

CONTINUITY METHOD FOR THE MABUCHI SOLITON ON THE EXTREMAL FANO MANIFOLDS

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ABSTRACT. We run the continuity method for Mabuchi's generalization of Kähler-Einstein metrics, assuming the existence of an extremal Kähler metric. It gives an analytic proof (without minimal model program) of the recent existence result obtained by Apostolov, Lahdili and Nitta. Our key observation is the boundedness of the energy functionals along the continuity method. The same argument can be applied to general g -solitons and g -extremal metrics.

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1. INTRODUCTION

The celebrated work of Chen-Donaldson-Sun [15] and Tian [41] states that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable. There are several generalization of Kähler-Einstein metrics, which may exist even if the Fano manifold is obstructed by Futaki's invariant [22]. One is the extremal Kähler metric introduced by Calabi [13]. It is defined in terms of the scalar curvature so that make sense for general polarized manifolds. In [37] Mabuchi introduced an analogous notion, called the Mabuchi soliton, in terms of the Ricci potential function. If the Futaki invariant vanishes, it is classically known that these two notions coincide with the Kähler-Einstein metric. As these are different metrics in general, one may only ask if the existence conditions are equivalent. This is the main issue discussed in this article.

The one direction is relatively clear; if the Fano manifold admits a Mabuchi soliton, there exists an extremal metric, as the second author pointed out in [35, Section 9.6] for

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example. See also [28, 43]. The recent result [2] states that the converse is also true if one assumes the condition on the Mabuchi constant (see Theorem 1.4). The proof in [2] is intricate as it exploits the deep result of the minimal model program in algebraic geometry.

The aim of the present paper is to provide a direct analytic proof for the existence of the Mabuchi soliton. The strategy is completely different from [2] and includes new arguments for the continuity method for the Mabuchi soliton which are a boundedness argument (Theorem 1.3) and a related compactness argument (Section 5.2). These new argument are inevitable since the monotonicity property of the energy along such path might not be expected as discussed after Theorem 1.3.

Our argument can be applied to general *g-soliton metric* in the sense of [9], [25]. In order to describe the general situation, let X be an n -dimensional Fano manifold and fix a maximal compact torus $T \subset \text{Aut}(X)$. Since the T -action lifts to the anti-canonical bundle, we have the moment map $\mu_\omega: X \rightarrow \mathfrak{t}^*$ for any T -invariant Kähler metric $\omega \in 2\pi c_1(X)$. Let P be the associated weight polytope as the image of μ_ω . We take a smooth positive function $g: P \rightarrow \mathbb{R}$ and denote the composition by $g_\omega := g \circ \mu_\omega$. We also assume that g is log concave¹.

Definition 1.1. A T -invariant Kähler metric $\omega \in 2\pi c_1(X)$ is called a *g-soliton* [25] if it satisfies

$$(1.1) \quad \text{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} \log g_\omega.$$

It is called a *g-extremal* (c.f. [38]) if the scalar curvature satisfies

$$(1.2) \quad S_\omega - n = 1 - g_\omega.$$

When g is an affine function, each g -soliton and g -extremal metric yields the Mabuchi soliton and the extremal metric. In the case $g = e^l$ for some affine function l , g -soliton yields the Kähler-Ricci soliton. The g -extremal metric is the same notion as the $(1, n + 1 - g)$ -weighted constant scalar curvature Kähler metric in the sense of [32].

As it was observed in Mabuchi's original paper [37], the constant

$$(1.3) \quad m_X := \sup_X (1 - g_\omega)$$

is independent of the metrics and $m_X < 1$ if there exists a g -soliton.

In order to construct a g -soliton, we introduce the following variant of the continuity method. Let us take a T -invariant reference metric $\omega_0 \in 2\pi c_1(X)$ and represent each T -invariant metric by a Kähler potential φ such that $\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$. The continuity

¹The log concavity assumption is only used in the regularity argument, which is based on [25], Proposition 3.8, for a solution of the continuity method (1.4) in section 3.2.1 and 5.2. One can also apply the argument in [19, 26]. We are not sure if the assumption can be relaxed.

path is then described as the Monge-Ampère equation:

$$(1.4) \quad g_{\omega_\varphi} \omega_\varphi^n = e^{-t\varphi + \rho_0} \omega_0^n \quad \text{for } t \in [0, 1].$$

The solution at $t = 1$ is nothing but a g -soliton. Such a continuity path is originated from [3], where the path is formulated for the Kähler-Einstein metrics. See also [42] for the Kähler-Ricci soliton, [34] for the Mabuchi soliton and [18] for more general soliton type metrics. Our main result is the following.

Theorem 1.2. *Let \mathcal{T} be the set of $t \in [0, 1]$ such that the equation (1.4) has a T -invariant solution. In general \mathcal{T} is non-empty and open. If one assume that X admits a g -extremal metric and that the Mabuchi constant enjoys $m_X < 1$, then $\mathcal{T} = [0, 1]$.*

Recall that the g -extremal metric can be characterized as a critical point of the g -Mabuchi functional M_g (see (2.24) for the definition). The following is a key to prove the closedness of \mathcal{T} .

Theorem 1.3. *Assume $m_X < 1$ and fix any $\varepsilon \in (0, 1)$. Let ω_t be a T -invariant solution of (1.4) at $t \in (\varepsilon, 1]$. There exists a constant $C > 0$ independent of ε and t satisfying*

$$M_g(\omega_t) \leq C\varepsilon^{-1}.$$

We emphasize that the critical point of M_g is a g -extremal metric which is different from the g -soliton unless $g \equiv 1$. When $g \equiv 1$, a g -extremal metric and a g -soliton coincide with a Kähler-Einstein metric, and Bando-Mabuchi [4] showed that $M_1(\omega_t)$ is monotonically decreasing in t . However, for general g , we cannot expect the monotonicity of $M_g(\omega_t)$. Our strategy for Theorem 1.3 is to give a quantitative bound. The difficulty of general g case is caused by the difference between M_g and another Mabuchi type functional F_g in the literatures. The energy F_g is defined as the free energy of the Monge-Ampère measure with g -density: $g_\omega \omega^n$, and the critical point is the g -soliton. See Remark 2.2.

The g -soliton is a critical point of the g -Ding functional (see (2.24) for the definition). Note that, unlike the Kähler-Einstein case ($g \equiv 1$ case), the currently known methods can not quantitatively compare the coercivity conditions for the two canonical energy functionals D_g and M_g , either.

As a consequence of Theorem 1.2, we have the following.

Theorem 1.4. *For any Fano manifold X the followings are equivalent.*

- (1) *X admits a g -soliton.*
- (2) *It satisfies $m_X < 1$ and is uniformly g -relatively D -stable.*
- (3) *It satisfies $m_X < 1$ and admits a g -extremal metric in $c_1(X)$.*

Equivalence of (1) and (2) was proved by [25]. Thus Theorem 1.2 completes the proof of Theorem 1.4. Our result establishes a variant of the Yau-Tian-Donaldson correspondence for g -extremal metrics on Fano manifolds in an analytic way, which is even new for

extremal metrics in the sense of Calabi. Note that the result in [2] showed the implication (3) \Rightarrow (2) in an algebraic way.

