Soliton resonances in four dimensional Wess-Zumino-Witten model

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We present two kinds of resonance soliton solutions on the ultrahyperbolic space $\mathbb{U}$ for the $G = U(2)$ Yang equation, which is equivalent to the anti-self-dual Yang-Mills (ASDYM) equation. We reveal and illustrate the solitonic behaviors in the four-dimensional Wess-Zumino-Witten (WZW_4) model through the sigma model action densities. The Yang equation is the equation of motion of the WZW_4 model. In the case of $\mathbb{U}$, the WZW_4 model describes a string field theory action of open $N = 2$ string theories. Hence, our solutions on $\mathbb{U}$ suggest the existence of the corresponding classical objects in the $N = 2$ string theories. Our solutions include multiple-pole solutions and V-shape soliton solutions. The V-shape solitons suggest annihilation and creation processes of two solitons and would be building blocks to classify the ASDYM solitons, like the role of Y-shape solitons in classification of the Kadomtsev-Petviashvili (line) solitons. We also clarify the relationship between the Cauchy matrix approach and the binary Darboux transformation in terms of quasideterminants. Our formalism can start with a simpler input data for the soliton solutions and hence might give a suitable framework for the classification of the ASDYM solitons.

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I. INTRODUCTION

Anti-self-dual Yang-Mills (ASDYM) equations have been attracting great interest for many years and established central positions in mathematics and physics. In gauge theories, instantons, global solutions to the ASDYM equation, have played crucial roles in revealing nonperturbative aspects of them [1] and have brought a new perspective to four-dimensional geometry (e.g., [2]). In integrable systems, the ASDYM equation is a master equation in the sense that it can be reduced to various lower-dimensional integrable equations, known for the Ward's conjecture [3]. Reduced equations include the Calogero-Bogoyavlenski-Shiff (CBS) equation and the Zakharov system in $(2+1)$ dimensions, the Korteweg–de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, and Toda equations in $(1+1)$ dimensions, and Painlevé equations in $(0+1)$ dimensions (see [4] and references therein). One recent example of the reduction is the Fokas-Lenells equation [5], which is related to the Kaup-Newell spectral problem and the derivative nonlinear Schrödinger equation family [6,7]. The ASDYM equation itself is integrable in various senses (e.g., [8–16]). Furthermore, there is another attractive feature of the ASDYM equation in string theories. The ASDYM equation is equivalent to the Yang equation [17,18], which is the equation of motion of the four-dimensional Wess-Zumino-Witten (WZW_4) model [19–22]. In the case of the split signature $(+,+,-,-)$, the WZW_4 model describes the space-time action of the open $N = 2$ string theory [23–25] and, therefore, solutions of the ASDYM can be realized and applicable for this string theory as classical physical objects.

Two of the authors have recently constructed soliton solutions of the ASDYM equations (based on the result of Darboux transformations [26,27]) and calculated the action densities of them in order to clarify the solitonic behaviors [28–33]. It is proven that the soliton solutions of them are described by Wronskian-type quasideterminants (called quasi-Wronskians) [34] and the ASDYM solitons behave quite similar to the KP (line) solitons in the sense of real-valued action densities: the one-soliton solutions are codimension-one solitons, whose action densities are localized on three-dimensional hyperplanes in four dimensions. The n-soliton solutions can be interpreted as a

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"nonlinear superposition" of n one-soliton solutions with phase shifts. These ASDYM solitons are considered as nonresonance solutions, called "line solitons" in this paper. Real-valued Kadomtsev-Petviashvili (KP) solitons are classified in terms of positive Grassmannians [35,36] and applied to shallow water waves [37]. We can hence expect that similar classification of ASDYM solitons would be possible and could be applied to reveal nonperturbative aspects of the open $N = 2$ string theory by analyzing the moduli spaces of them.

On the other hand, another two of the authors have successfully applied the Cauchy matrix approach (CMA) to the ASDYM equations to construct Grammian-type solutions [38,39]. In the CMA, the input data are simpler than those in the Darboux transformation and would be more suitable for the classification of the ASDYM solitons. For the (scalar) KP equation, it is proven that Wronskian and Grammian soliton solutions are equivalent [40], and it is worth clarifying whether this is true of the ASDYM solitons or not. The CMA closely relates to the binary Darboux transformation [26], direct linearization methods [41], and the bi-differential calculus [42]. It is also worthwhile to rewrite their formalisms in terms of quasideterminants so as to clarify the relationship between the CMA and the binary Darboux transformation because quasideterminants usually simplify the calculations and make it easier to get to the essence.

In this paper, we construct resonance solutions in the context of the WZW_4 model. In integrable systems, resonance solutions can usually be realized as limits of multiline solitons, such as large phase-shift limits. In fact, phase shifts come from "intermediate states," as discussed in Sec. IV B. Hence, we can expect that in the large phase-shift limit of the two-soliton, the intermediate state can be observed as a one-soliton. This expectation is actually correct for complex valued solutions, however by putting the reality condition on the solutions, the amplitudes of the intermediate states become zero. This fact implies that in the large phase-shift limit, our two-soliton solutions decompose into two V-shape solitons, not the composite of two Y-shape solitons. Our result is consistent with the fact that there actually exist V-shape solitons in the CBS equation [43] and the Zakharov system [44], both of which can be obtained from the ASDYM equation by dimensional reduction, while the KP equation cannot. Existence of V-shape solitons of the ASDYM equation suggests a quite different aspect from the KP solitons, and classification of the ASDYM solitons might be achieved based on non-Grassmannian geometry. On the other hand, by tuning orientations of the two-line solitons, we get another type of resonance solution, the double-pole solution [45], which describes the interaction between two solitary waves of equal amplitude [46]. By observing the limit processes, we find suitable input data for the multiple-pole solution, where the spectral parameter matrix is in a Jordan normal form.

In order to make interpretations of the solutions in the WZW_4 model, we restrict our discussion to the case of the split signature and of unitary group valued solutions (simply called unitary solutions). The WZW_4 model consists of two terms: the nonlinear sigma model (NLσM) term and the Wess-Zumino term. For the ASDYM solitons, the main contribution of solitonic behavior can be captured by the NLσM term because the Wess-Zumino term identically vanishes in the asymptotic region. Furthermore, in a dimensionally reduced system, the Hamiltonian of the Wess-Zumino term always vanishes everywhere [32]. Therefore, calculating the NLσM term is sufficient for the illustration of solitonic behavior. On the other hand, the unitary solutions lead to real-valued nonsingular NLσM terms and, hence, the action densities can be plotted specifically by *Mathematica*. We also clarify the relationship between the CMA and the binary Darboux transformation in terms of quasideterminants. Our formalism can start with simpler input data for the soliton solutions and, hence, might give a suitable framework for the classification of the ASDYM solitons.

This paper is organized as follows. In Sec. II, we give a brief review of the WZW_4 model. In Sec. III, we reformulate the CMA in terms of quasideterminants. We extend the dispersion relation in the CMA so that the relation to the binary Darboux transformation is clarified in Sec. III D. Finally, we get unitary quasi-Grammian solutions. In Sec. IV, we present exact one- and two-soliton solutions by the formulation of quasi-Grammian. The new input data are much simpler than those used in the Darboux transformations, which are formulated by quasi-Wronskian. We calculate the NLσM action density for one- and two-solitons, respectively, and find that our results coincide with the ones calculated from quasi-Wronskian solutions [32]. In Sec. IV C, we discuss some particular limits of the solutions, which lead to resonance processes. In Sec. V, we construct the multiple-pole solution from the modified input data, where the spectral parameter matrix is in a Jordan normal form. In Sec. VI, the distributions of action densities for various resonance solutions are demonstrated as 2D slice figures by using *Mathematica*. These figures show that the double-pole solutions actually have nontrivial behavior of resonances or bound states, and V-shape solitons emerge in the large phase-shift limits of the two-soliton. Section VII is devoted to the conclusion and discussion. The Appendix includes some miscellaneous formulas, detailed calculations, and necessary proofs.

II. FOUR-DIMENSIONAL WESS-ZUMINO-WITTEN MODEL

In this section, we present a brief review of the WZW_4 model. For more details, refer to [32]. For convenience of discussion, firstly, we introduce a four-dimensional space $\mathbb{M}_4$ with complex coordinates $(z, \tilde{z}, w, \tilde{w})$ and the flat metric [4], defined by

$$ds^2 := g_{mn}dz^m dz^n := 2(dzd\tilde{z} - dwd\tilde{w}), \tag{2.1}$$

where $m, n \in \{1,2,3,4\}$ and $(z^1, z^2, z^3, z^4) := (z, \tilde{z}, w, \tilde{w})$.

For our purpose in this paper, we focus our discussion only on the four-dimensional flat real space with the split signature, which is called the ultrahyperbolic space, denoted by $\mathbb{U}$. With the local coordinates (x^1, x^2, x^3, x^4) of $\mathbb{U}$, the metric can be chosen as follows:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu := (dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2, \tag{2.2}$$

where $\mu, \nu \in \{1,2,3,4\}$. This real space $\mathbb{U}$ can be realized from $\mathbb{M}_4$ by imposing the following reality condition on $(z, \tilde{z}, w, \tilde{w})$:

$$(\mathbb{U}) \begin{pmatrix} z & w \\ \tilde{w} & \tilde{z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -(x^2 - x^4) & x^1 - x^3 \end{pmatrix}, \tag{2.3}$$

where $z, \tilde{z}, w, \tilde{w} \in \mathbb{R}$.[1] If we replace the reality condition (2.3) with $\tilde{z} = \bar{z}$ and $\tilde{w} = -\bar{w}$, we can discuss the Euclidean case [32].

Now, we consider the action of the WZW_4 model defined on $\mathbb{U}$, denoted by S_{WZW_4}. The WZW_4 action consists of two parts, that is, the NLσM term S_σ and the Wess-Zumino term S_{WZ}, which are explicitly given by

$$\begin{aligned} S_{\mathrm{WZW}_4} &:= S_\sigma + S_{\mathrm{WZ}} \\ &:= \frac{i}{4\pi}\int_{\mathbb{U}} \omega \wedge \mathrm{Tr}[(\partial J)J^{-1} \wedge (\tilde{\partial} J)J^{-1}] \\ &\quad - \frac{i}{12\pi}\int_{\mathbb{U}} A \wedge \mathrm{Tr}[(dJ)J^{-1}]^3, \end{aligned} \tag{2.4}$$

where the dynamical variable J is a smooth map from $\mathbb{U}$ to $G = GL(N, \mathbb{C})$, and ω is a two-form on $\mathbb{U}$ given by

$$\omega = \frac{i}{2}(dz \wedge d\tilde{z} - dw \wedge d\tilde{w}). \tag{2.5}$$

The differential one-form A is chosen as $A = (i/4)(zd\tilde{z} - \tilde{z}dz - wd\tilde{w} + \tilde{w}dw)$ so that $\omega = dA$. The cubic term $[(dJ)J^{-1}]^3$ is an abbreviation of the wedge product of three one-forms: $[(dJ)J^{-1}]^3 = (dJ)J^{-1} \wedge (dJ)J^{-1} \wedge (dJ)J^{-1}$. The exterior derivatives are defined as

$$d := \partial + \tilde{\partial}, \qquad \partial := dw\partial_w + dz\partial_z, \qquad \tilde{\partial} := d\tilde{w}\partial_{\tilde{w}} + d\tilde{z}\partial_{\tilde{z}}. \tag{2.6}$$

The equation of motion of the WZW_4 model is exactly the Yang equation,

$$\tilde{\partial}[\omega \wedge (\partial J)J^{-1}] = 0 \Leftrightarrow \partial_{\tilde{z}}[(\partial_z J)J^{-1}] - \partial_{\tilde{w}}[(\partial_w J)J^{-1}] = 0, \tag{2.7}$$

which gives an equivalent expression for the anti-self-dual Yang-Mills equation.

The WZW_4 action can be expressed in terms of the real coordinates. For the NLσM action, we have

$$\begin{aligned} S_\sigma &= \frac{-1}{16\pi}\int_{\mathbb{U}} \mathrm{Tr}[(\partial_m J)J^{-1}(\partial^m J)J^{-1}]dz \wedge d\tilde{z} \wedge dw \wedge d\tilde{w} \\ &= \frac{-1}{16\pi}\int_{\mathbb{U}} \mathrm{Tr}[(\partial_\mu J)J^{-1}(\partial^\mu J)J^{-1}]dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4, \end{aligned} \tag{2.8}$$

where $\partial^m := g^{mn}\partial_n$ and $\partial^\mu := \eta^{\mu\nu}\partial_\nu$, in which g^{mn} and $\eta^{\mu\nu}$ are the inverse matrices of g_{mn} and $\eta_{\mu\nu}$, respectively. For the component representation of the Wess-Zumino action, one can refer to [32].

As mentioned in Sec. I, we will only calculate the NLσM action density because the main contribution can be captured by this term well. The NLσM action density is real-valued when J is unitary because $(\partial_\mu J)J^{-1}$ is anti-Hermitian with pure imaginary eigenvalues in this case.

III. QUASI-GRAMMIAN SOLUTIONS IN THE WZW_4 MODEL

In this section, we briefly review the CMA for the Yang equation [38,39], and then extend the dispersion relations to a general version. For the ultrahyperbolic space $\mathbb{U}$, we show that such dispersion relations can be regarded as initial linear systems in binary Darboux transformations. Finally, we reduce two linear systems to a single one to construct unitary solutions.

