda_j \mathbf{v}_j$ are orthogonal for $i \neq j$,

$$\left(A\mathbf{v}_i\right)^T \left(A\mathbf{v}_j\right) = v_i^T A^T A\mathbf{v}_j = v_i^T \left(\lambda_j \mathbf{v}_j\right) = 0$$

- Thus $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is an orthogonal set.

# The Singular Values of an *m* by *n* Matrix (2 of 3)

- Since the lengths of the vectors $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ are the singular values of *A*, and since there are *r* nonzero singular values, $A\mathbf{v}_i \neq \mathbf{0}$ if and only if $1 \leq i \leq r$.
- So $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ are linearly independent vectors, and they are in Col *A*.
- Finally, for any **y** in Col *A*—say, $\mathbf{y} = A\mathbf{x}$—we can write $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, and

$$\mathbf{y} = A\mathbf{x} = c_1A\mathbf{v}_1 + \cdots + c_rA\mathbf{v}_r + c_{r+1}A\mathbf{v}_{r+1} + \cdots + c_nA\mathbf{v}_n$$

# The Singular Values of an *m* by *n* Matrix (3 of 3)

$$= c_1 A\mathbf{v}_1 + \cdots + c_r A\mathbf{v}_r + 0 + \cdots + 0$$

- Thus **y** is in Span $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$, which shows that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an (orthogonal) basis for Col *A*.
- Hence rank *A* = dim Col *A* = *r*.