We give some remarks for the case when g is affine which is one of the most interesting case. As it was shown by [43], the condition $m_X < 1$ follows from the stability. Note that the assumption $m_X < 1$ in (3) is necessary, as the second author pointed out in [35, Section 9.5]. In fact there exists a Fano 3-fold which admits an extremal metric in $2\pi c_1(X)$ but $m_X \geq 1$. By contrast, there exist Fano manifolds admitting no extremal metric (thus admitting no Mabuchi soliton) in the first Chern class. For example, one can find Fano 3-folds whose automorphisms are the additive group \mathbb{C}^+ in [40], and these are obstructed by the result [33], Lemma 1. More recently, a toric Fano 10-fold admitting no extremal metric in the first Chern class was found in [30].

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2. PRELIMINARY

2.1. g -solitons and g -extremal metrics. Throughout this paper we consider an n -dimensional Fano manifold X . We fix a maximal compact torus $T \subset \text{Aut}(X)$ and a T -invariant Kähler metric $\omega_0 \in 2\pi c_1(X)$. Let

$$(2.1) \quad \mathcal{H}(X, \omega_0)^T := \left\{ \varphi \in C^\infty(X, \mathbb{R}) : \omega_\varphi := \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ and } \varphi \text{ is } T\text{-invariant} \right\}$$

be the set of T -invariant Kähler potentials. Here each $\sigma \in T_{\mathbb{C}}$ sends φ to $\sigma[\varphi]$ which is defined by $\sigma^*\omega_\varphi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\sigma[\varphi]$. Note that $\sigma[\varphi]$ is defined only up to addition of a function, although this never matters in the following argument. The Ricci potential is a smooth function ρ_φ characterized by the property

$$(2.2) \quad \text{Ric}(\omega_\varphi) - \omega_\varphi = \sqrt{-1}\partial\bar{\partial}\rho_\varphi \quad \text{and} \quad \int_X (e^{\rho_\varphi} - 1)\omega_\varphi^n = 0.$$

We say that ω_φ is Kähler-Einstein if $\rho_\varphi = 0$. In terms of the scalar curvature $S_\varphi := \text{Tr}_{\omega_\varphi} \text{Ric}(\omega_\varphi)$, it is equivalent to say $S_\varphi = n$.

For a convenient let us fix an isomorphism $T \simeq (\mathbb{S}^1)^r$ with some non-negative integer r . For each $\alpha = 1, \dots, r$, let ξ_α be the holomorphic vector field on X generating the action of the α -th factor of $T_{\mathbb{C}} \simeq (\mathbb{C}^*)^r$. Once the metric $\varphi \in \mathcal{H}(X, \omega_0)^T$ is fixed, each vector

$\xi \in \mathfrak{t}_{\mathbb{C}}$ defines the Hamilton function $\theta_{\xi}(\varphi) \in C^{\infty}(X; \mathbb{R})$ which is characterized by the properties such that

$$(2.3) \quad i_{\xi}\omega_{\varphi} = \sqrt{-1}\bar{\partial}\theta_{\xi}(\varphi) \quad \text{and} \quad \int_X \theta_{\xi}(\varphi)\omega_{\varphi}^n = 0.$$

We set $\theta_{\alpha} := \theta_{\xi_{\alpha}}$. The moment map $\mu_{\varphi}: X \rightarrow \mathbb{R}^r$ is then described as $\mu_{\varphi}(x) = (\theta_1(x), \dots, \theta_r(x))$. It is well-known that the image $P := \mu_{\varphi}(X)$ defines a convex polytope in \mathbb{R}^r and independent of the choice of $\varphi \in \mathcal{H}(X, \omega_0)^T$.

We will also fix a smooth positive function $g: P_X \rightarrow \mathbb{R}$ and put $g_{\varphi} := g \circ \mu_{\varphi}$. We also assume that g is log concave. In what follows it is always normalized such that

$$(2.4) \quad \int_X (g_{\varphi} - 1)\omega_{\varphi}^n = 0$$

holds. Thanks to the compactness of X there exists a uniform constant $C > 0$ independent of the choice of $\varphi \in \mathcal{H}(X, \omega_0)^T$ such that

$$(2.5) \quad C^{-1} \leq g_{\varphi} \leq C$$

Definition 2.1. A T -invariant Kähler metric $\omega \in 2\pi c_1(X)$ is called a g -soliton [25] if it satisfies

$$(2.6) \quad \text{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}\log g_{\omega},$$

or equivalently, $e^{\rho_{\varphi}} = g_{\varphi}$. It is called g -extremal (c.f. [38]) if the scalar curvature satisfies

$$(2.7) \quad S_{\omega} - n = 1 - g_{\omega}.$$

The definition of g -soliton generalizes the notion of the Kähler-Einstein metric, the Kähler-Ricci soliton, and the Mabuchi soliton, as it was explained in the introduction. The g -extremal Kähler metric gives a generalization of extremal Kähler metrics. The notion of general weighted soliton is originated in [36]. In the work [9] consideration of general weighted density is motivated by the optimal transport theory.

Let us consider the particular case when g is an affine function. In this case ω_{φ} is g -soliton if and only if $e^{\rho_{\varphi}} - 1$ is contained in the space of (the above normalized) Hamilton functions of $T_{\mathbb{C}}$. We fix a Levi subgroup of $\text{Aut}_0(X)$, or may assume that $\text{Aut}_0(X)$ is reductive in our purpose. The vector field η corresponds to $\theta_{\eta}(\varphi) = 1 - e^{\rho_{\varphi}}$ is algebraically characterized by the property

$$(2.8) \quad \text{Fut}(\xi) + \langle \xi, \eta \rangle = 0,$$

where $\text{Fut}(\xi)$ denotes the Futaki invariant [22]. Similarly g -extremality defines a vector field with the property $\theta_{\eta}(\varphi) = S(\omega_{\varphi}) - n$. As it was classically known in [23, 37], these two notions are equivalent and we call η *the extremal vector field*. One can also check that

η is contained in the center of the Levi subgroup we fixed. As the first obstruction to the g -soliton metric, Mabuchi [37] introduced the invariant

$$(2.9) \quad m_X := \sup_{x \in X} \theta_\eta(\varphi)(x),$$

which is actually independent of the choice of $\varphi \in \mathcal{H}(X, \omega_0)^T$. The existence of the Mabuchi soliton forces $m_X < 1$, as the positivity of $e^{\rho_\varphi} = 1 - \theta_\eta(\varphi)$.