A. A brief review of the Cauchy matrix approach ($G = GL(2, \mathbb{C})$)

The CMA for the Yang equation was developed by two of the authors and collaborators in recent research [38,39]. This method starts by introducing the Sylvester equation

$$KM(r,s) - M(r,s)L = rs^T, \tag{3.1}$$

where K and L are constant matrices and r, s satisfies the following dispersion relations:

$$\partial_{x_j} r = K^j r\sigma_3, \qquad \partial_{x_j} s = -(L^T)^j s\sigma_3, \qquad j \in \mathbb{Z}. \tag{3.2}$$

Solutions $M(r,s)$ of the Sylvester equation are dressed Cauchy matrices.[2] The sizes of these involved matrices are

[1] We note that there is another real slice $\tilde{z} = \bar{z}$ and $\tilde{w} = \bar{w}$ to realize the split signature, however, we do not consider this case in this paper because the unitarity condition of J leads to trivial action densities [32].

[2] The Cauchy matrix is a matrix in the form of $(\frac{1}{k_i - l_j})_{1\le i,j\le N}$, while the dressed Cauchy matrix is in the form of $(\frac{\rho(k_i)\sigma(l_j)}{k_i - l_j})_{1\le i,j\le N}$.

given so that the above matrix multiplications are well defined. Here, we consider the case of $K, L \in \mathbb{C}_{N\times N}$, $r, s \in \mathbb{C}_{N\times 2}[x], M(r,s) \in \mathbb{C}_{N\times N}[x]$, $\sigma_3 \coloneqq \mathrm{diag}(1,-1)$, and $x \coloneqq (\cdots, x_{-1}, x_0, x_1, \cdots)$. The number N relates to the number of solitons, as is discussed in Sec. IV.

Defining $u, v \in \mathbb{C}_{2\times 2}[x]$ as

$$u = -s^T M(r,s)^{-1} r, \qquad v = I - s^T M(r,s)^{-1} K^{-1} r, \quad (3.3)$$

we can prove a differential recurrence relation (see Appendix A in [39]),

$$(\partial_{x_{j+1}} v) v^{-1} = \partial_{x_j} u, \qquad j \in \mathbb{Z}, \quad (3.4)$$

and, therefore, for any given $n, m \in \mathbb{Z}$, through the compatibility $\partial_{x_m}(\partial_{x_n} u) = \partial_{x_n}(\partial_{x_m} u)$, v satisfies

$$\partial_{x_m}[(\partial_{x_{n+1}} v) v^{-1}] - \partial_{x_n}[(\partial_{x_{m+1}} v) v^{-1}] = 0, \quad (3.5)$$

which is the Yang equation if we assign $(x_{n+1}, x_m, x_{m+1}, x_n)$ to $(z, \tilde{z}, w, \tilde{w})$.

In fact, u and v possess the structure of quasi-Grammians (Grammian-like quasideterminants [47]). To explain this, firstly, we give several equivalent definitions of quasideterminants for the following $(N+k)\times(N+k)$ matrix with partition:

$$\begin{vmatrix} A_{N\times N} & B_{N\times k} \\ C_{k\times N} & \boxed{d_{k\times k}} \end{vmatrix} \coloneqq d - CA^{-1}B \in \mathbb{C}_{k\times k}, \quad (3.6)$$

for any positive integers k, N. In particular, $k = 2$ implies

$$\begin{vmatrix} A & B_1 & B_2 \\ C_1 & \boxed{d_{11}} & \boxed{d_{12}} \\ C_2 & \boxed{d_{21}} & \boxed{d_{22}} \end{vmatrix} = \begin{pmatrix} \begin{vmatrix} A & B_1 \\ C_1 & \boxed{d_{11}} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & \boxed{d_{12}} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_2 & \boxed{d_{21}} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & \boxed{d_{22}} \end{vmatrix} \end{pmatrix}$$
$$= \frac{1}{|A|} \begin{pmatrix} \begin{vmatrix} A & B_1 \\ C_1 & d_{11} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_1 & d_{12} \end{vmatrix} \\ \begin{vmatrix} A & B_1 \\ C_2 & d_{21} \end{vmatrix} & \begin{vmatrix} A & B_2 \\ C_2 & d_{22} \end{vmatrix} \end{pmatrix}, \quad (3.7)$$

where

$$d = d_{2\times 2} \coloneqq \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \qquad B = B_{N\times 2} \coloneqq (B_1, B_2),$$
$$C = C_{2\times N} \coloneqq \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (3.8)$$

The last equality in (3.7) is due to the commutative limit of quasideterminants (see proposition 2.2 and equality (2.12) in [30]).

Now, (3.3) can be rewritten in the form of quasideterminants as

$$u = \begin{vmatrix} M(r,s) & r \\ s^T & \boxed{0} \end{vmatrix}, \qquad v = \begin{vmatrix} M(r,s) & K^{-1} r \\ s^T & \boxed{I} \end{vmatrix}. \quad (3.9)$$

Note that the dressed Cauchy matrix $M(r,s)$ has a Grammian-like structure [47] and can be represented as a difference of two Gram matrices, as follows. Let us represent each column of r, s explicitly as

$$r \coloneqq (r_1, r_2), \qquad s \coloneqq (s_1, s_2), \qquad r_1, r_2, s_1, s_2 \in \mathbb{C}_{N\times 1}[x]. \quad (3.10)$$

From [38,39], the derivative of $M(r,s)$ can be rewritten as

$$\begin{aligned} \partial_{x_n} M(r,s) &= \sum_{\ell=0}^{n-1} K^{n-\ell-1} r \sigma_3 s^T L^\ell \\ &= \sum_{\ell=0}^{n-1} K^{n-\ell-1} (r_1, r_2) \mathrm{diag}(1,-1) (s_1, s_2)^T L^\ell \\ &= \sum_{\ell=0}^{n-1} K^{n-\ell-1} r_1 s_1^T L^\ell - \sum_{\ell=0}^{n-1} K^{n-\ell-1} r_2 s_2^T L^\ell \\ &= \partial_{x_n} M_1(r,s) - \partial_{x_n} M_2(r,s), \end{aligned} \quad (3.11)$$

where

$$\partial_{x_n} M_1(r,s) \coloneqq \sum_{\ell=0}^{n-1} K^{n-\ell-1} r_1 s_1^T L^\ell,$$
$$\partial_{x_n} M_2(r,s) \coloneqq \sum_{\ell=0}^{n-1} K^{n-\ell-1} r_2 s_2^T L^\ell. \quad (3.12)$$

The two relations in (3.12) can be understood as the following. Suppose that we have two Sylvester equations,

$$KM_1 - M_1 L = r_1 s_1^T, \qquad KM_2 - M_2 L = r_2 s_2^T, \quad (3.13)$$

while $r = (r_1, r_2)$ and $s = (s_1, s_2)$ satisfy the dispersion relation (3.2). Then, following [38,39], we can derive (3.12) from (3.13).

Therefore, v defined in (3.9) is a quasi-Grammian solution of the Yang equation [Eq. (3.5)] in the sense that $M_1(r,s)$ and $M_2(r,s)$ are the Grammian-like matrices and so is $M(r,s)$.

B. An extension of the Cauchy matrix approach [$G = GL(2,\mathbb{C})$]

In this section, we generalize the CMA in Sec. III A by modification of the dispersion relation (3.2) in an extended form. Let us consider the Sylvester equation for $\tilde{M}(\tilde{r}, \tilde{s}) \in \mathbb{C}_{N\times N}[x]$,

$$K\tilde{M}(\tilde{r},\tilde{s})-\tilde{M}(\tilde{r},\tilde{s})L=\tilde{r}\tilde{s}^T, \tag{3.14}$$

where $\tilde{r},\tilde{s}\in\mathbb{C}_{N\times 2}[x]$ satisfy the following differential recurrence relations:

$$\partial_{x_{j+1}}\tilde{r}=K\partial_{x_j}\tilde{r}, \qquad \partial_{x_{j+1}}\tilde{s}=L^T\partial_{x_j}\tilde{s}, \qquad j\in\mathbb{Z}. \tag{3.15}$$

Solutions of the Sylvester equation [Eq. (3.14)] are also represented by dressed Cauchy matrices. Note that any solutions $(\tilde{r},\tilde{s})$ of the previous dispersion relation (3.2) satisfy the differential one (3.15). Once we set $\partial_{x_0}\tilde{r}=\tilde{r}\sigma_3$ and $\partial_{x_0}\tilde{s}=-\tilde{s}\sigma_3$, we can get back to (3.2) from (3.15).

Under this setting, our goal is to show that (3.9) can be generalized to

$$\tilde{u}=\begin{vmatrix}\tilde{M}(\tilde{r},\tilde{s}) & \tilde{r}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}, \qquad \tilde{v}=\begin{vmatrix}\tilde{M}(\tilde{r},\tilde{s}) & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{I}\end{vmatrix} \tag{3.16}$$

so that $\tilde{v}$ still satisfies the Yang equation [Eq. (3.5)]. Here, we use the tilde notation to distinguish the generalized solution from the original one.

Firstly, we find that the dressed Cauchy matrix $\tilde{M}(\tilde{r},\tilde{s})$ still has the Grammian-like structure because its derivative can be expressed by the following sum of scalar products:

Lemma 3.1. For K and L that do not share eigenvalues, under the conditions (3.14) and (3.15), the Cauchy matrix $\tilde{M}(\tilde{r},\tilde{s})$ satisfies

$$\begin{aligned}\partial_{x_{j+1}}\tilde{M}(\tilde{r},\tilde{s})&=(\partial_{x_j}\tilde{r})\tilde{s}^T+[\partial_{x_j}\tilde{M}(\tilde{r},\tilde{s})]L\\ &=K[\partial_{x_j}\tilde{M}(\tilde{r},\tilde{s})]-\tilde{r}(\partial_{x_j}\tilde{s}^T), \qquad j\in\mathbb{Z}.\end{aligned} \tag{3.17}$$

Proof. The proof is given in Appendix A. ∎

Note that $\tilde{M}(\tilde{r},\tilde{s})$ satisfies the requirement of the following Lemma 3.2 and, therefore, the derivative of (3.16) can be rewritten in the form of (3.19).

Lemma 3.2. Derivative formula of quasi-Grammian [47]. Let A be a $N\times N$ matrix, B be a $N\times 1$ column matrix, C be a $1\times N$ row matrix, and d be a 1×1 matrix. If the derivative of matrix A can be expressed as

$$\partial A=\sum_{\ell=1}^{k}E_\ell F_\ell, \tag{3.18}$$

where E_ℓ and F_ℓ stand for certain square matrices, then we have the following derivative formula of the quasideterminant:

$$\begin{aligned}\partial\begin{vmatrix}A & B\\ C & \boxed{d}\end{vmatrix}&=\partial d+\begin{vmatrix}A & B\\ \partial C & \boxed{0}\end{vmatrix}+\begin{vmatrix}A & \partial B\\ C & \boxed{0}\end{vmatrix}\\ &\quad+\sum_{\ell=1}^{k}\begin{vmatrix}A & E_\ell\\ C & \boxed{0}\end{vmatrix}\begin{vmatrix}A & B\\ F_\ell & \boxed{0}\end{vmatrix}.\end{aligned} \tag{3.19}$$

Now, we can apply (3.19) to verify that $\tilde{v}$ in (3.16) satisfies the Yang equation [Eq. (3.5)]. Our arguments are summarized as the following Theorem 3.3.

Theorem 3.3. Under the conditions (3.14) and (3.15), $\tilde{u}$ and $\tilde{v}$ in (3.16) satisfy the differential recurrence relation

$$(\partial_{x_{j+1}}\tilde{v})\tilde{v}^{-1}=\partial_{x_j}\tilde{u}, \qquad j\in\mathbb{Z}, \tag{3.20}$$

and, therefore, such $\tilde{v}$ is a solution of the Yang equation

$$\partial_{x_m}[(\partial_{x_{n+1}}\tilde{v})\tilde{v}^{-1}]-\partial_{x_n}[(\partial_{x_{m+1}}\tilde{v})\tilde{v}^{-1}]=0 \tag{3.21}$$

for any given $n,m\in\mathbb{Z}$.

