Linear Combinations

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \cdots, c_p$,

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

  is called a **linear combination** of $\mathbf{v}_1, \cdots, \mathbf{v}_p$ with **weights or coefficients** $c_1, \cdots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.

From **Matrix** Equation to **Vector** Equation

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$A \quad \mathbf{x} = \mathbf{b}$$

- A matrix equation can be converted into a vector equation:

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

Existence of Solution for $A\mathbf{x} = \mathbf{b}$

- Consider its vector equation:

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \quad \mathbf{a}_2 x_2 \quad + \quad \mathbf{a}_3 x_3 = \quad \mathbf{b}$$

- When does the solution exist for $A\mathbf{x} = \mathbf{b}$?

Span

- **Definition**: Given a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_p \in \mathbb{R}^n$, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is defined as the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.
- That is, Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \cdots + c_p\mathbf{v}_p$$

  with arbitrary scalars $c_1, \cdots, c_p$.
- Span $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is also called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.