2.2. Energy functionals. Let us briefly review the geometry of the space $\mathcal{H}^T = \mathcal{H}(X, \omega_0)^T$ and the canonical energy functionals there. As it was first observed by Mabuchi, the tangent space of \mathcal{H}^T at φ is naturally identified with the smooth functions space $C^\infty(X : \mathbb{R})^T$ which equips the canonical L^p metric

$$(2.10) \quad \|u\|_p := \left[\frac{1}{V} \int_X |u|^p \omega_\varphi^n \right]^{\frac{1}{p}},$$

where $V = \int_X \omega_\varphi^n$ is the volume. Recent development of variational approach to the Monge-Ampère equation reveals that the topology of \mathcal{H}^T induced by the L^1 -metric is rather important. (See [8], [11] for this point.) We denote by d_1 the distance function of the above L^1 -metric structure. It is a bit confusing but note that the d_1 -convergence is much stronger than the ordinal convergence $\int_X |\varphi_j - \varphi| \omega_0^n \rightarrow 0$ of L^1 functions. The d_1 -topology is closely related to the Monge-Ampère energy

$$(2.11) \quad E(\varphi) = \frac{1}{(n+1)V} \sum_{i=0}^n \int_X \varphi \omega_0^i \wedge \omega_\varphi^{n-i},$$

which is pluripotential generalization of the Dirichlet energy and indeed characterized by the exterior derivative property

$$(2.12) \quad d_\varphi E = V^{-1} \omega_\varphi^n.$$

In fact the energy $E(\varphi) \in [-\infty, \infty)$ is defined for an arbitrary ω_0 -plurisubharmonic (psh in short) function φ . Extending the classical idea of [5], in [10] the authors also extended the definition of the Monge-Ampère product $V^{-1} \omega_\varphi^n$ to general ω_0 -psh functions, as a measure $\text{MA}(\varphi)$ which contains no mass on any pluripolar set. The differential formula (2.12) then still holds in the appropriate sense. See the textbook [24] for the exposition. According to the seminal work of Darvas [16, 17], the space of finite energy ω_0 -psh functions

$$(2.13) \quad \mathcal{E}^1(X, \omega_0)^T := \left\{ \varphi \in \text{PSH}(X, \omega_0)^T : \int_X \omega_\varphi^n = V, \quad \int_X \varphi \omega_\varphi^n > -\infty \right\}$$

precisely gives the completion of $\mathcal{H}(X, \omega_0)^T$ endowed with the distance d_1 .

According to the theory of [9], one can also define $V^{-1} g_\varphi \omega_\varphi^n$ as a non-pluripolar probability measure $\text{MA}_g(\varphi)$, for general ω_0 -psh φ and for a fixed positive function $g : P \rightarrow \mathbb{R}$ which

was discussed in the previous subsection. At the same time they defined the g -twisted Monge-Ampère energy which enjoys the property

$$(2.14) \quad d_\varphi E_g = V^{-1} g_\varphi \omega_\varphi^n,$$

at least for smooth φ . If one sets $g \equiv 1$ it recovers the original Monge-Ampère energy. It is then straightforward to extend the definition of the classical energy I and J of Aubin to this setting. For a smooth φ , they are written down as

$$(2.15) \quad I_g(\varphi) = \frac{1}{V} \int_X \varphi (g_0 \omega_0^n - g_\varphi \omega_\varphi^n),$$

$$(2.16) \quad J_g(\varphi) = \frac{1}{V} \int_X \varphi g_0 \omega_0^n - E_g(\varphi).$$

A simple integration by parts yields $0 \leq I \leq (n+1)(I - J) \leq nI$. According to (2.5), one has the uniform estimate $\frac{1}{C}I \leq I_g \leq CI$, $\frac{1}{C}J \leq J_g \leq CJ$ and $\frac{1}{C}(I - J) \leq (I_g - J_g) \leq C(I - J)$. The reader can consult with [25] for the detail computations.

Now we introduce the canonical energy functionals D and M which are originally introduced to study the Kähler-Einstein and the constant scalar curvature Kähler metric. The principal term of each energy is the L -functional and the relative entropy (of the two probability measure μ, μ_0), defined respectively as

$$(2.17) \quad L(\varphi) = -\log \frac{1}{V} \int_V e^{-\varphi + \rho_0} \omega_0^n \quad \text{and} \quad \text{Ent}(\mu|\mu_0) = \int_X \log \left(\frac{\mu}{\mu_0} \right) \mu.$$

The D-energy and the Mabuchi's K-energy are defined as

$$(2.18) \quad D(\varphi) := L(\varphi) - E(\varphi) \quad \text{and}$$

$$(2.19) \quad M(\varphi) := \text{Ent}(\text{MA}(\varphi)|\text{MA}(0)) + \frac{1}{V} \int_X \varphi \omega_\varphi^n - E(\varphi) + \frac{1}{V} \int_X \rho_0 (\omega_0^n - \omega_\varphi^n).$$

It is not so hard to check that the critical points of each energy give the Kähler-Einstein and cscK metrics, respectively. According to [6], these two energies can be related from the thermodynamical point of view. The reader can also consult with [8]. The Legendre duality between the functions and measures introduces the pluricomplex energy of the probability measure μ as

$$(2.20) \quad E^*(\mu) := \sup_{u \in \mathcal{E}(X, \omega_0)} \left[E(u) - \int_X u d\mu \right] \in (-\infty, \infty].$$

Then the Helmholtz free energy of the given probability measure μ is defined to be

$$(2.21) \quad F(\mu) := \text{Ent}(\mu|\mu_0) - E^*(\mu).$$

In this context D and M appear in the form

$$(2.22) \quad \text{Ent}(\mu|\mu_0) = \sup_{f \in C^0(X; \mathbb{R})} \left[\int_X f d\mu - \log \int_X e^f d\mu_0 \right],$$

$$(2.23) \quad M(\varphi) = F(\text{MA}(\varphi)),$$

where the background measure is chosen to be $\mu_0 = \text{MA}(0)$. The above relation is essential in showing the equivalence of the coercivity of two energies D and M .

Let us discuss the g -twisted version of the canonical energies, which are defined as

$$(2.24) \quad D_g = D + E - E_g \quad \text{and} \quad M_g = M + E - E_g.$$

Critical points of D_g and M_g gives g -solitons and g -extremal metric, respectively.

Remark 2.2. One can also define the g -twisted free energy $F_g(\varphi) := F(\text{MA}_g(\varphi))$ which is explicitly written down to the form

$$(2.25) \quad F_g(\varphi) := \text{Ent}(\text{MA}_g(\varphi) | \text{MA}_g(0)) + \frac{1}{V} \int_X \varphi g_\varphi \omega_\varphi^n - E_g(\varphi) + \frac{1}{V} \int_X \rho_0 (g_0 \omega_0^n - g_\varphi \omega_\varphi^n).$$

The functional plays an important role in the study of g -solitons, however, the critical point of F_g does not give the extremal metrics but g -solitons. See [25] for the detail. The fact $F_g \neq M_g$ for general g causes difficulty in comparing D_g with M_g . This is why we need to develop another approach in showing (3) \Rightarrow (1) of Theorem 1.4.