Proof. For simplicity, we use $\tilde{M}$ to denote $\tilde{M}(\tilde{r},\tilde{s})$ in the following proof. From Lemma 3.1, Lemma 3.2, and Eq. (3.15), we have

$$\begin{aligned}\partial_{x_{j+1}}\tilde{v}&=\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \partial_{x_{j+1}}\tilde{s}^T & \boxed{0}\end{vmatrix}+\begin{vmatrix}\tilde{M} & K^{-1}(\partial_{x_{j+1}}\tilde{r})\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\\ &\quad+\begin{vmatrix}\tilde{M} & \partial_{x_j}\tilde{r}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\\ &\quad+\begin{vmatrix}\tilde{M} & \partial_{x_j}\tilde{M}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ L & \boxed{0}\end{vmatrix}\\ &=\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \partial_{x_j}\tilde{s}^TL & \boxed{0}\end{vmatrix}+\begin{vmatrix}\tilde{M} & \partial_{x_j}\tilde{r}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{I}\end{vmatrix}\\ &\quad+\begin{vmatrix}\tilde{M} & \partial_{x_j}\tilde{M}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ L & \boxed{0}\end{vmatrix}.\end{aligned} \tag{3.22}$$

From the definition of quasideterminants and the Sylvester equation [Eq. (3.1)], we have

$$\begin{aligned}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \partial_{x_j}\tilde{s}^TL & \boxed{0}\end{vmatrix}&=\begin{vmatrix}\tilde{M} & (\tilde{M}L)\tilde{M}^{-1}K^{-1}\tilde{r}\\ \partial_{x_j}\tilde{s}^T & \boxed{0}\end{vmatrix}\\ &=\begin{vmatrix}\tilde{M} & (K\tilde{M}-\tilde{r}\tilde{s}^T)\tilde{M}^{-1}K^{-1}\tilde{r}\\ \partial_{x_j}\tilde{s}^T & \boxed{0}\end{vmatrix}\\ &=\begin{vmatrix}\tilde{M} & \tilde{r}\\ \partial_{x_j}\tilde{s}^T & \boxed{0}\end{vmatrix}+\begin{vmatrix}\tilde{M} & \tilde{r}\\ \partial_{x_j}\tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{0}\end{vmatrix}\\ &=\begin{vmatrix}\tilde{M} & \tilde{r}\\ \partial_{x_j}\tilde{s}^T & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{I}\end{vmatrix}.\end{aligned} \tag{3.23}$$

Through similar calculation, we have

$$\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ L & \boxed{0}\end{vmatrix}=\begin{vmatrix}\tilde{M} & \tilde{r}\\ I & \boxed{0}\end{vmatrix}\begin{vmatrix}\tilde{M} & K^{-1}\tilde{r}\\ \tilde{s}^T & \boxed{I}\end{vmatrix}. \tag{3.24}$$

Now, we can conclude that

$$\partial_{x_{j+1}}\tilde{v} = \left\{ \begin{vmatrix} \tilde{M} & \tilde{r} \\ \partial_{x_j}\tilde{s}^T & \boxed{0} \end{vmatrix} + \begin{vmatrix} \tilde{M} & \partial_{x_j}\tilde{r} \\ \tilde{s}^T & \boxed{0} \end{vmatrix} + \begin{vmatrix} \tilde{M} & \partial_{x_j}\tilde{M} \\ \tilde{s}^T & \boxed{0} \end{vmatrix} \begin{vmatrix} \tilde{M} & \tilde{r} \\ I & \boxed{0} \end{vmatrix} \right\} \begin{vmatrix} \tilde{M} & K^{-1}\tilde{r} \\ \tilde{s}^T & \boxed{I} \end{vmatrix}$$
$$= \left\{ \partial_{x_j} \begin{vmatrix} \tilde{M} & \tilde{r} \\ \tilde{s}^T & \boxed{0} \end{vmatrix} \right\} \begin{vmatrix} \tilde{M} & K^{-1}\tilde{r} \\ \tilde{s}^T & \boxed{I} \end{vmatrix} = (\partial_{x_j}\tilde{u})\tilde{v}. \quad (3.25)$$

Here, we have used Lemma 3.2 again with respect to $\tilde{u}$. ■

C. Unitary solutions in the WZW$_4$ model

For the physical purpose of clarifying the WZW$_4$ model, in this section we aim to find the unitary solutions of (3.21) on the ultrahyperbolic space $\mathbb{U}$, whose metric can be realized by (2.3). Let us assign the coordinates $(x_{n+1}, x_m, x_{m+1}, x_n)$ in (3.21) to $(z, \tilde{z}, w, \tilde{w}) \in \mathbb{R}^4$. The differential recurrences in (3.15) now reduce to

$$\begin{cases} \partial_z \tilde{r} = K\partial_{\tilde{w}}\tilde{r} \\ \partial_w \tilde{r} = K\partial_{\tilde{z}}\tilde{r} \end{cases}, \begin{cases} \partial_z \tilde{s} = L^T\partial_{\tilde{w}}\tilde{s} \\ \partial_w \tilde{s} = L^T\partial_{\tilde{z}}\tilde{s} \end{cases}, \qquad z, \tilde{z}, w, \tilde{w} \in \mathbb{R}, \quad (3.26)$$

which can, in fact, be regarded as a special class of the linear systems in the context of the binary Darboux transformations [26],

$$\begin{cases} [\partial_z - (\partial_z J)J^{-1}]\theta - (\partial_{\tilde{w}}\theta)\Lambda = 0, \\ [\partial_w - (\partial_w J)J^{-1}]\theta - (\partial_{\tilde{z}}\theta)\Lambda = 0, \end{cases}$$
$$\begin{cases} [\partial_z - (\partial_z J^{-\dagger})J^{\dagger}]\rho - (\partial_{\tilde{w}}\rho)\Xi = 0, \\ [\partial_w - (\partial_w J^{-\dagger})J^{\dagger}]\rho - (\partial_{\tilde{z}}\rho)\Xi = 0, \end{cases} \quad \text{where } z, \tilde{z}, w, \tilde{w} \in \mathbb{R}, \qquad J^{-\dagger} := (J^{-1})^{\dagger}, \quad (3.27)$$

which compose the Lax pair of the Yang equation. This coincides with (3.26) through the following identification:

$$J = I, \qquad (\theta, \Lambda) = (\tilde{s}^T, L), \qquad (\rho, \Xi) = (\tilde{r}^{\dagger}, K^{\dagger}). \quad (3.28)$$

The linear systems in (3.27) in the case of $J = I$ are used as initial linear systems when the binary Darboux transformations are applied. Therefore, we have the following theorem on the ultrahyperbolic space $\mathbb{U}$:

Theorem 3.4. (cf. [26]) Let us consider the Sylvester equation

$$\Xi^{\dagger}\Omega(\theta, \rho) - \Omega(\theta, \rho)\Lambda = \rho^{\dagger}\theta, \quad (3.29)$$

where (θ, Λ) and (ρ, Ξ) satisfy the linear systems in (3.27). If we define $\hat{U}$ and $\hat{J}$ by

$$\hat{U} := \begin{vmatrix} \Omega(\theta, \rho) & \rho^{\dagger} \\ \theta & \boxed{U} \end{vmatrix}, \qquad \hat{J} := \begin{vmatrix} \Omega(\theta, \rho) & \Xi^{-\dagger}\rho^{\dagger} \\ \theta & \boxed{I} \end{vmatrix} J, \quad (3.30)$$

where U satisfies

$$\partial_{\tilde{z}}U = (\partial_w J)J^{-1}, \qquad \partial_{\tilde{w}}U = (\partial_z J)J^{-1}, \quad (3.31)$$

then $\hat{U}$ and $\hat{J}$ satisfy the differential relations

$$\partial_{\tilde{z}}\hat{U} = (\partial_w \hat{J})\hat{J}^{-1}, \qquad \partial_{\tilde{w}}\hat{U} = (\partial_z \hat{J})\hat{J}^{-1}. \quad (3.32)$$

That is, $\hat{J}$ satisfies the Yang equation

$$\partial_{\tilde{z}}[(\partial_z \hat{J})\hat{J}^{-1}] - \partial_{\tilde{w}}[(\partial_w \hat{J})\hat{J}^{-1}] = 0 \quad (3.33)$$

on the ultrahyperbolic space $\mathbb{U}$.

Proof. We can check that Lemma 3.1 still holds for linear systems in (3.27), that is, the Cauchy matrix Ω satisfies the following relations:

$$\begin{cases} \partial_z\Omega = \Xi^{\dagger}(\partial_{\tilde{w}}\Omega) - \rho^{\dagger}(\partial_{\tilde{w}}\theta) = (\partial_{\tilde{w}}\Omega)\Lambda + (\partial_{\tilde{w}}\rho^{\dagger})\theta \\ \partial_w\Omega = \Xi^{\dagger}(\partial_{\tilde{z}}\Omega) - \rho^{\dagger}(\partial_{\tilde{z}}\theta) = (\partial_{\tilde{z}}\Omega)\Lambda + (\partial_{\tilde{z}}\rho^{\dagger})\theta \end{cases}. \quad (3.34)$$

Since the proof is quite similar to the proof in Theorem 3.3, we just skip all the details here. ■

Note that if we choose J to be unitary and $(\rho, \Xi) = (\theta, \Lambda)$, the second system of equations of (3.27) is now identical to the first one, that is, (θ, Λ) satisfies the following linear system:

$$\begin{cases} [\partial_z - (\partial_z J)J^{-1}]\theta - (\partial_{\tilde{w}}\theta)\Lambda = 0 \\ [\partial_w - (\partial_w J)J^{-1}]\theta - (\partial_{\tilde{z}}\theta)\Lambda = 0 \end{cases}. \quad (3.35)$$

Under this setting, we can obtain a class of unitary solutions as the following corollary:

Corollary 3.5. Let J be unitary and $\Omega(\theta, \theta)$ be the dressed Cauchy matrix satisfying the Sylvester equation

$$\Lambda^{\dagger}\Omega(\theta, \theta) - \Omega(\theta, \theta)\Lambda = \theta^{\dagger}\theta. \quad (3.36)$$

Then, we can obtain unitary solutions in the form of

$$\hat{J} = \begin{vmatrix} \Omega(\theta,\theta) & \Lambda^{-\dagger}\theta^{\dagger} \\ \theta & \boxed{I} \end{vmatrix} J. \tag{3.37}$$

Proof. By using the fact that $\Omega^{\dagger} = -\Omega$ [see Eq. (3.36)] and direct calculation, we have

$$\begin{aligned}\hat{J}^{\dagger}\hat{J} &= J^{\dagger}(I - \theta\Lambda^{-1}\Omega^{-\dagger}\theta^{\dagger})(I - \theta\Omega^{-1}\Lambda^{-\dagger}\theta^{\dagger})J \\ &= J^{\dagger}[I - \theta\Lambda^{-1}\Omega^{-\dagger}\theta^{\dagger} - \theta\Omega^{-1}\Lambda^{-\dagger}\theta^{\dagger} + \theta\Lambda^{-1}\Omega^{-\dagger} \\ &\quad \times (\Lambda^{\dagger}\Omega - \Omega\Lambda)\Omega^{-1}\Lambda^{-\dagger}\theta^{\dagger}]J \\ &= J^{\dagger}[I - \theta(I + \Lambda^{-1}\Omega^{-\dagger}\Omega\Lambda)\Omega^{-1}\Lambda^{-\dagger}\theta^{\dagger}]J \\ &= J^{\dagger}J = I. \end{aligned} \tag{3.38}$$

Similarly, we also have $JJ^{\dagger} = I$. Therefore, $\hat{J}$ is unitary. ∎

A strategy to construct exact solutions is summarized as follows. Firstly, we solve the initial linear systems to get (θ, Λ),

$$\partial_z\theta = (\partial_{\tilde{w}}\theta)\Lambda, \qquad \partial_w\theta = (\partial_{\tilde{z}}\theta)\Lambda. \tag{3.39}$$

The solutions (θ, Λ) are called the input data. Secondly, we solve the Sylvester equation [Eq. (3.36)] by the input data (θ, Λ) to get Ω. Finally, we can get a unitary solution $\hat{J}$ via (3.37).

IV. EXACT SOLITONS IN THE WZW$_4$ MODEL

In this section, we give several exact soliton solutions in the WZW$_4$ model and calculate the corresponding action densities of the NLσM term for one-solitons in Sec. IV A and for two-solitons in Sec. IV B. We clarify that the quasi-Wronskian solution [32] and quasi-Grammian solutions (3.37) give the same action densities for one- and two-soliton solutions. In Sec. IV C, we discuss some limits of the two-soliton solutions, some of which lead to resonance solutions.

A. Exact one-solitons

A set of input data (θ, Λ) for one-solitons can be given by solving (3.39),

$$\theta := \begin{pmatrix} a_1^2 e^{\xi_1} \\ b_1^2 e^{-\xi_1} \end{pmatrix}, \qquad \Lambda := (\lambda_1), \tag{4.1}$$

where

$$\xi_1 := \lambda_1\alpha_1 z + \beta_1\tilde{z} + \lambda_1\beta_1 w + \alpha_1\tilde{w}, \qquad a_1, b_1, \lambda_1, \alpha_1, \beta_1 \in \mathbb{C}. \tag{4.2}$$

The corresponding solution of the Sylvester equation is

$$\Omega = \frac{|a_1|^2 e^{\bar{\xi}_1 + \xi_1} + |b_1|^2 e^{-(\bar{\xi}_1 + \xi_1)}}{\bar{\lambda}_1 - \lambda_1}, \tag{4.3}$$

where $\bar{\lambda}$ stands for the complex conjugate of λ. The resulting NLσM action density can be calculated via (3.37) as

$$\mathcal{L}_{\sigma} = -\frac{1}{16\pi}\mathrm{Tr}\left[(\partial_{\mu}\hat{J})\hat{J}^{-1}(\partial^{\mu}\hat{J})\hat{J}^{-1}\right] = \frac{d_{11}}{8\pi}\mathrm{sech}^2\tilde{X}_1, \tag{4.4}$$

where

$$d_{11} = \frac{(\alpha_1\bar{\beta}_1 - \beta_1\bar{\alpha}_1)(\lambda_1 - \bar{\lambda}_1)^3}{|\lambda_1|^2}, \tag{4.5a}$$

$$\tilde{X}_1 = X_1 + \log\delta_1 := \xi_1 + \bar{\xi}_1 + \log\delta_1, \qquad \delta_1 = |a_1|^2/|b_1|^2. \tag{4.5b}$$

The result of $\mathcal{L}_{\sigma}$ is exactly the same as the quasi-Wronskian one-soliton solution in [32], where the input data θ and Λ are both 2×2 matrices, rather than (4.1).

The resulting Wess-Zumino action density $\mathcal{L}_{\mathrm{WZ}}$ identically vanishes [32]. Hence, the WZW$_4$ action density is the same as $\mathcal{L}_{\sigma}$ in (4.4).

