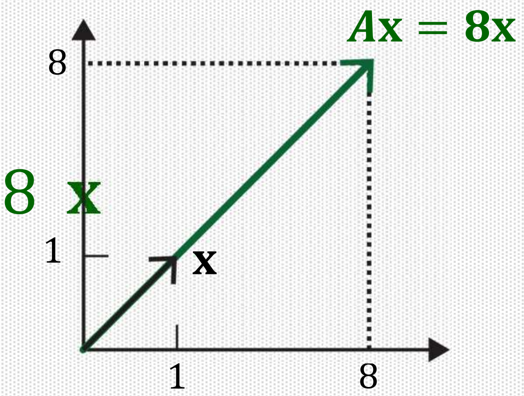
Eigenvectors and Eigenvalues

- **Definition**: An **eigenvector** of a **square** matrix $A \in \mathbb{R}^{n \times n}$ is a **nonzero** vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ
  In this case, λ is called an **eigenvalue** of A, and
  such an $\mathbf{x}$ is called an ***eigenvector corresponding to λ***.

Transformation Perspective

- Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- If $\mathbf{x}$ is an eigenvector, then $T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$, which means the output vector has **the same direction** as $\mathbf{x}$, but the length is scaled by a factor of λ.
- **Example:** For $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since

$$T(\mathbf{x}) = A\mathbf{x} = \underbrace{\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \begin{bmatrix} 8 \\ 8 \end{bmatrix} = \underbrace{8}_{8} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$



Computational Advantage

- Which computation is faster between $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $8\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Eigenvectors and Eigenvalues

- The equation $A\mathbf{x} = \lambda\mathbf{x}$ can be re-written as

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- λ is an eigenvalue of an $n \times n$ matrix A if and only if this equation has a **nontrivial** solution (since $\mathbf{x}$ should be a nonzero vector).

Eigenvectors and Eigenvalues

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- The set of *all* solutions of the above equation is the **null space** of the matrix $(A - \lambda I)$, which we call the **eigenspace** of A **corresponding to λ**.
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ, satisfying the above equation.

Example: Eigenvalues and Eigenvectors

- **Example**: Show that 8 is an eigenvalue of a matrix $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ and find the corresponding eigenvectors.

- **Solution**: The scalar 8 is an eigenvalue of A if and only if
  the equation $(A - 8I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution:

$$(A - 8I)\mathbf{x} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any nonzero scalar c,
  which is Span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Example: Eigenvalues and Eigenvectors

- In the previous example, -3 is also an eigenvalue:
$$(A + 3I)\mathbf{x} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\mathbf{x} = \mathbf{0}$$
- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}$ for any nonzero scalar c,
  which is Span $\left\{\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}\right\}$.