We here prepare for the later use an explicit difference between M_g and D_g .

Lemma 2.3. *For any $\varphi \in \mathcal{H}^T$,*

$$M_g(\varphi) - D_g(\varphi) = -\frac{1}{V} \int_X \rho_\varphi \omega_\varphi^n + \frac{1}{V} \int_X \rho_0 \omega_0^n.$$

In particular, $M_g(\varphi) - D_g(\varphi) \geq \frac{1}{V} \int_X \rho_0 \omega_0^n$.

Proof. By the definition of D_g and M_g , it suffice to show the equality for $g \equiv 1$. The equality for $g \equiv 1$ was firstly observed by Ding-Tian [20]. The inequality follows from the fact that $e^x \geq 1 + x$ and the normalization of the Ricci potential. \square

Nextly we introduce the coercivity property of the energy, which corresponds to algebraic (so-called uniform) stability and ensures the existence of a critical point. See [29] for the validity of the following definition which is modified by a (possibly non-maximal) torus.

Definition 2.4. We say that a functional $F: \mathcal{H}^T \rightarrow \mathbb{R}$ is $T_{\mathbb{C}}$ -coercive if there exist uniform constants $\delta, C > 0$ such that for any $\varphi \in \mathcal{H}^T$

$$(2.26) \quad F(\varphi) \geq \delta \inf_{\sigma \in T_{\mathbb{C}}} J(\sigma[\varphi]) - C$$

holds, where $\sigma[\varphi]$ is defined by $\sigma^* \omega_\varphi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \sigma[\varphi]$.

Note that we can also use the functional $I_g - J_g$ to define the coercivity, since J and $I_g - J_g$ are equivalent, as it was explained before.

Theorem 2.5. *A Fano manifold X admits a g -soliton in \mathcal{H}^T if and only if D_g is $T_{\mathbb{C}}$ -coercive. It admits a g -extremal metric in \mathcal{H}^T if and only if M_g is $T_{\mathbb{C}}$ -coercive.*

Proof. For the g -soliton, this is due to [25], Theorem 3.5. See also [38]. For the g -extremal metric, we can adapt the argument of [38] which generalizes He's extension [27] of Chen-Chen's result [14]. We also refer the proof of [2] Theorem 5. Indeed more general weighted constant scalar curvature Kähler metric case is treated in [1, 26]. \square

In view of the inequality in Lemma 2.3 and above theorems, we can conclude that if a Fano manifold admits a g -soliton it also admits a g -extremal metric. This proves the direction (1) \Rightarrow (3) in Theorem 1.4.

3. CONTINUITY METHOD FOR g -SOLITONS

3.1. g -twisted Laplacian. We start from discussing the linearized equation for the continuity method. Let $\Delta_\varphi := -\bar{\partial}^* \bar{\partial}$ be the negative Laplacian with respect to the Kähler metric $\varphi \in \mathcal{H}^T$, acting on smooth functions space $C^\infty(X; \mathbb{R})$. The adjoint operator $\bar{\partial}^*$ is taken with respect to the natural Hermitian inner product

$$(3.1) \quad \langle u, v \rangle_\varphi = \frac{1}{V} \int_X u \bar{v} \omega_\varphi^n.$$

Define the g -twisted Laplacian for $\varphi \in \mathcal{H}^T$ acting on functions as

$$(3.2) \quad \Delta_{g,\varphi} f = -g_\varphi^{-1} \bar{\partial}^* (g_\varphi \bar{\partial} f).$$

Note that $\Delta_{g,\varphi}$ is the one-half of the weighted Laplacian used by Di Nezza-Jubert-Lahdili [19] and Boucksom-Jonsson-Trusiani [12].

Lemma 3.1. *We have the following properties for $\Delta_{g,\varphi}$.*

- (1) *The operator $\Delta_{g,\varphi}$ is elliptic and self-adjoint with respect to the g -twisted Hermitian inner product*

$$(3.3) \quad \langle u, v \rangle_{g,\varphi} := \frac{1}{V} \int_X u \bar{v} g_\varphi \omega_\varphi^n.$$

- (2) *The kernel of $\Delta_{g,\varphi}$ consists of the constant functions.*

- (3) *Let $g_{i\bar{j}}$ be the metric tensor of ω_φ . Then*

$$\Delta_{g,\varphi} f = \Delta_\varphi f + \langle \bar{\partial} \log g_\varphi, \bar{\partial} f \rangle = \Delta_\varphi f + g^{i\bar{j}} \partial_{\bar{j}} f \partial_i \log g_\varphi.$$

- (4) *For any $\varphi \in \mathcal{H}^T$, let φ_t be a path in \mathcal{H}^T with $\varphi_0 = \varphi$ and $\frac{d}{dt} \Big|_{t=0} \varphi_t =: u$. Then*

$$(3.4) \quad \frac{d}{dt} \Big|_{t=0} (g_{\varphi_t} \omega_{\varphi_t}^n) = (\Delta_{g,\varphi} u) g_\varphi \omega_\varphi^n.$$

Proof. The claims (1), (2) and (3) follow from the definition of $\Delta_{g,\varphi}$ immediately. We give a proof of (4). Recall that ξ_1, \dots, ξ_r are the generating vector fields for $T_{\mathbb{C}} \simeq (\mathbb{C}^*)^r$. Let $(\theta_1, \dots, \theta_r)$ be the standard coordinate of the moment polytope P . In view of (3), it suffice to show

$$(3.5) \quad \frac{d}{dt} \Big|_{t=0} g_{\varphi_t} = \langle \bar{\partial} g_\varphi, \bar{\partial} u \rangle.$$

By simple calculations, we get

$$\left. \frac{d}{dt} \right|_{t=0} g_{\varphi_t} = \sum_{\alpha=1}^r \frac{\partial g_{\varphi}}{\partial \theta_{\alpha}} \xi_{\alpha}(u) \quad \text{and} \quad \langle \bar{\partial} g_{\varphi}, \bar{\partial} u \rangle = \sum_{\alpha=1}^r \frac{\partial g_{\varphi}}{\partial \theta_{\alpha}} \overline{\xi_{\alpha}(u)}.$$

Since any metric in \mathcal{H}^T is invariant under the action of the imaginary part of ξ_{α} , we have $\xi_{\alpha}(u) = \overline{\xi_{\alpha}(u)}$. This completes the proof. \square

3.2. Continuity method. In this subsection we prove the set \mathcal{T} in Theorem 1.2 is non-empty and open. Recall we consider the continuity method (1.4) to construct a g -soliton. Let \mathcal{T} be the set of $t \in [0, 1]$ such that the equation (1.4) has a solution in \mathcal{H}^T . For this purpose, we consider the following modified equation

$$(3.6) \quad g_{\varphi} \omega_{\varphi}^n = e^{-t\varphi - E_g(\varphi) + \rho_0} \omega_0^n \quad \text{for } t \in [0, 1].$$

Let \mathcal{T}^* be the set of $t \in [0, 1]$ such that the equation (3.6) has a solution in \mathcal{H}^T . For any $t \in (0, 1]$ there is a one-to-one correspondence between the solution of (1.4) and the solution of (3.6). The linearized operator associated from (1.4), however, is not invertible at $t = 0$ as discussed below.