B. Exact two-solitons

A set of input data for two-solitons can be given by the following 2×2 matrix pair (θ, Λ):

$$\theta := \begin{pmatrix} a_1^2 e^{\xi_1} & a_2^2 e^{\xi_2} \\ b_1^2 e^{-\xi_1} & b_2^2 e^{-\xi_2} \end{pmatrix}, \qquad \Lambda := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{4.6}$$

where

$$\begin{aligned} &\xi_j := \lambda_j\alpha_j z + \beta_j\tilde{z} + \lambda_j\beta_j w + \alpha_j\tilde{w}, \\ &a_j, b_j, \lambda_j, \alpha_j, \beta_j \in \mathbb{C}, \quad j = 1, 2. \end{aligned} \tag{4.7}$$

The NLσM action density can be calculated as (see Appendix B for details)

$$\mathcal{L}_\sigma = -\frac{1}{16\pi}\mathrm{Tr}[(\partial_\mu \hat{J})\hat{J}^{-1}(\partial^\mu \hat{J})\hat{J}^{-1}] = \frac{\left\{\begin{array}{l} c_1 c_2[d_{11}\cosh^2\tilde{X}_2 + d_{22}\cosh^2\tilde{X}_1] \\ +c_1 c_3\left[d_{12}\cosh^2\left(\frac{\tilde{X}_1+\tilde{X}_2-i\tilde{\Theta}_{12}}{2}\right) + \bar{d}_{12}\cosh^2\left(\frac{\tilde{X}_1+\tilde{X}_2+i\tilde{\Theta}_{12}}{2}\right)\right] \\ -c_2 c_3\left[f_{12}\sinh^2\left(\frac{\tilde{X}_1-\tilde{X}_2-i\tilde{\Theta}_{12}}{2}\right) + \bar{f}_{12}\sinh^2\left(\frac{\tilde{X}_1-\tilde{X}_2+i\tilde{\Theta}_{12}}{2}\right)\right]\end{array}\right\}}{2\pi[c_1\cosh(\tilde{X}_1+\tilde{X}_2) + c_2\cosh(\tilde{X}_1-\tilde{X}_2) + c_3\cos\tilde{\Theta}_{12}]^2}, \tag{4.8}$$

where

$$c_1 := (\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2), \qquad c_2 := (\lambda_1-\bar{\lambda}_2)(\bar{\lambda}_1-\lambda_2), \qquad c_3 := (\lambda_1-\bar{\lambda}_1)(\lambda_2-\bar{\lambda}_2), \tag{4.9a}$$

$$d_{jk} := \frac{(\alpha_j\bar{\beta}_k-\beta_j\bar{\alpha}_k)(\lambda_j-\bar{\lambda}_k)^3}{\lambda_j\bar{\lambda}_k}, \qquad f_{jk} := \frac{(\alpha_j\beta_k-\beta_j\alpha_k)(\lambda_j-\lambda_k)^3}{\lambda_j\lambda_k}, \qquad j,k = 1,2, \tag{4.9b}$$

$$\tilde{X}_j := X_j + \log\delta_j := \xi_j + \bar{\xi}_j + \log\delta_j, \qquad \delta_j := |a_j|^2/|b_j|^2, \tag{4.9c}$$

$$\tilde{\Theta}_{12} := \Theta_1 - \Theta_2 + \phi, \qquad \Theta_j := -i(\xi_j - \bar{\xi}_j), \quad j = 1,2, \tag{4.9d}$$

$$\phi := 2\mathrm{Arg}(a_1\bar{a}_2/b_1\bar{b}_2) = 2\mathrm{Arg}(a_1 b_2/a_2 b_2). \tag{4.9e}$$

The resulting $\mathcal{L}_\sigma$ is exactly the same as the case of the quasi-Wronskian two-soliton solution in [32]. The input data here are much simpler than those in [32] because the input data in [32] consist of two sets of 2×2 matrix pairs, (θ_j, Λ_j), $j = 1, 2$.

Note that the NLσM action density $\mathcal{L}_\sigma$ is nonsingular and real-valued. The distribution of $\mathcal{L}_\sigma$ behaves like KP two-solitons because we can show that the asymptotic limits of $\mathcal{L}_\sigma$ are

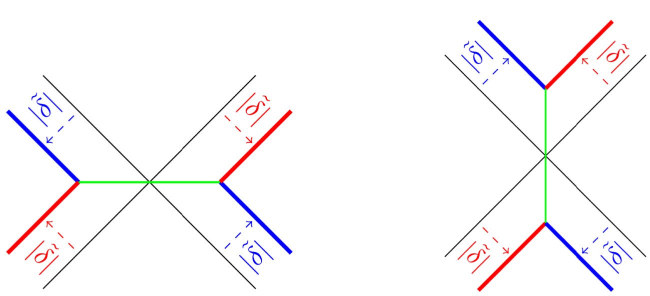
$$-8\pi\mathcal{L}_\sigma \longrightarrow \begin{cases} d_{11}\mathrm{sech}^2(\tilde{X}_1 \pm \tilde{\delta}) & \text{if } X_1 \text{ is finite and } X_2 \to \pm\infty \\ d_{22}\mathrm{sech}^2(\tilde{X}_2 \pm \tilde{\delta}) & \text{if } X_2 \text{ is finite and } X_1 \to \pm\infty \end{cases}, \tag{4.10}$$

where the phase-shift factor is given by

$$\tilde{\delta} := \frac{1}{2}\log\left(\frac{c_1}{c_2}\right) = \log\frac{|\lambda_1-\lambda_2|}{|\lambda_1-\bar{\lambda}_2|}. \tag{4.11}$$

In Fig. 1 [also refer to Fig. 8(b)], we can see the physical meaning of phase shift. All figures in this paper are 2D slice plotted by taking $(w,\tilde{w}) = (0,0)$.

The red line and blue line denote two-line solitons and, as we can see, the V-shape soliton is composed of a half red line soliton and half blue line soliton. The green line describes the distance between the vertices of the two V-shape solitons, which becomes longer and longer as $\tilde{\delta}$ increases.



(a) (b)

FIG. 1. 2D slice of two-soliton NLσM action density for $(w,\tilde{w}) = (0,0)$. (a) The case of $\tilde{\delta} > 0$. (b) The case of $\tilde{\delta} < 0$.

We note that when λ_1 is real, that is, $\lambda_1 = \bar{\lambda}_1$, the two-soliton reduces to a one-soliton case: $\mathcal{L}_\sigma = \frac{d_{11}}{8\pi}\mathrm{sech}^2\tilde{X}_1$, because of $c_3 = d_{22} = 0$. Similarly, when λ_2 is real, $\mathcal{L}_\sigma = \frac{d_{22}}{8\pi}\mathrm{sech}^2\tilde{X}_2$ because of $c_3 = d_{11} = 0$.

C. Resonance limits of two-solitons

Let us discuss some limits of the two-solitons, where the absolute value of phase shift (4.11) goes to infinity, that is, the cases of (1) $\lambda_2 \to \lambda_1$ and (2) $\lambda_2 \to \bar{\lambda}_1$. These processes usually lead to resonance states.

Naive discussion of cases (1) and (2) shows that

$$\mathcal{L}_\sigma \to \begin{cases} (1){:}0, & \text{if } \lambda_2 \to \lambda_1 (c_1, f_{12} \to 0) \\ (2){:}0, & \text{if } \lambda_2 \to \bar{\lambda}_1 (c_2, d_{12} \to 0), \end{cases} \tag{4.12}$$

which is correct for almost all cases because the numerator of (4.8) tends to zero. However, we need to be more careful when dealing with the case when the denominator becomes zero. For the case (1), there is a particular choice for the parameters: $\alpha_1 = \alpha_2 := \alpha, \beta_1 = \beta_2 := \beta$. In this case, the two-line solitons have equal "amplitude": $d_{11} = d_{22}$ [cf. Eq. (4.10)]. For simplicity, we assume the initial phase terms $\log \delta_j$ and ϕ to be zero [i.e., $a_j = b_j, j = 1, 2$ in (4.8)]. By using the fact that $c_1 = c_2 + c_3$ and some properties of hyperbolic functions, the denominator of (4.8) can be rewritten as

$$2\pi\left[c_2 \cosh(\xi_1 + \bar{\xi}_1)\cosh(\xi_2 + \bar{\xi}_2) + c_3 \cosh(\xi_1 + \bar{\xi}_2)\cosh(\bar{\xi}_1 + \xi_2)\right]^2. \tag{4.13}$$

Clearly, (4.13) tends to zero as $\lambda_2 \to \lambda_1 (c_1 \to 0 \Rightarrow c_2 \to -c_3, \xi_1 \to \xi_2)$. Now, we find that $\mathcal{L}_\sigma \to 0/0$ in the above case (1). Dividing a common factor c_1^2 in both the denominator and numerator of (4.8) and using some properties of hyperbolic functions, the denominator (4.13)/c_1^2 can be rewritten as

$$2\pi\left[\begin{array}{l}\cosh^2 X_1 |\cosh\xi|^2 - \cosh X_1 \sinh X_1 \sinh(\xi + \bar{\xi}) \\ +(c_2 \cosh^2 X_1 + c_3 \sinh^2 X_1)\left|\frac{\sinh\xi}{\lambda}\right|^2\end{array}\right]^2, \tag{4.14}$$

where for fixed λ_1 and ξ_1, we define $\lambda := \lambda_2 - \lambda_1$, $\xi := \xi_2 - \xi_1 = \lambda(\alpha z + \beta w)$. Note that as $\lambda \to 0$, we have

$$\cosh\xi \to 1, \qquad \sinh(\xi + \bar{\xi}) \to 0, \tag{4.15a}$$

$$c_2 \cosh^2 X_1 + c_3 \sinh^2 X_1 \to |\lambda_1 - \overline{\lambda_1}|^2 (\cosh^2 X_1 - \sinh^2 X_1) = |\lambda_1 - \overline{\lambda_1}|^2, \tag{4.15b}$$

$$\frac{\sinh\xi}{\lambda} \to \dot{\xi}\cosh\xi|_{\lambda=0} = \dot{\xi}, \qquad \dot{\xi} := \partial_\lambda \xi = \alpha z + \beta w. \tag{4.15c}$$

Therefore, (4.14) converges to

$$2\pi[|\lambda_1 - \bar{\lambda}_1|^2 |\dot{\xi}_1|^2 + \cosh^2 X_1]^2 = \frac{\pi}{2}[1 + 2|\lambda_1 - \bar{\lambda}_1|^2 |\dot{\xi}_1|^2 + \cosh(2X_1)]^2 \tag{4.16}$$

as $\lambda_2 \to \lambda_1$, where $\dot{\xi}_1 := \partial_{\lambda_1}\xi_1 = \alpha z + \beta w$. As we will see later in the next section, (4.16) matches with the denominator of (5.11) exactly. In Sec. V, we will focus on solutions of this type ($\lambda_2 \to \lambda_1$) and construct them systematically.

Finally, let us discuss the intermediate state described by green lines in Fig. 1 in the setting where $\hat{J}$ is not unitary, which implies that $\mathcal{L}_\sigma$ are not real-valued in general, even in the resonance limits.

In this case, we have to consider the unreduced non-unitary solution (3.30) in Theorem 3.4 [not the unitary solution (3.37) in Corollary 3.5], where there are additional spectral parameters μ_1 and μ_2, which come from Ξ (see Appendix C for details). Then, the vanishing asymptotics in the cases (1) and (2) of (4.12) become nontrivial due to the additional degree of freedom. Under the assumption $\Theta_{12} = 2n\pi$ or $(2n+1)\pi$, the infinite phase-shift limits are

$$\mathcal{L}_\sigma \to \begin{cases} (1)\ \text{When } \Theta_{12} = (2n+1)\pi: & \\ \frac{-\tilde{f}_{12}}{8\pi}\operatorname{sech}^2\left(\frac{\tilde{X}_1 - \tilde{X}_2}{2}\right), & \text{if } \lambda_2 \to \lambda_1 (c_1, f_{12} \to 0), \\ \frac{-f_{12}}{8\pi}\operatorname{sech}^2\left(\frac{\tilde{X}_1 - \tilde{X}_2}{2}\right), & \text{if } \mu_2 \to \mu_1 (c_1, \tilde{f}_{12} \to 0), \\ (2)\ \text{When } \Theta_{12} = 2n\pi: & \\ \frac{d_{12}}{8\pi}\operatorname{sech}^2\left(\frac{\tilde{X}_1 + \tilde{X}_2}{2}\right), & \text{if } \lambda_2 \to \mu_1 (c_2, \tilde{d}_{12} \to 0), \\ \frac{\tilde{d}_{12}}{8\pi}\operatorname{sech}^2\left(\frac{\tilde{X}_1 + \tilde{X}_2}{2}\right), & \text{if } \mu_2 \to \lambda_1 (c_2, d_{12} \to 0). \end{cases} \tag{4.17}$$

Hence, in the infinite phase-shift limit, we can find the intermediate soliton states (described by green lines in Fig. 1) in the expected directions. Therefore, in the complex valued settings, solutions of this type suggest the existence of Y-shape solitons [48], which are building blocks in the classification of KP line solitons [35,36].

On the other hand, in the real-valued settings, this intermediate soliton state vanishes for the NLσM action density, as mentioned in (4.12), and for the Wess-Zumino action density as well, as is seen in Appendix D.4 in [32] [where (D.18) and (D.19) vanish in the $\lambda_1 \to \bar{\lambda}_2$ limit; then, $b = 0$ and $\mathcal{D}_{12} = \mathcal{D}_{21} = \mathscr{d}_{12} = \mathscr{d}_{21} = 0$, see also (D.15)]. This is a reasonable result because this intermediate state describes a one-soliton distribution, and the Wess-Zumino action density identically vanishes, as mentioned in Sec. IV A. Hence, the effect of the Wess-Zumino action density appears only around the two vertices of two V-shape solitons.

D. Multisoliton solutions

In the same way, we can easily construct N-soliton solutions by setting the input data of the $N \times N$ diagonal matrix Λ and $2 \times N$ matrix θ, which satisfy (3.39). For the $N = 3$ case, we have

$$\theta \coloneqq \begin{pmatrix} a_1^2 e^{\xi_1} & a_2^2 e^{\xi_2} & a_3^2 e^{\xi_3} \\ b_1^2 e^{-\xi_1} & b_2^2 e^{-\xi_2} & b_3^2 e^{-\xi_3} \end{pmatrix}, \quad \Lambda \coloneqq \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{4.18}$$

where the parameters are defined as in (4.7) for $j \in \{1, 2, 3\}$.