3.2.1. Solution at $t = 0$. According to [9], Theorem 1.2, there exists a T -invariant continuous solution φ of $g_{\varphi} \omega_{\varphi}^n = e^{\rho_0} \omega_0$. The regularity argument [25], Proposition 3.8 (by using the log concavity assumption for g) shows φ is smooth. One can also apply the argument of [19, 26]. Up to addition of a constant φ satisfies (3.6) at $t = 0$. Therefore $0 \in \mathcal{T}^*$.

3.2.2. The openness of \mathcal{T}^* at $t = 0$. For any $\varphi \in \mathcal{H}^T$, define

$$\Phi^*(\varphi) = \log \left(\frac{g_{\varphi} \omega_{\varphi}^n}{\omega_0^n} \right) + E_g(\varphi) - \rho_0.$$

By lemma 3.1, the linearized operator at $t = 0$ is $\delta\Phi^*(u) = \Delta_{g,\varphi} u + V^{-1} \int_X u g_{\varphi} \omega_{\varphi}^n$, where u is any variation at a solution φ of (3.6) at $t = 0$. Since the kernel of $\delta\Phi^*$ is trivial, the implicit function theorem shows the openness of \mathcal{T}^* at $t = 0$.

3.2.3. The openness of $\mathcal{T} \cap (0, 1)$. For $\varphi \in \mathcal{H}^T$, define

$$\Phi(\varphi) = \log \left(\frac{g_{\varphi} \omega_{\varphi}^n}{\omega_0^n} \right) + t\varphi - \rho_0.$$

By Lemma 3.1, the linearized operator is $\delta\Phi(u) = \Delta_{g,\varphi} u + tu$, where u is any variation at a solution φ of (1.4) at t . The openness of $\mathcal{T} \cap (0, 1)$ follows from the next proposition combined with the implicit function theorem.

Proposition 3.2. *Let $\varphi_t \in \mathcal{H}^T$ be the solution of (1.4) at $t \in (0, 1)$. The first eigenvalue of the operator $\Delta_{g,\varphi_t} + t$ is negative.*

Proof. Let $g_{i\bar{j}}$ be the metric tensor for ω_{φ_t} and λ be the first non-zero eigenvalue of Δ_{g, φ_t} , and $u \in C^\infty(X, \mathbb{R})$ be the eigenvector. Put $f = \log g_{\varphi_t}$ for simplicity. Applying $\nabla_{\bar{k}}$ to the equation $\Delta_{g, \varphi_t} u = \lambda u$ we have

$$(3.7) \quad \lambda \nabla_{\bar{k}} u = g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \nabla_{\bar{k}} u - R_{\bar{k}}^{\bar{p}} \nabla_{\bar{p}} u + g^{i\bar{j}} \nabla_{\bar{j}} u \nabla_{\bar{k}} \nabla_i f + g^{i\bar{j}} \nabla_{\bar{k}} \nabla_{\bar{j}} f \nabla_i u$$

$$(3.8) \quad < g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} \nabla_{\bar{k}} u - t \nabla_{\bar{k}} u + g^{i\bar{j}} \nabla_{\bar{k}} \nabla_{\bar{j}} u \nabla_i f$$

$$(3.9) \quad = e^{-f} \nabla_i \left(e^f \nabla_{\bar{j}} \nabla_{\bar{k}} u \right) - t \nabla_{\bar{k}} u.$$

In the above inequality we used the equation (1.4) in the form

$$(3.10) \quad \text{Ric}(\omega_{\varphi_t}) - \sqrt{-1} \partial \bar{\partial} \log g_{\varphi_t} = (1-t) \omega_0 + t \omega_{\varphi_t} > t \omega_{\varphi_t}.$$

If one multiplies $g^{m\bar{l}} \nabla_m u g_{\varphi_t} \omega_{\varphi_t}^n$ to the both sides then a simple integration by part shows

$$\lambda \int_X |\bar{\nabla} u|^2 g_{\varphi_t} \omega_{\varphi_t}^n < - \int_X |\bar{\nabla} \bar{\nabla} u|^2 g_{\varphi_t} \omega_{\varphi_t}^n - t \int_X |\bar{\nabla} u|^2 g_{\varphi_t} \omega_{\varphi_t}^n.$$

Therefore $\lambda + t < 0$. \square

Combining the above results together, we conclude that \mathcal{T} is non-empty and open.

4. ESTIMATE OF M_g ALONG THE CONTINUITY METHOD

In this section we prove Theorem 1.3. Let $\varphi_t \in \mathcal{H}^T$ be a solution of the equation (1.4) at t . In order to obtain the upper bound of $M_g(\varphi_t)$, it suffices to control $D_g(\varphi_t) - \int_X \rho_\varphi \omega_\varphi^n$, in view of Lemma 2.3. We put $\omega_t := \omega_{\varphi_t}$, $\rho_t := \rho_{\varphi_t}$, $g_t := g_{\varphi_t}$ and $\Delta_t := \Delta_{\varphi_t}$ for simplicity.

Lemma 4.1. *Let $\varphi_t \in \mathcal{H}^T$ be a solution of (1.4) at t . One has the relation*

$$(4.1) \quad \rho_t + (1-t)\varphi_t - \log g_t - L(\varphi_t) = 0.$$

Proof. From the equation (1.4) and definition of the Ricci potential, we observe

$$(4.2) \quad 0 = -\sqrt{-1} \partial \bar{\partial} \log g_{\varphi_t} + \text{Ric}(\omega_t) - \text{Ric}(\omega_0) + \sqrt{-1} \partial \bar{\partial} (\rho_0 - t\varphi_t)$$

$$(4.3) \quad = \sqrt{-1} \partial \bar{\partial} \left(\rho_t + (1-t)\varphi_t - \log g_{\varphi_t} \right).$$

Thus $\rho_t + (1-t)\varphi_t - \log g_{\varphi_t} = c_t$ for some constant c_t depending on t . The normalization of ρ_t and the equation (1.4) forces

$$(4.4) \quad c_t = -\log \frac{1}{V} \int_X e^{-(1-t)\varphi_t} g_{\varphi_t} \omega_t^n = -\log \frac{1}{V} \int_X e^{-\varphi_t + \rho_0} \omega_0^n.$$

The right-hand side is the L -functional introduced by (2.17). \square

Integrating the both sides of Lemma 4.1, one obtains

$$(4.5) \quad -\frac{1}{V} \int_X \rho_t \omega_t^n \leq \frac{1-t}{V} \int_X \varphi_t \omega_t^n - \log(\inf_X g) - L(\varphi_t).$$

Thanks to the assumption $g > 0$, the first term is bounded from above, as follows.

Lemma 4.2. *For a fixed $\varepsilon \in (0, 1)$ the solution φ_t at $t > \varepsilon$ satisfies*

$$(4.6) \quad \frac{1}{V} \int_X \varphi_t \omega_t^n \leq \frac{e^{\sup \rho_0 - 1}}{\inf_X g} \varepsilon^{-1}.$$

Proof. Using the equation (1.4) of the continuity method, we transrate the volume form into the form

$$\begin{aligned} \frac{1}{V} \int_X \varphi_t \omega_t^n &\leq \frac{1}{V} \int_{\{\varphi_t > 0\}} \varphi_t g_t^{-1} e^{-t\varphi_t + \rho_0} \omega_0^n \\ &\leq \frac{e^{\sup \rho_0}}{V \inf_X g} \int_{\{\varphi_t > 0\}} \varphi_t e^{-t\varphi_t} \omega_0^n. \end{aligned}$$

This completes the proof, since for any $t > \varepsilon$ the function $\mathbb{R}_{>0} \ni x \mapsto xe^{-tx}$ is bounded from above by the constant $(e\varepsilon)^{-1}$. \square

Although the third term $L(\varphi_t)$ in the right hand side in (4.5) is not bounded from above, this is cancelled by the same term in D_g . The remaining part is in fact uniformly bounded from above regardless of m_X or ε .

Lemma 4.3. *The functional $I_g - J_g$ is non-decreasing along the continuity path (1.4).*

Proof. The same result for $I - J$ is proved in [4] page 28. If one differentiates (1.4), Lemma 3.1 implies

$$(4.7) \quad \Delta_{g,t} \dot{\varphi}_t + t \dot{\varphi}_t + \varphi_t = 0.$$

So we may compute as

$$(4.8) \quad \frac{d}{dt}(I_g - J_g)(\varphi_t) = -\frac{1}{V} \int_X \varphi_t \Delta_{g,t} \dot{\varphi}_t g_t \omega_t^n$$

$$(4.9) \quad = \frac{1}{V} \int_X (\Delta_{g,t} \dot{\varphi}_t + t \dot{\varphi}_t) \Delta_{g,t} \dot{\varphi}_t g_t \omega_t^n$$

$$(4.10) \quad = \frac{1}{V} \int_X (\Delta_{g,t} \dot{\varphi}_t + t \dot{\varphi}_t)^2 \omega_t^n - \frac{t}{V} \int_X \dot{\varphi}_t (\Delta_{g,t} \dot{\varphi}_t + t \dot{\varphi}_t) g_t \omega_t^n.$$

The second term in the last line is non-negative by Proposition 3.2. \square

Proposition 4.4. *Along the solution φ_t , the g -twisted Monge-Ampère energy is non-negative: $E_g(\varphi_t) \geq 0$.*

Proof. We first show that one can extend the solution φ_t at t to a family of solutions φ_s for $s \in [0, t]$. Let \mathcal{T}_t be the set of $s \in [0, t]$ such that the equation (1.4) has a solution in \mathcal{H}^T . We already showed in Section 3.2 that \mathcal{T}_t is non-empty and open. By Lemma 4.3, $(I_g - J_g)(\varphi_s) \leq (I_g - J_g)(\varphi_t)$ for any $s \in \mathcal{T}_t$. Thus we may apply the argument of Proposition 5.3 to show that \mathcal{T}_t is closed.

In order to show $E_g(\varphi_t) \geq 0$, it is sufficient to prove the formula

$$(4.11) \quad E_g(\varphi_t) = \frac{1}{t} \int_0^t \left(I_g(\varphi_s) - J_g(\varphi_s) \right) ds,$$

because $I_g - J_g$ is non-negative. This is also well-known to the experts but we give a proof of (4.11) for the convenience to the reader. From the general expression

$$(4.12) \quad I_g(\varphi) - J_g(\varphi) = E_g(\varphi) - \frac{1}{V} \int_X \varphi g_\varphi \omega_\varphi^n,$$

it is reduced to the differential formula

$$(4.13) \quad t \frac{d}{dt} E_g(\varphi_t) = -\frac{1}{V} \int_X \varphi_t g_t \omega_t^n.$$

Let us differentiate the Monge-Ampère energy along the continuity path. We denote $\dot{\rho}_t := \frac{d}{dt} \rho_t$ and $\dot{\varphi}_t := \frac{d}{dt} \varphi_t$ for simplicity. If one differentiates the identity of Lemma 4.1 first, he observes

$$(4.14) \quad \dot{\rho}_t = \varphi_t - (1-t)\dot{\varphi}_t - \langle \bar{\partial} \log g_t, \bar{\partial} \dot{\varphi}_t \rangle + \frac{1}{V} \int_X \dot{\varphi}_t e^{\rho_t} \omega_t^n.$$

On the other hand, the definition of the Ricci potential yields

$$(4.15) \quad \dot{\rho}_t = \Delta_t \dot{\varphi}_t - \dot{\varphi}_t + \frac{1}{V} \int_X \dot{\varphi}_t e^{\rho_t} \omega_t^n,$$

where Δ_t is the geometric Laplacian with respect to ω_t . Using the two equations (4.14) and (4.15), one can compute as

$$(4.16) \quad \frac{d}{dt} E_g(\varphi_t) = \frac{1}{V} \int_X \dot{\varphi}_t g_t \omega_t^n$$

$$(4.17) \quad = \frac{1}{V} \int_X \left(-\varphi_t + (1-t)\dot{\varphi}_t + \langle \bar{\partial} \log g_t, \bar{\partial} \dot{\varphi}_t \rangle + \Delta_t \dot{\varphi}_t \right) g_t \omega_t^n$$

$$(4.18) \quad = -\frac{1}{V} \int_X \varphi_t g_t \omega_t^n + (1-t) \frac{d}{dt} E_g(\varphi_t).$$

In the last line we used Lemma 3.1, (3), (4) which tells

$$(4.19) \quad 0 = \frac{d}{dt} \int_X g_t \omega_t^n = \int_X \left(\Delta_t \dot{\varphi}_t + \langle \bar{\partial} \log g_t, \bar{\partial} \dot{\varphi}_t \rangle \right) g_t \omega_t^n.$$

It shows (4.13). \square

Summarizing the above argument, we obtain the following: Fix any $\varepsilon \in (0, 1)$. Let $\varphi_t \in \mathcal{H}^T$ be a solution of the equation (1.4) at $t > \varepsilon$. Then one has the estimate

$$M_g(\varphi_t) \leq \frac{e^{\sup \rho_0 - 1}}{\inf g} \varepsilon^{-1} - \log(\inf_X g) + \frac{1}{V} \int_X \rho_0 \omega_0^n.$$

This completes the proof of Theorem 1.3.