We note that the input data are much simpler than those for quasi-Wronskian solutions [32]. More specifically, we can construct multisoliton solutions with arbitrary soliton numbers without applying the iterations of Darboux transformation.

V. MULTIPLE-POLE SOLUTIONS

As we indicated in the previous section, there is a type of nontrivial solution called the double-pole solution, which could be derived by taking the limit $\lambda_2 \to \lambda_1$ (with the constraint $\alpha_1 = \alpha_2, \beta_1 = \beta_2$) on the two-solitons. In this limit, both the normal vectors and the amplitudes of the two-line solitons converge to the same value [cf. (4.7), (4.9c), and (4.10)]. Nevertheless, we will see in Sec. V B that in the 2D slice $[(w, \tilde{w}) = (0, 0)]$, the two-solitons converge not to parallel two-line solitons but to two "curved solitons" along with a linear function background due to resonance effects in the interacting region. More details will be provided in Sec. V B.

In this section, we aim to construct the double-pole solutions from the two-soliton solutions (4.6) by taking some nontrivial transformations and using L'Hôpital's rule, and then calculate the corresponding NLσM action density.

A. Double-pole solutions

In this section, we show that the double-pole solution can be considered as the $\lambda_2 \to \lambda_1$ limit of the two-soliton solution (4.6). For convenience of the later discussion, we introduce the following transformation:

$$(\theta', \Lambda') = (\theta P^{-1}, P \Lambda P^{-1}), \qquad \Omega'(\theta', \theta') = P^{-\dagger} \Omega(\theta, \theta) P^{-1}. \tag{5.1}$$

Direct calculation shows that the linear system and the Sylvester equation are form invariant under (5.1). More explicitly, we have

$$\begin{cases} [\partial_z - (\partial_z J) J^{-1}] \theta' - (\partial_{\tilde{w}} \theta') \Lambda' = 0 \\ [\partial_w - (\partial_w J) J^{-1}] \theta' - (\partial_{\tilde{z}} \theta') \Lambda' = 0 \end{cases} \tag{5.2}$$

and

$$(\Lambda')^\dagger \Omega'(\theta', \theta') - \Omega'(\theta', \theta') \Lambda' = (\theta')^\dagger \theta'. \tag{5.3}$$

One iteration of the binary Darboux transformation (3.37) shows

$$\begin{aligned} \hat{J}' &= \begin{vmatrix} \Omega'(\theta', \theta') & (\Lambda')^{-\dagger} (\theta')^\dagger \\ \theta' & \boxed{I} \end{vmatrix} J \\ &= \begin{vmatrix} P^\dagger \Omega(\theta, \theta) P^{-1} & P^{-\dagger} \Lambda^{-\dagger} \theta^\dagger \\ \theta P^{-1} & \boxed{I} \end{vmatrix} J \\ &= \begin{vmatrix} \Omega(\theta, \theta) & \Lambda^{-\dagger} \theta^\dagger \\ \theta & \boxed{I} \end{vmatrix} J = \hat{J}, \end{aligned} \tag{5.4}$$

that is, $\hat{J}$ is invariant under (5.1). Due to this fact, an equivalent expression of the two-soliton solution can be obtained by choosing

$$P = \begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 1 & \lambda_1 - \lambda_2 \end{pmatrix} \tag{5.5}$$

in (5.1). In this case,

$$\theta' = \begin{pmatrix} \frac{a_1^2 e^{\xi_1}}{\lambda_1 - \lambda_2} - \frac{a_2^2 e^{\xi_2}}{(\lambda_1 - \lambda_2)^2} & \frac{a_2^2 e^{\xi_2}}{\lambda_1 - \lambda_2} \\ \frac{b_1^2 e^{-\xi_1}}{\lambda_1 - \lambda_2} - \frac{b_2^2 e^{-\xi_2}}{(\lambda_1 - \lambda_2)^2} & \frac{b_2^2 e^{-\xi_2}}{\lambda_1 - \lambda_2} \end{pmatrix}, \qquad \Lambda' = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{pmatrix}, \tag{5.6}$$

where $\xi_j, j = 1, 2$ are defined in (4.7).

Now, let us show that the double-pole solution can be derived by taking the limit on (θ', Λ'). By choosing $a_1^2 = b_1^2 = 1$, $a_2^2 = b_2^2 = \lambda_1 - \lambda_2$, and $\alpha_1 = \alpha_2, \beta_1 = \beta_2$, we have

$$\theta' = \begin{pmatrix} \frac{e^{\xi_1} - e^{\xi_2}}{\lambda_1 - \lambda_2} & e^{\xi_2} \\ \frac{e^{-\xi_1} - e^{-\xi_2}}{\lambda_1 - \lambda_2} & e^{-\xi_2} \end{pmatrix}. \tag{5.7}$$

By taking the limit $\lambda_2 \to \lambda_1$ and using L'Hôpital's rule, we have

$$\lim_{\lambda_2 \to \lambda_1} \theta' = \begin{pmatrix} \dot{\xi}_1 e^{\xi_1} & e^{\xi_1} \\ -\dot{\xi}_1 e^{-\xi_1} & e^{-\xi_1} \end{pmatrix}, \qquad \lim_{\lambda_2 \to \lambda_1} \Lambda' = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}, \tag{5.8}$$

where $\dot{\xi}_1 \coloneqq \partial_{\lambda_1} \xi_1$. For simplicity, we omit the lower index and rewrite the input data for the double-pole solution as the following 2×2 matrix pair (θ, Λ):

$$\theta := \begin{pmatrix} \dot{\xi}e^{\xi} & e^{\xi} \\ -\dot{\xi}e^{-\xi} & e^{-\xi} \end{pmatrix}, \qquad \Lambda := \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \tag{5.9}$$

where

$$\xi := \lambda\alpha z + \beta\tilde{z} + \lambda\beta w + \alpha\tilde{w}, \qquad \dot{\xi} := \partial_\lambda \xi = \alpha z + \beta w, \qquad \lambda, \alpha, \beta, z, \tilde{z}, w, \tilde{w} \in \mathbb{C}. \tag{5.10}$$

In fact, such types of solutions with Jordan block matrices Λ are obtained in Appendix C. 3 in [38].

The corresponding NLσM action density can be calculated as (see Appendix D for details)

$$\begin{aligned} \mathcal{L}_\sigma &= -\frac{1}{16\pi}\mathrm{Tr}[(\partial_m \hat{J})\hat{J}^{-1}(\partial^m \hat{J})\hat{J}^{-1}] \\ &= \frac{-(\alpha\bar{\beta}-\bar{\alpha}\beta)(\lambda-\bar{\lambda})^3 \left\{ \begin{aligned} &-(\lambda+\bar{\lambda})^2[1+\cosh(2X)] \\ &+2(\lambda-\bar{\lambda})^2|\lambda|^2|\dot{\xi}|^2\cosh(2X) \\ &+(\lambda-\bar{\lambda})[(\lambda+\bar{\lambda})(\lambda\dot{\xi}-\bar{\lambda}\bar{\dot{\xi}})+|\lambda|^2(\dot{\xi}-\bar{\dot{\xi}})]\sinh(2X) \end{aligned} \right\}}{4\pi|\lambda|^4[1+2|\lambda-\bar{\lambda}|^2|\dot{\xi}|^2+\cosh(2X)]^2}, \end{aligned} \tag{5.11}$$

where

$$X := \xi + \bar{\xi}, \tag{5.12}$$

$$\begin{aligned} \xi &= \lambda\alpha z + \beta\tilde{z} + \lambda\beta w + \alpha\tilde{w} \\ &= \frac{1}{\sqrt{2}}\{(\lambda\alpha+\beta)x^1 + (\lambda\beta-\alpha)x^2 \\ &\quad + (\lambda\alpha-\beta)x^3 + (\lambda\beta+\alpha)x^4\}. \end{aligned} \tag{5.13}$$

We remark that in the $\lambda_2 \to \lambda_1$ limit, the Wess-Zumino action density also goes to $0/0$, as is seen in Appendix D.4 in [32] [where the numerators in (D.16) $\sim$ (D.19) vanish in the $\lambda_2 \to \lambda_1$ limit; then, $a = 0$ and $\mathcal{E}_{12} = \mathcal{E}_{21} = e_{12} = \tilde{e}_{12} = 0$, see also (D.15)]. Hence, there would be the same kind of contribution from the Wess-Zumino action density only in the interacting region. (In the asymptotic region, there is no contribution because $\mathcal{L}_{\mathrm{WZ}} = 0$ in any direction.)

B. Asymptotic analysis of double-pole solutions

In this section, we analyze the asymptotic behavior of (5.11). First of all, we introduce

$$\begin{aligned} Z_\pm &:= X \mp \log|\dot{\xi}| \\ &= (\lambda\alpha + \bar{\lambda}\bar{\alpha})z + (\beta+\bar{\beta})\tilde{z} + (\lambda\beta + \bar{\lambda}\bar{\beta})w \\ &\quad + (\alpha+\bar{\alpha})\tilde{w} \mp \log|\alpha z + \beta w| \end{aligned} \tag{5.14}$$

and consider the following two asymptotic regions:

$$\begin{cases} (1)\ R_+:\ X \to +\infty, \log|\dot{\xi}| \to +\infty \text{ such that } Z_+ \text{ is finite} \\ (2)\ R_-: X \to -\infty, \log|\dot{\xi}| \to +\infty \text{ such that } Z_- \text{ is finite} \end{cases}. \tag{5.15}$$

Note that $Z_\pm = \text{constant}$ describes the logarithmic curves living on a linear function background.

By dividing a common factor $|\dot{\xi}|^2 e^{2X}$ in both the denominator and numerator of (5.11) and considering case (1) of (5.15), since $|\lambda\dot{\xi} - \bar{\lambda}\bar{\dot{\xi}}| = 2\,\mathrm{Im}\,|\lambda\dot{\xi}| \le 2|\lambda||\dot{\xi}|$ implies $|\lambda\dot{\xi} - \bar{\lambda}\bar{\dot{\xi}}| = O(|\dot{\xi}|)$, we find that (5.11) is dominated by the following expression[3]:

$$16\pi\mathcal{L}_\sigma = \frac{A + O(|\dot{\xi}|^{-1})}{[(|\dot{\xi}|^{-1}e^X + 4|\lambda-\bar{\lambda}|^2|\dot{\xi}|e^{-X})/2 + O(|\dot{\xi}|^{-1}e^{-X})]^2}, \tag{5.16}$$

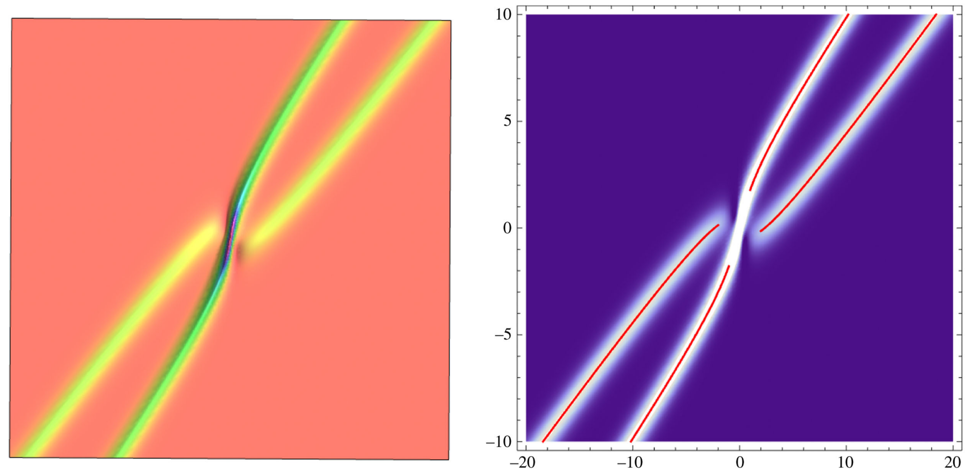
$$= \frac{d_{11} + O(|\dot{\xi}|^{-1})}{\cosh^2[X - \log|\dot{\xi}| - \log(2|\lambda-\bar{\lambda}|)] + O(|\dot{\xi}|^{-2})}, \tag{5.17}$$

$$\begin{aligned} &\longrightarrow d_{11}\mathrm{sech}^2[X - \log|\dot{\xi}| - \log(2|\lambda-\bar{\lambda}|)] \\ &= d_{11}\mathrm{sech}^2(Z_+ - \delta) \text{ on } R_+, \end{aligned} \tag{5.18}$$

where $A := -16(\alpha\bar{\beta} - \bar{\alpha}\beta)(\lambda-\bar{\lambda})^5/|\lambda|^2$, d_{11} is defined in (4.9b) with $\lambda_1 = \lambda$, and the phase-shift factor is $\delta := \log(2|\lambda-\bar{\lambda}|)$.

Similarly, by dividing another common factor $|\dot{\xi}|^2 e^{-2X}$ in both the denominator and numerator of (5.11) and consider

[3] We briefly introduce the definition of the big O notation. If there exist x_0, c such that $|f(x)| \le c|g(x)|$, for all $x > x_0$, then we say $f(x) = O(g(x))$.