5. THE CLOSEDNESS OF \mathcal{T}

In this last section we complete the proof of Theorem 1.2. It remains to show the closedness of \mathcal{T} , when there exists an extremal Kähler metric. Thanks to Theorem 2.5, we may assume the coercivity of M_g . As in the previous section, we will write as $\omega_t := \omega_{\varphi_t}$, $\rho_t := \rho_{\varphi_t}$ and $g_t := g_{\varphi_t}$.

5.1. Control of $I_g - J_g$. If the coercivity of M_g is assumed, then $\inf_{\sigma \in T_{\mathbb{C}}}(I_g - J_g)(\sigma[\varphi_t])$ is bounded by Theorem 1.3. The first step is to show that one can in fact control $(I_g - J_g)(\varphi_t)$ along the continuity path φ_t . It extends [34], Lemma 3.3 to the general g case.

The following lemma rephrases the equivalence of the two expressions for the (modified) Futaki invariant, in terms of the energies.

Lemma 5.1. *Let $\xi \in \mathfrak{t}_{\mathbb{C}}$ and $\{\sigma_s\}_{s \in \mathbb{R}}$ the one-parameter subgroup generated by the real part $\operatorname{Re}(\xi)$. Fix any $\varphi \in \mathcal{H}^T$. Then,*

$$(5.1) \quad \frac{d}{ds} M_g(\sigma_s[\varphi]) = \frac{d}{ds} D_g(\sigma_s[\varphi]).$$

If the functional M_g is bounded from below on \mathcal{H}^T , then the both sides vanish identically.

Proof. Since $\sigma_s^* \omega_\varphi = \omega_{\sigma_s[\varphi]}$ and $\sigma_s^* \rho_\varphi = \rho_{\sigma_s[\varphi]}$, (5.1) follows from Lemma 2.3. We put $\psi := \sigma_s[\varphi]$ and starts from the expression

$$(5.2) \quad \frac{d}{ds} D_g(\sigma_s[\varphi]) = \frac{1}{V} \int_X \operatorname{Re}(\theta_\xi(\psi))(e^{\rho_\psi} - g_\psi) \omega_\psi^n,$$

where $\theta_\xi(\psi)$ is the Hamilton function. If we differentiates the defining equation of the Ricci potential along ξ , we observe

$$(5.3) \quad \Delta_\psi \theta_\xi(\psi) + \theta_\xi(\psi) + \xi \rho_\psi = \frac{1}{V} \int_X \theta_\xi(\psi) e^{\rho_\psi} \omega_\psi^n$$

(see e.g. [39], Lemma 3.1). Therefore we derive

$$(5.4) \quad \frac{d}{ds} D_g(\sigma_s[\varphi]) = \frac{1}{V} \int_X \operatorname{Re}(\xi)(\rho_\psi - \log g_\psi) g_\psi \omega_\psi^n.$$

The right-hand side is the real part of the g -twisted Futaki invariant (see [25], Definition 4.1). In particular, it does not depend on the choice of ψ and s . If M_g is bounded from below, therefore, it identically vanishes. \square

Proposition 5.2. *Assume the functional M_g is bounded from below on \mathcal{H}^T . Let $\varphi_t \in \mathcal{H}^T$ be a solution of (1.4) at $t \in [0, 1)$. Then we have*

$$\inf_{\sigma \in T_{\mathbb{C}}} (I_g - J_g)(\sigma[\varphi_t]) = (I_g - J_g)(\varphi_t).$$

Proof. In the same notation as the above lemma, a direct computation using (4.12) shows

$$(5.5) \quad \left. \frac{d}{ds} \right|_{s=0} (I_g - J_g)(\sigma_s[\varphi_t]) = -\frac{1}{V} \int_X \left. \frac{d}{ds} \right|_{s=0} \sigma_s[\varphi_t] (\Delta_{g, \varphi_t} \varphi_t) g_t \omega_t^n$$

$$(5.6) \quad = \frac{1}{V} \int_X \langle \bar{\partial} \operatorname{Re}(\theta_\xi(\varphi_t)), \bar{\partial} \varphi_t \rangle g_t \omega_t^n.$$

If one takes notice of the definition of the Hamilton function and Lemma 4.1, it is equivalent to

$$(5.7) \quad -\frac{1}{(1-t)V} \int_X \operatorname{Re}(\xi)(\rho_t - \log g_t) g_t \omega_t^n = -\frac{1}{1-t} \left. \frac{d}{ds} \right|_{s=0} D_g(\sigma_s[\varphi]).$$

According to Lemma 5.1, it implies that $I_g - J_g$ is critical at $\sigma = \text{id}$.

We will show that for any fixed $\varphi \in \mathcal{H}^T$, the function $s \mapsto f(s) := (I_g - J_g)(\sigma_s[\varphi])$ is properly convex (so that the infimum is achieved by $\sigma = \text{id}$). Similar argument for the J , D , and M has appeared in [28], Theorem 2.5. See also [25], Lemma 3.5 and 4.6 for the detail of the fiber integration below. We consider the complex variable $s \in \mathbb{C}$ and define $\Omega(z, s) = \sigma_s^*(\omega_\varphi(z))$ as a smooth semipositive form on the product space $X \times \mathbb{C}$. If regards the pulled-back form $p_1^*\omega_0$ by the first projection as a reference form, one has a potential function $\Psi(z, s)$ such that $\Omega = p_1^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\Psi$. In terms of the fiber integration the second variation of the Monge-Ampère energy (Recall that the first variation was given by (2.12)) is expressed as

$$(5.8) \quad \sqrt{-1}\partial\bar{\partial} E_g(\Psi) = \frac{1}{V} \int_X g_\Psi \Omega^{n+1},$$

where the operator ∂ is taken for $n+1$ variables (z, s) . It actually vanishes since $\Omega^{n+1} = \sigma_s^*(\omega_\varphi^{n+1}) = 0$. On the other hand, the integration of Ψ against the g -twisted Monge-Ampère measure on the product space is computed as

$$(5.9) \quad -\sqrt{-1}\partial\bar{\partial} \int_X \Psi g_\Psi \Omega^n = - \int_X \sqrt{-1}\partial\bar{\partial} \Psi \wedge g_\Psi \Omega^n$$

$$(5.10) \quad = - \int_X g_\Psi \Omega^{n+1} + \int_X p_1^*\omega_0 \wedge g_\Psi \Omega^n$$

$$(5.11) \quad = \int_X p_1^*\omega_0 \wedge g_\Psi \Omega^n.$$

The last term is obviously non-negative. It implies that $(I_g - J_g)(\sigma_s[\varphi])$ is subharmonic in s . Properness of $f(s)$ follows from the fact that the slope at infinity of $f(s)$ is positive unless $\text{Re}(\xi) = 0$. \square