(a) (b)

FIG. 2. Plots of the 2D slice of the double-pole soliton NLσM action density with $\lambda_1 = -1 + i, \alpha = 0.5 - i, \beta = -0.7 - 1.4i$, $(z, \tilde{z}) \in [-20, 20] \times [-10, 10]$, and $(w, \tilde{w}) = (0, 0)$. (a) Shape of the 2D slice of NLσM action density. (b) Density plot with four red curves of $Z_\pm \mp \delta = 0$.

case (2) of (5.15), we obtain

$$16\pi\mathcal{L}_\sigma = \frac{A + O(|\dot{\xi}|^{-1})}{[(4|\lambda - \bar{\lambda}|^2|\dot{\xi}|e^X + |\dot{\xi}|^{-1}e^{-X})/2 + O(|\dot{\xi}|^{-1}e^X)]^2}, \tag{5.19}$$

$$= \frac{d_{11} + O(|\dot{\xi}|^{-1})}{\cosh^2[X + \log|\dot{\xi}| + \log(2|\lambda - \bar{\lambda}|)] + O(|\dot{\xi}|^{-2})}, \tag{5.20}$$

$$\begin{aligned} &\longrightarrow d_{11}\mathrm{sech}^2[X + \log|\dot{\xi}| + \log(2|\lambda - \bar{\lambda}|)] \\ &= d_{11}\mathrm{sech}^2(Z_- + \delta)\,\mathrm{on}\, R_-. \end{aligned} \tag{5.21}$$

Note that (5.18) and (5.21) involve a logarithm function term, which is quite different from (4.10), although (5.18), (5.21), and (4.10) are quite similar in form at first glance. Therefore, we find that the two peaks of action density in (5.11) are localized on two curved hypersurfaces $Z_\pm \mp \delta = 0$. This is illustrated by taking a 2D slice $(w, \tilde{w}) = (0, 0)$ in Fig. 2, where $Z_\pm = (\lambda\alpha + \bar{\lambda}\bar{\alpha})z + (\beta + \bar{\beta})\tilde{z} \mp \log|\alpha| \mp \log|z|$.

C. Multiple-pole solutions

In a similar way, we can construct multiple-pole (of degree N) solutions by setting the input data of an $N \times N$ Jordan block matrix Λ (instead of a diagonal matrix). For such solutions, the input data are given by

$$\theta = \begin{pmatrix} \frac{\partial_\lambda^{N-1}(e^\xi)}{(N-1)!} & \cdots & \partial_\lambda(e^\xi) & e^\xi \\ \frac{\partial_\lambda^{N-1}(e^{-\xi})}{(N-1)!} & \cdots & \partial_\lambda(e^{-\xi}) & e^{-\xi} \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}. \tag{5.22}$$

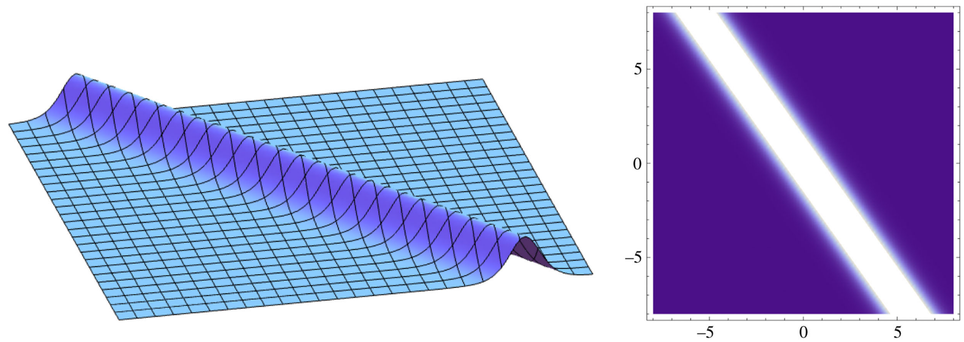
For example, if we take $N = 3$, the input data for the triple-pole solution will be

$$\theta = \begin{pmatrix} \frac{1}{2}[\ddot{\xi} + (\dot{\xi})^2]e^\xi & \dot{\xi}e^\xi & e^\xi \\ \frac{1}{2}[-\ddot{\xi} + (\dot{\xi})^2]e^{-\xi} & -\dot{\xi}e^{-\xi} & e^{-\xi} \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \tag{5.23}$$

where $\ddot{\xi} \coloneqq \partial_\lambda^2\xi$, and the parameters are defined as in (4.7) and (5.12). This can be realized as the $\lambda_2 \to \lambda_1$ and $\lambda_3 \to \lambda_1$ limits [cf. Eq. (4.18)] of three-line solitons with equal amplitude.

VI. EXAMPLES AND FIGURES

In Secs. IV B and V A, the NLσM action densities of the two-soliton solution and double-pole solution are shown. Their connection via a limiting procedure is revealed in Sec. IV C. In this section, we explain the resonance limits of two-soliton solutions by illustrating fruitful figures in terms of the explicit result (4.8), where the double-pole interaction (5.11) and V-shape soliton will be revealed.



(a) (b)

FIG. 3. Plots of the 2D slice of two-soliton NLσM action density with $\lambda_1 = 0.5 + 0.5i, \alpha_1 = 0.5 - 0.5i, \beta_1 = -0.7 - 1.4i$, $(z, \tilde{z}) \in [-8, 8] \times [-8, 8]$, and $(w, \tilde{w}) = (0, 0)$. (a) Shape of the 2D slice of NLσM action density. (b) Density plot of the 2D slice of NLσM action density.

Since $\mathcal{L}_\sigma$ is a four-dimensional function of $(z, \tilde{z}, w, \tilde{w})$, to illustrate the shape, we can draw a two-dimensional slice of it with flexible $(z, \tilde{z})$ and fixed $(w, \tilde{w})$. For convenience, we will choose $(w, \tilde{w}) = (0, 0)$ in later illustrations.

A. One-soliton solution

We start with the one-soliton case, and the corresponding NLσM action density is given by (4.4). The shape and density plot of the two-dimensional slice are illustrated in Fig. 3.

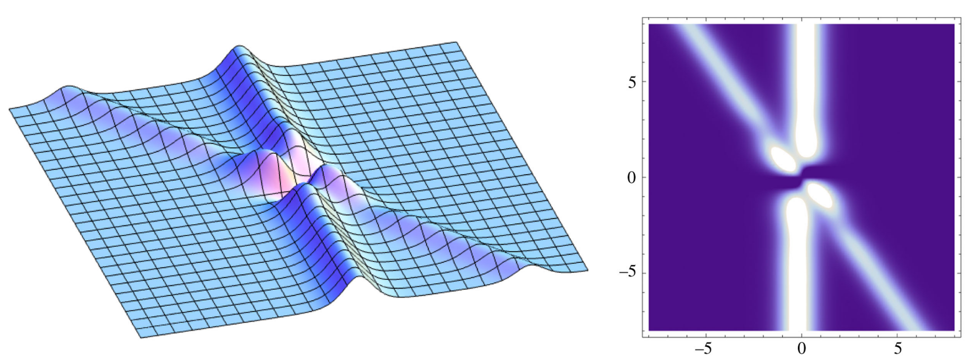
B. Two-soliton solution

For two-soliton solutions, the shape and density plots of the two-dimensional slice of NLσM action density are illustrated in Fig. 4, where we can clearly see there is a gap in the interacting region. This fact coincides with our conjecture that the two-soliton may be decomposed to two V-shape solitons by taking large phase-shift limits.

C. Double-pole solution

Now, we consider the case of double-pole solutions. According to the result in (5.11), the shape and density plots of the two-dimensional slice of NLσM action density are illustrated in Fig. 5. By taking the $\lambda_2 \to \lambda_1$ limit of two-line solitons together with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, the gap in the two-soliton interaction vanishes and becomes a hill. Note that in general, in the double-pole solution, the branches are asymptotically governed by the logarithmic function of z (e.g., see [49]). This is true of our case, as is seen in Sec. V B.

In Sec. V A, we claimed that the double-pole solution can be regarded as a limit of the two-soliton solution by choosing parameters (5.8). On the other hand, this fact can be verified by illustrating various two-soliton action density figures with λ_2 going to λ_1. In Fig. 6, we illustrate different plots of the two-soliton, with $|\lambda_2 - \lambda_1|$ becoming extremely small. This result coincides with the plot in Fig. 5.



(a) (b)

FIG. 4. Plots of the 2D slice of two-soliton NLσM action density with $\lambda_1 = -1 + i, \lambda_2 = 0.5 + 0.5i, \alpha_1 = \alpha_2 = 0.5 - 0.5i$, $\beta_1 = \beta_2 = -0.7 - 1.4i$, $(z, \tilde{z}) \in [-8, 8] \times [-8, 8]$, and $(w, \tilde{w}) = (0, 0)$. (a) Shape of the 2D slice of NLσM action density. (b) Density plot of the 2D slice of NLσM action density.

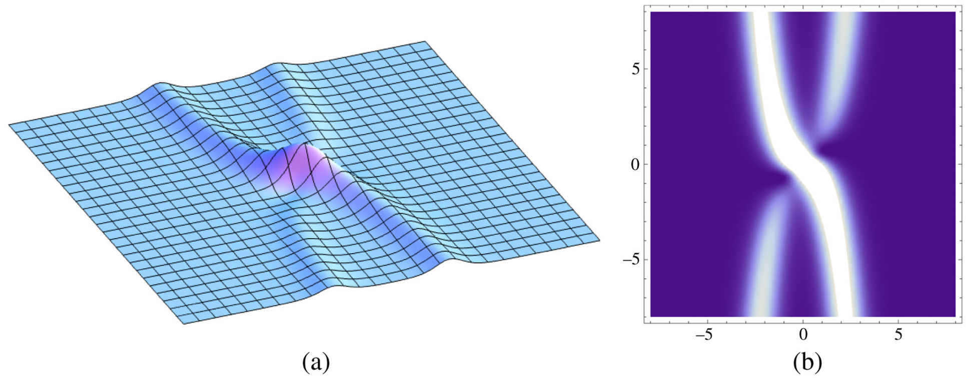


FIG. 5. Plots of the 2D slice of double-pole soliton NLσM action density with $\lambda_1 = -1 + i, \alpha = 0.5 - 0.5i, \beta = -0.7 - 1.4i$, $(z, \tilde{z}) \in [-8, 8] \times [-8, 8]$, and $(w, \tilde{w}) = (0, 0)$. (a) Shape of the 2D slice of NLσM action density. (b) Density plot of the 2D slice of NLσM action density.

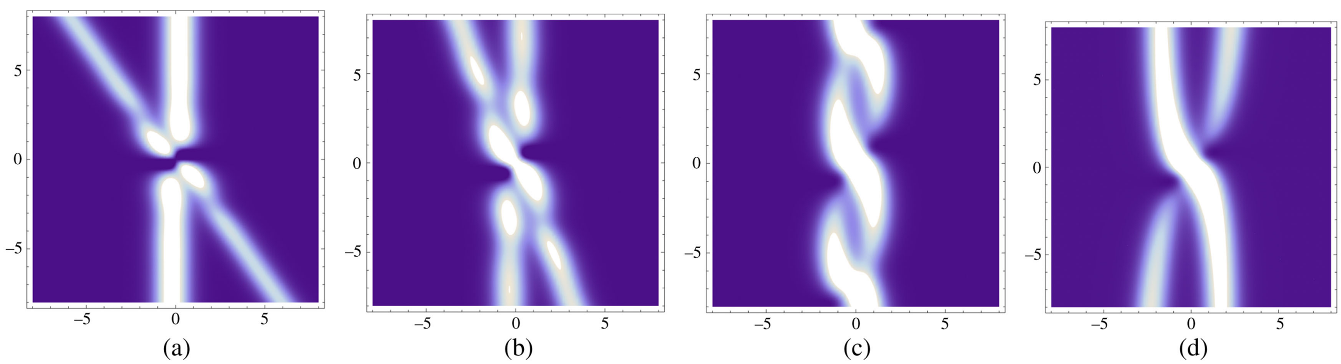


FIG. 6. Density plots of the 2D slice of two-soliton NLσM action density with $\alpha_1 = \alpha_2 = 0.5 - 0.5i, \beta_1 = \beta_2 = -0.7 - 1.4i$, $(z, \tilde{z}) \in [-8, 8] \times [-8, 8]$, and $(w, \tilde{w}) = (0, 0)$. (a) Plot with $\lambda_1 = -1 + i$ and $\lambda_2 = 0.5 + 0.5i$. (b) Plot with $\lambda_1 = -1 + i$ and $\lambda_2 = 0.5i$. (c) Plot with $\lambda_1 = -1 + i$ and $\lambda_2 = -0.6 + 0.5i$. (d) Plot with $\lambda_1 = -1 + i$ and $\lambda_2 = -0.9 + 0.8i$.

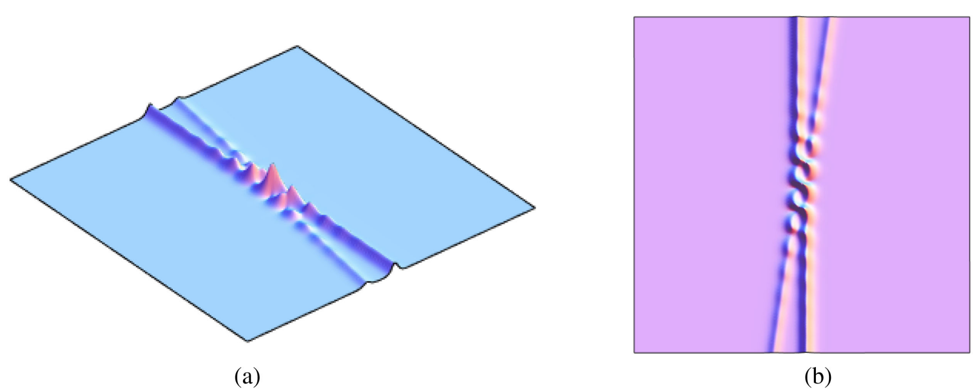


FIG. 7. Shape of the two-soliton interaction in Fig. 6(c), with $(z, \tilde{z}) \in [-50, 50] \times [-25, 25]$. (a) Oblique projection. (b) Top view.

Notice that in Fig. 6(c) [see also in Fig. 7, as Fig. 6(c) is a zoom in of Fig. 7], the two-soliton interaction behaves like a DNA double-helix structure, which exhibits periodical behavior in the interaction area.

D. V-shape solution

As we mentioned in Sec. IV B, the V-shape solution appears when we take the large phase-shift limit of a two-soliton. From (4.11), the phase-shift factor $\tilde{\delta}$ grows when

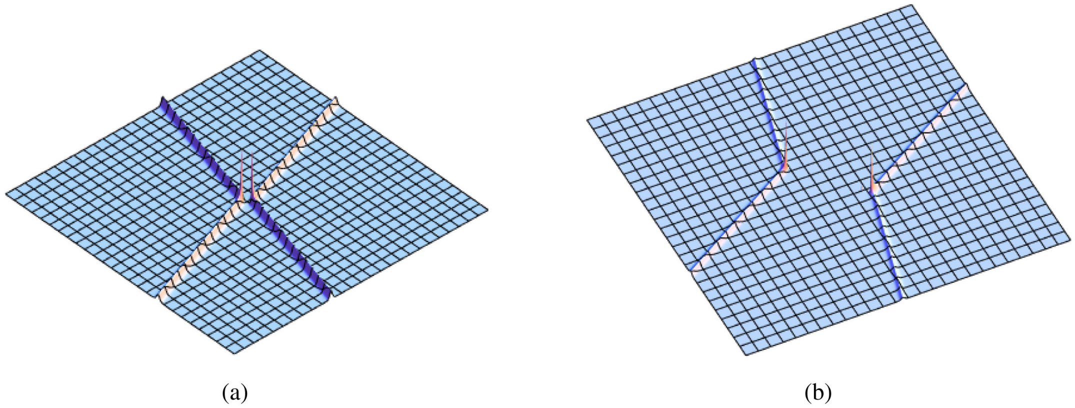


FIG. 8. Plots of the 2D slice of V-shape NLσM action density with $\alpha_1 = \alpha_2 = 0.5–0.7i, \beta_1 = -0.5 - 1.5i, \beta_2 = 1.5 + 0.5i$, $(z, \tilde{z}) \in [-20, 20] \times [-20, 20]$, and $(w, \tilde{w}) = (0, 0)$. (a) The case of $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1–1.5i$. (b) The case of $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1–1.9999999i$.

we take the limit $\lambda_2 \to \bar{\lambda}_1$. In the case (2) of (4.12), the NLσM action density $\mathcal{L}_\sigma$ becomes zero when $\lambda_2 \to \bar{\lambda}_1$ and, during this procedure, the gap in the two-soliton interaction grows larger, hence two V-shape solitons will emerge.

VII. CONCLUSION AND DISCUSSION

In this paper, we discovered resonance soliton solutions in the WZW$_4$ model in ultrahyperbolic space $\mathbb{U}$ from the perspective of the CMA and the binary Darboux transformation. We found that the exact solutions in [38] can be rewritten as the quasi-Grammian form, so that some quasideterminant techniques can be applied to investigate various relations between the CMA [38] and the binary Darboux transformation. According to our present research, the CMA appears to be a particular class of the binary Darboux transformation in [26], and we established a generalized CMA, which admits simple input data for the soliton solutions. For the one-soliton and two-soliton cases, we found that the quasi-Grammian and quasi-Wronskian soliton solutions gave the same action densities. It would be worthwhile to investigate whether this is a general result for n-solitons.

One highlight of this paper is that we found new classical solutions in the WZW$_4$ model by observing resonance limits from two-soliton solutions. One resonance solution was the double-pole solution. The NLσM action density of this solution (5.11) revealed a new nonlinear phenomenon, different from the two-soliton case (4.8). Furthermore, by considering the large phase-shift limit of two-solitons, we discovered V-shape solitons in pairs. This suggests pair annihilations or creations of two-line solitons in the open $N = 2$ string theory.

In the classification of the KP line solitons, Y-shape resonance solitons are building blocks of the soliton interaction diagrams and are essential. Likewise, we can expect that the V-shape solitons would be building blocks to classify the ASDYM solitons, including the resonance processes. Furthermore, V-shape solitons also exist in the CBS equation [50] and Zakharov systems [51], both of which can be derived from the ASDYM equation by dimensional reduction [52,53]. Therefore, the hidden symmetry behind the ASDYM equation would be described by the 2-toroidal algebra [20–22] because the toroidal algebras are the hidden symmetry behind the CBS equation [50] and Zakharov systems [51].

On the other hand, the hidden symmetry behind the KP equations is described by $\mathrm{W}_{1+\infty}$ algebras [54–57]. There seems to be no solutions of either V-shape solitons or smooth multiple-pole solitons for the KP equations (exact speaking the KPII equations).[4] The KP equations have not yet been derived from the ASDYM equations by reduction.[5] Therefore, our present results would suggest new aspects different from the KP solitons and lead to a new classification theory of the ASDYM solitons.

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[4]There are two kinds of KP equations: the KPI and KPII equations. The latter describe shallow water waves and, therefore, are popular in the study of integrable systems. In this paper, we refer to the KPII equations only. For the KPI equation, there exists smooth multiple-pole solutions [58].

[5]If we allow operator-valued gauge fields, the KP equation can be reduced from the ASDYM equation [59]. However, in this case, it lacks any twistor descriptions and, hence, is not considered as an example of the Ward conjecture.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PROOF OF LEMMA 3.1

For the Sylvester equation [Eq. (3.1)] and the differential recurrence (3.15), we have

$$K(\partial_{x_j}M)-(\partial_{x_j}M)L=(\partial_{x_j}r)s^T+r(\partial_{x_j}s^T) \tag{A1}$$

and

$$\begin{aligned}K(\partial_{x_{j+1}}M)-(\partial_{x_{j+1}}M)L&=(\partial_{x_{j+1}}r)s^T+r(\partial_{x_{j+1}}s^T)\\&=K(\partial_{x_j}r)s^T+r(\partial_{x_j}s^T)L.\end{aligned} \tag{A2}$$

By calculating $K\times(\mathrm{A.1})+(\mathrm{A.1})\times L$ and using (A2), one obtains

$$\begin{aligned}K^2(\partial_{x_j}M)-(\partial_{x_j}M)L^2&=K(\partial_{x_j}r)s^T+Kr(\partial_{x_j}s^T)\\&\quad+(\partial_{x_j}r)s^TL+r(\partial_{x_j}s^T)L\\&=K(\partial_{x_{j+1}}M)-(\partial_{x_{j+1}}M)L\\&\quad+(\partial_{x_j}r)s^TL+Kr(\partial_{x_j}s^T),\end{aligned} \tag{A3}$$

which implies

$$\begin{aligned}&K[(\partial_{x_{j+1}}M)+r(\partial_{x_j}s^T)-K(\partial_{x_j}M)]\\&\quad-[(\partial_{x_{j+1}}M)-(\partial_{x_j}r)s^T-(\partial_{x_j}M)L]L=0.\end{aligned} \tag{A4}$$

Let us define a new matrix C as

$$\begin{aligned}C&:=\partial_{x_{j+1}}M+r(\partial_{x_j}s^T)-K(\partial_{x_j}M)\\&\overset{(\mathrm{A.1})}{=}\partial_{x_{j+1}}M-(\partial_{x_j}r)s^T-(\partial_{x_j}M)L.\end{aligned} \tag{A5}$$

For the Sylvester equation $KC-CL=0$, where K and L do not have the same eigenvalues, $C=0$ is the unique solution. Thus,

$$\begin{aligned}\partial_{x_{j+1}}M&=(\partial_{x_j}r)s^T+(\partial_{x_j}M)L\\&=K(\partial_{x_j}M)-r(\partial_{x_j}s^T).\end{aligned} \tag{A6}$$

■

APPENDIX B: CALCULATION OF THE TWO-SOLITON

Substituting (4.6) into the Sylvester equation [Eq. (3.36)], we can obtain the entries of $\Omega:=(\Omega_{jk})$ as

$$\Omega_{jk}=\frac{(\bar{a}_ja_k)^2e^{\bar{\xi}_j+\xi_k}+(\bar{b}_jb_k)^2e^{-(\bar{\xi}_j+\xi_k)}}{\bar{\lambda}_j-\lambda_k},\qquad j,k=1,2. \tag{B1}$$

The determinant of Ω is

$$|\Omega|=\frac{\left\{\begin{array}{l}c_1[|a_1|^4|a_2|^4e^{X_1+X_2}+|b_1|^4|b_2|^4e^{-(X_1+X_2)}]\\+c_2[|a_1|^4|b_2|^4e^{X_1-X_2}+|a_2|^4|b_1|^4e^{-(X_1-X_2)}]\\+c_3[(a_1\bar{a}_2\bar{b}_1b_2)^2e^{i\Theta_{12}}+(\bar{a}_1a_2b_1\bar{b}_2)^2e^{-i\Theta_{12}}]\end{array}\right\}}{c_2c_3}, \tag{B2}$$

where

$$\begin{aligned}&c_1:=(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2),\qquad c_2:=(\lambda_1-\bar{\lambda}_2)(\bar{\lambda}_1-\lambda_2),\\&c_3:=(\lambda_1-\bar{\lambda}_1)(\lambda_2-\bar{\lambda}_2),\end{aligned} \tag{B3a}$$

$$X_j:=\xi_j+\bar{\xi}_j,\qquad \Theta_j:=-i(\xi_j-\bar{\xi}_j),\qquad \Theta_{jk}:=\Theta_j-\Theta_k. \tag{B3b}$$

Substituting (4.6) and (B1) into (3.37) and then applying (3.7) and (B2), we have

$$\hat{J}:=\frac{1}{\Delta}\begin{pmatrix}\Delta_{11}&\Delta_{12}\\\Delta_{21}&\Delta_{22}\end{pmatrix}, \tag{B4}$$

where

$$\Delta \coloneqq \overline{\lambda_1}\,\overline{\lambda_2}\,c_2 c_3 |\Omega|,$$

$$\Delta_{11} \coloneqq \left\{ \begin{array}{c} c_1[\lambda_1\lambda_2|a_1|^4|a_2|^4 e^{X_1+X_2} + \bar{\lambda}_1\bar{\lambda}_2|b_1|^4|b_2|^4 e^{-(X_1+X_2)}] \\ +c_2[\lambda_1\bar{\lambda}_2|a_1|^4|b_2|^4 e^{X_1-X_2} + \bar{\lambda}_1\lambda_2|a_2|^4|b_1|^4 e^{-(X_1-X_2)}] \\ +c_3[|\lambda_1|^2(a_1\bar{a}_2\bar{b}_1 b_2)^2 e^{i\Theta_{12}} + |\lambda_2|^2(\bar{a}_1 a_2 b_1\bar{b}_2)^2 e^{-i\Theta_{12}}] \end{array} \right\}, \qquad \Delta_{22} = \bar{\Delta}_{11},$$

$$\Delta_{12} \coloneqq \left\{ \begin{array}{c} (a_2\bar{b}_2)^2[c_4|a_1|^4 e^{X_1+i\Theta_2} - \bar{c}_4|b_1|^4 e^{-X_1+i\Theta_2}] \\ +(a_1\bar{b}_1)^2[c_5|a_2|^4 e^{X_2+i\Theta_1} - \bar{c}_5|b_2|^4 e^{-X_2+i\Theta_1}] \end{array} \right\}, \qquad \Delta_{21} \coloneqq -\bar{\Delta}_{12},$$

$$c_4 \coloneqq \bar{\lambda}_1(\lambda_1-\lambda_2)(\lambda_1-\bar{\lambda}_2)(\lambda_2-\bar{\lambda}_2), \qquad c_5 \coloneqq \bar{\lambda}_2(\lambda_1-\lambda_2)(\lambda_1-\bar{\lambda}_1)(\bar{\lambda}_1-\lambda_2). \tag{B5}$$

The NLσM action density is

$$\mathcal{L}_\sigma = -\frac{1}{16\pi}\mathrm{Tr}[(\partial_\mu \hat{J})\hat{J}^{-1}(\partial^\mu \hat{J})\hat{J}^{-1}]$$
$$= \frac{\left\{ \begin{array}{l} c_1 c_2\left[d_{11}\left(\frac{|a_2|^2}{|b_2|^2}e^{X_2} + \frac{|b_2|^2}{|a_2|^2}e^{-X_2}\right)^2 + d_{22}\left(\frac{|a_1|^2}{|b_1|^2}e^{X_1} + \frac{|b_1|^2}{|a_1|^2}e^{-X_1}\right)^2\right] \\ +c_1c_3\left[\begin{array}{l} d_{12}\left(\frac{\bar{a}_1 a_2}{\bar{b}_1 b_2}e^{\frac{X_1+X_2-i\Theta_{12}}{2}} + \frac{\bar{b}_1 b_2}{\bar{a}_1 a_2}e^{-(\frac{X_1+X_2-i\Theta_{12}}{2})}\right)^2 \\ +\bar{d}_{12}\left(\frac{a_1\bar{a}_2}{b_1\bar{b}_1}e^{\frac{X_1+X_2+i\Theta_{12}}{2}} + \frac{b_1\bar{b}_2}{a_1\bar{a}_2}e^{-(\frac{X_1+X_2+i\Theta_{12}}{2})}\right)^2 \end{array}\right] \\ -c_2c_3\left[\begin{array}{l} f_{12}\left(\frac{\bar{a}_1\bar{b}_2}{\bar{a}_2\bar{b}_1}e^{\frac{X_1-X_2-i\Theta_{12}}{2}} - \frac{\bar{a}_2\bar{b}_1}{\bar{a}_1\bar{b}_2}e^{-(\frac{X_1-X_2-i\Theta_{12}}{2})}\right)^2 \\ +\bar{f}_{12}\left(\frac{a_1 b_2}{a_2 b_1}e^{\frac{X_1-X_2+i\Theta_{12}}{2}} - \frac{a_2 b_1}{a_1 b_2}e^{-(\frac{X_1-X_2+i\Theta_{12}}{2})}\right)^2 \end{array}\right] \end{array} \right\}}{2\pi\left[\begin{array}{l} c_1\left(\frac{|a_1|^2|a_2|^2}{|b_1|^2|b_2|^2}e^{X_1+X_2} + \frac{|b_1|^2|b_2|^2}{|a_1|^2|a_2|^2}e^{-(X_1+X_2)}\right) \\ +c_2\left(\frac{|a_1|^2|b_2|^2}{|a_2|^2|b_1|^2}e^{X_1-X_2} + \frac{|a_2|^2|b_1|^2}{|a_1|^2|b_2|^2}e^{-(X_1-X_2)}\right) \\ +c_3\left(\frac{a_1\bar{a}_2\bar{b}_1 b_2}{\bar{a}_1 a_2 b_1\bar{b}_2}e^{i\Theta_{12}} + \frac{\bar{a}_1 a_2 b_1\bar{b}_2}{a_1\bar{a}_2\bar{b}_1 b_2}e^{-i\Theta_{12}}\right) \end{array}\right]^2}, \tag{B6}$$

where

$$d_{jk} \coloneqq \frac{(\alpha_j\bar{\beta}_k - \beta_j\bar{\alpha}_k)(\lambda_j-\bar{\lambda}_k)^3}{\lambda_j\bar{\lambda}_k}, \qquad f_{jk} \coloneqq \frac{(\alpha_j\beta_k - \beta_j\alpha_k)(\lambda_j-\lambda_k)^3}{\lambda_j\lambda_k}. \tag{B7}$$

For simplicity, we define

$$\delta_1 \coloneqq \frac{|a_1|^2}{|b_1|^2}, \qquad \delta_2 \coloneqq \frac{|a_2|^2}{|b_2|^2}, \qquad \delta_3 \coloneqq \frac{a_1\bar{a}_2}{b_1\bar{b}_2}, \qquad \delta_4 \coloneqq \frac{a_1 b_2}{a_2 b_1}, \tag{B8}$$

which implies

$$|\delta_3|^2 = \delta_1\delta_2, \qquad |\delta_4|^2 = \frac{\delta_1}{\delta_2}, \qquad \frac{\delta_3}{|\delta_3|} = \frac{\delta_4}{|\delta_4|}. \tag{B9}$$

Therefore, we can rewrite δ_3 and δ_4 in the polar form

$$\delta_3 = (\delta_1\delta_2)^{\frac{1}{2}} e^{\frac{i\phi}{2}}, \qquad \delta_4 = \left(\frac{\delta_1}{\delta_2}\right)^{\frac{1}{2}} e^{\frac{i\phi}{2}}. \tag{B10}$$

APPENDIX C: NONUNITARY TWO-SOLITON

Below are miscellaneous results on nonunitary two-soliton solutions with complex valued action density.

In this case, we consider the unreduced nonunitary solution (3.30) in Theorem 3.4. The input data are not only (θ, Λ) in (4.6) but also (ρ, Ξ), given by

$$\rho = \begin{pmatrix} a_1^2 e^{\eta_1} & a_2^2 e^{\eta_2} \\ b_1^2 e^{-\eta_1} & b_2^2 e^{-\eta_2} \end{pmatrix}, \qquad \Xi = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \tag{C1}$$

where

$$\eta_j \coloneqq \mu_j \bar{\alpha}_j z + \bar{\beta}_j \tilde{z} + \mu_j \bar{\beta}_j w + \bar{\alpha}_j \tilde{w}, \quad j = 1, 2. \tag{C2}$$

The NLσM action density for the nonunitary $\hat{J}$ in Sec. IV C is

$$\begin{aligned} \mathcal{L}_\sigma &= -\frac{1}{16\pi} \mathrm{Tr}[(\partial_\mu \hat{J})\hat{J}^{-1}(\partial^\mu \hat{J})\hat{J}^{-1}] \\ &= \frac{\left\{ \begin{array}{l} c_1 c_2 [d_{11} \cosh^2 \tilde{X}_2 + d_{22} \cosh^2 \tilde{X}_1] \\ + c_1 c_3 \left[d_{12} \cosh^2 \left(\frac{\tilde{X}_1 + \tilde{X}_2 - i\tilde{\Theta}_{12}}{2} \right) + \tilde{d}_{12} \cosh^2 \left(\frac{\tilde{X}_1 + \tilde{X}_2 + i\tilde{\Theta}_{12}}{2} \right) \right] \\ - c_2 c_3 \left[f_{12} \sinh^2 \left(\frac{\tilde{X}_1 - \tilde{X}_2 - i\tilde{\Theta}_{12}}{2} \right) + \tilde{f}_{12} \sinh^2 \left(\frac{\tilde{X}_1 - \tilde{X}_2 + i\tilde{\Theta}_{12}}{2} \right) \right] \end{array} \right\}}{2\pi [c_1 \cosh(\tilde{X}_1 + \tilde{X}_2) + c_2 \cosh(\tilde{X}_1 - \tilde{X}_2) + c_3 \cos \tilde{\Theta}_{12}]^2}, \end{aligned} \tag{C3}$$

where

$$c_1 \coloneqq (\lambda_1 - \lambda_2)(\mu_1 - \mu_2), \qquad c_2 \coloneqq (\lambda_1 - \mu_2)(\mu_1 - \lambda_2), \qquad c_3 \coloneqq (\lambda_1 - \mu_1)(\lambda_2 - \mu_2), \tag{C4a}$$

$$d_{12} \coloneqq \frac{(\alpha_1 \bar{\beta}_2 - \beta_1 \bar{\alpha}_2)(\lambda_1 - \mu_2)^3}{\lambda_1 \mu_2}, \qquad f_{12} \coloneqq \frac{(\alpha_1 \beta_2 - \beta_1 \alpha_2)(\lambda_1 - \lambda_2)^3}{\lambda_1 \lambda_2}, \tag{C4b}$$

$$\tilde{d}_{12} \coloneqq \frac{(\alpha_1 \bar{\beta}_2 - \beta_1 \bar{\alpha}_2)(\lambda_1 - \mu_2)^3}{\lambda_1 \mu_2}, \qquad \tilde{f}_{12} \coloneqq \frac{(\bar{\alpha}_1 \bar{\beta}_2 - \bar{\beta}_1 \bar{\alpha}_2)(\mu_1 - \mu_2)^3}{\mu_1 \mu_2}, \tag{C4c}$$

$$\tilde{X}_j \coloneqq \xi_j + \eta_j + \log \delta_j, \qquad \delta_j \coloneqq |a_j|^2/|b_j|^2, \tag{C4d}$$

$$\tilde{\Theta}_{12} \coloneqq \Theta_1 - \Theta_2 + \phi, \qquad \Theta_j \coloneqq -i(\xi_j - \eta_j), \quad j = 1, 2, \tag{C4e}$$

$$\phi \coloneqq 2\mathrm{Arg}(a_1 \bar{a}_2 / b_1 \bar{b}_2) = 2\mathrm{Arg}(a_1 b_2 / a_2 b_2). \tag{C4f}$$

APPENDIX D: DATA OF THE DOUBLE-POLE SOLUTION

Substituting (5.9) into (3.36), we get the explicit form of $\Omega \coloneqq (\Omega_{ij})$, where

$$\begin{aligned} \Omega_{11} &= \frac{-2}{(\lambda - \bar{\lambda})^3} \{ [(\lambda - \bar{\lambda})^2 |\xi'|^2 - 2] \cosh(\xi + \bar{\xi}) + (\lambda - \bar{\lambda})(\xi' - \bar{\xi}') \sinh(\xi + \bar{\xi}) \}, \\ \Omega_{12} &= \frac{-2}{(\lambda - \bar{\lambda})^2} [(\lambda - \bar{\lambda}) \bar{\xi}' \sinh(\xi + \bar{\xi}) + \cosh(\xi + \bar{\xi})], \\ \Omega_{21} &= \frac{-2}{(\lambda - \bar{\lambda})^2} [(\lambda - \bar{\lambda}) \xi' \sinh(\xi + \bar{\xi}) - \cosh(\xi + \bar{\xi})], \\ \Omega_{22} &= \frac{-2}{\lambda - \bar{\lambda}} \cosh(\xi + \bar{\xi}). \end{aligned} \tag{D1}$$

The determinant of Ω is

$$|\Omega| = \frac{4}{(\lambda-\bar{\lambda})^4}[(\lambda-\bar{\lambda})^2|\xi'|^2 - \cosh^2(\xi+\bar{\xi})]. \tag{D2}$$

Substituting (5.9), (D1), and (D2) into (3.37), we get the explicit form of the J matrix

$$\hat{J} \coloneqq \frac{1}{\Delta}\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix},$$

where

$$\begin{aligned}
&\Delta \coloneqq 2\bar{\lambda}^2\left\{\frac{1}{2}[1+\cosh(2(\xi+\bar{\xi}))] - (\lambda-\bar{\lambda})^2|\xi'|^2\right\}\\
&\Delta_{11} \coloneqq \left\{\begin{array}{l}(\lambda-\bar{\lambda})^2[(\lambda\xi'-\bar{\lambda}\bar{\xi}') - 2|\lambda|^2|\xi'|^2]\\ +\frac{(\lambda^2+\bar{\lambda}^2)}{2}[1+\cosh(2(\xi+\bar{\xi}))] + \frac{(\lambda^2-\bar{\lambda}^2)}{2}\sinh(2(\xi+\bar{\xi}))\end{array}\right\}, \qquad \Delta_{22} = \bar{\Delta}_{11},\\
&\Delta_{12} \coloneqq (\lambda-\bar{\lambda})^2(\bar{\lambda}\bar{\xi}'e^{2\xi} + \lambda\xi' e^{-2\bar{\xi}}) + (\lambda^2-\bar{\lambda}^2)\cosh(\xi+\bar{\xi})e^{\xi-\bar{\xi}}, \qquad \Delta_{21} = -\bar{\Delta}_{12}.\\
&\partial_z\Delta = -2\bar{\lambda}^2\{(\lambda+\bar{\lambda})^2(\alpha\bar{L}'+\bar{\alpha}L') + (\lambda\alpha+\bar{\lambda}\,\bar{\alpha})\sinh[2(L+\bar{L})]\},\\
&\partial_{\bar{z}}\Delta = -2\lambda^2(\beta+\bar{\beta})\sinh[2(L+\bar{L})],\\
&\partial_w\Delta = \partial_z\Delta|_{\alpha\to\beta}, \qquad \partial_{\bar{w}}\Delta = \partial_{\bar{z}}\Delta|_{\beta\to\alpha}.\\
&\partial_z\Delta_{11} = \left\{\begin{array}{l}(\lambda+\bar{\lambda})^2[(\lambda\alpha-\bar{\lambda}\,\bar{\alpha}) - 2|\lambda|^2(\alpha\bar{L}'+\bar{\alpha}L')]\\ +(\lambda\alpha+\bar{\lambda}\,\bar{\alpha})[(\lambda^2+\bar{\lambda}^2)\sinh[2(L+\bar{L})] + (\lambda^2-\bar{\lambda}^2)\cosh[2(L+\bar{L})]]\end{array}\right\}\\
&\partial_{\bar{z}}\bar{\Delta}_{11} = (\beta+\bar{\beta})[(\lambda^2+\bar{\lambda}^2)\sinh[2(L+\bar{L})] - (\lambda^2-\bar{\lambda}^2)\cosh[2(L+\bar{L})]],\\
&\partial_w\Delta_{11} = \partial_z\Delta_{11}|_{\alpha\to\beta}, \qquad \partial_{\bar{w}}\bar{\Delta}_{11} = \partial_{\bar{z}}\bar{\Delta}_{11}|_{\beta\to\alpha}.\\
&\partial_z\Delta_{12} = \left\{\begin{array}{l}(\lambda+\bar{\lambda})^2[(\bar{\lambda}\,\bar{\alpha}+2|\lambda|^2\alpha\bar{L}')e^{2L} + (\lambda\alpha-2|\lambda|^2\bar{\alpha}L')e^{-2\bar{L}}]\\ +(\lambda^2-\bar{\lambda}^2)[(\lambda\alpha+\bar{\lambda}\,\bar{\alpha})\sinh[L+\bar{L}] + (\lambda\alpha-\bar{\lambda}\,\bar{\alpha})\cosh[L+\bar{L}]]e^{L-\bar{L}}\end{array}\right\},\\
&\partial_{\bar{z}}\bar{\Delta}_{12} = \left\{\begin{array}{l}2(\lambda+\bar{\lambda})^2(\lambda\bar{\beta}L'e^{2\bar{L}} - \bar{\lambda}\beta\bar{L}'e^{-2L})\\ -(\lambda^2-\bar{\lambda}^2)[(\beta+\bar{\beta})\sinh(L+\bar{L}) - (\beta-\bar{\beta})\cosh(L+\bar{L})]e^{-(L-\bar{L})}\end{array}\right\},\\
&\partial_w\Delta_{12} = \partial_z\Delta_{12}|_{\alpha\to\beta}, \qquad \partial_{\bar{w}}\bar{\Delta}_{12} = \partial_{\bar{z}}\bar{\Delta}_{12}|_{\beta\to\alpha}.
\end{aligned}$$

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