5.2. Closedness of \mathcal{T} . Let us now prove the closedness of \mathcal{T} so that conclude Theorem 1.2.

Proposition 5.3. *Suppose that the functional M_g is coercive on \mathcal{H}^T . Take $t_j \in \mathcal{T}$ which converges to $t_\infty \in \mathbb{R}$, and $\varphi_j \in \mathcal{H}^T$ as the solution of (1.4) at $t = t_j$. Replaced with a subsequence if necessary, φ_j converges to some $\varphi_\infty \in \mathcal{E}^1(X, \omega_0)^T$ in the d_1 -topology and the limit function φ_∞ is in fact a smooth solution of (1.4) at $t = t_\infty$. It implies $t_\infty \in \mathcal{T}$.*

Proof. In the sequel all the constants C are uniform in j . We write $\omega_j := \omega_{\varphi_j}$ and $g_j := g_{\varphi_j}$.

As we have already mentioned, the functionals I , J , and $I_g - J_g$ are interchangeable with each other. According to Theorem 1.3 and Proposition 5.2, the coercivity of M_g implies the uniform bound $J(\varphi_j) \leq C$.

Uniform bound of the I -functional, combined with Lemma 4.2 yields $\int_X \varphi_j \omega_0^n \leq C$. Then by the standard Green function argument we may obtain the uniform bound of $\sup_X \varphi_j$. Let G_{ω_0} be the Green function for the background metric ω_0 , which satisfies

$\inf_X G_{\omega_0} \geq -B$ for some positive constant B . Since $\Delta_{\omega_0} \varphi_j \geq -n$, the representaion formula implies

$$(5.12) \quad \sup_X \varphi_j \leq \frac{1}{V} \int_X \varphi_i(y) \omega_0^n(y) + \sup_{x \in X} \int_X (G_{\omega_0}(x, y) + B)(-2\Delta_{\omega_0} \varphi_i(y)) \omega_i^n(y) \leq C.$$

The lower bound $\sup_X \varphi_i \geq -C$ is rather clear since

$$(5.13) \quad \int_X e^{-t_j \varphi_j + \rho_0} \omega_0^n = \int_X g_j \omega_j^n = \int_X \omega_0^n.$$

Thus we obtain

$$(5.14) \quad \left| \sup_X \varphi_i \right| \leq C.$$

The above Green function argument implies the general inequality $\sup_X \varphi \leq V^{-1} \int_X \varphi \omega_0^n + C$ so the uniform estimate of $I(\varphi_i)$ now implies $|\int_X \varphi_j \omega_j^n| < C$. If look back to the expression of M_g (which is bounded from above by Theorem 1.3), we obtain the uniform bound of the relative entropy

$$(5.15) \quad \text{Ent}(\text{MA}(\varphi_j) | \text{MA}(0)) \leq C.$$

This is a key to extract a convergent subsequence. Indeed by the fundamental compactness result [8], Theorem 2.17 and Proposition 2.6, there exists a convergent subsequence and the limit in $(\mathcal{E}^1)^T = \mathcal{E}^1(X, \omega_0)^T$. As an abuse of notation, we denote the subsequence by the same symbol φ_j . According to [17], the convergence $\varphi_j \rightarrow \varphi_\infty$ in d_1 -topology implies $\varphi_j \rightarrow \varphi_\infty$ in $L^1(X, \omega_0^n)$, $E(\varphi_j) \rightarrow E(\varphi_\infty)$, and vice versa. Finiteness of the Monge-Ampère energy forces the singularities of the psh functions very mild. Indeed, we can apply the uniform version of the Skoda integrability theorem ([44], Corollary 3.2) to the sequence so that

$$(5.16) \quad \int_X e^{-pt_j \varphi_j} \omega_0^n, \quad \int_X e^{-pt_\infty \varphi_\infty} \omega_0^n \leq C$$

holds for any $p > 1$. The effective version of Demailly-Kollàr's semi-continuity theorem for log canonical thresholds ([21], Main Theorem 0.2, (2)). See also [8], Proposition 1.4) shows the convergence $e^{-t_j \varphi_j} \rightarrow e^{-t_\infty \varphi_\infty}$ in $L^p(\omega_0^n)$ for all $p > 1$. It follows the convergence of probability measures

$$(5.17) \quad g_j \omega_j^n = e^{-t_j \varphi_j + \rho_0} \omega_0^n \rightarrow e^{-t_\infty \varphi_\infty + \rho_0} \omega_0^n.$$

We also claim the convergence of probability measures

$$(5.18) \quad g_j \omega_j^n = \text{MA}_g(\varphi_j) \rightarrow \text{MA}_g(\varphi_\infty).$$

Note that the limit measure is a priori singular and g_{φ_∞} itself does not make sense. See [9] for the detail. It is convenient to use the symmetric I -functional

$$(5.19) \quad I(\psi_1, \psi_2) := \int_X (\psi_1 - \psi_2)(\omega_{\psi_1}^n - \omega_{\psi_2}^n)$$

which is defined for $\psi_1, \psi_2 \in \mathcal{E}^1(X, \omega_0)$. As [7], Lemma 3.13, we observe

$$\begin{aligned} \sup_{(\mathcal{E}^1)_B^T} \left| \int_X u g_{\psi_1} \omega_{\psi_1}^n - \int_X u g_{\psi_2} \omega_{\psi_2}^n \right| &\leq C \sup_{(\mathcal{E}^1)_B^T} \left| \int_X u \omega_{\psi_1}^n - \int_X u \omega_{\psi_2}^n \right| \\ &\leq CI(\psi_1, \psi_2)^{1/2} \end{aligned}$$

for the weak compact set

$$(5.20) \quad (\mathcal{E}^1)_B^T := \left\{ \varphi \in \mathcal{E}^1(X, \omega_0)^T : \sup_X \varphi \leq 0, E(\varphi) \geq -B \right\}.$$

Following [7], Proposition 5.6, we may exploit the above inequality to control the (g -twisted) pluricomplex energy of the probability measure

$$(5.21) \quad E_g^*(\mu) := \sup_{u \in (\mathcal{E}^1)^T} \left[E_g(u) - \int_X u d\mu \right].$$

Indeed, the convergence $\varphi_i \rightarrow \varphi_\infty$ in d_1 -topology is equivalent to $I(\varphi_j, \varphi_\infty) \rightarrow 0$. Using the above inequality, we deduce $E_g^*(\text{MA}_g(\varphi_j)) \rightarrow E_g^*(\text{MA}_g(\varphi_\infty))$, which in particular implies the weak convergence (5.18).

From (5.17) and (5.18), we obtain the weak solution which satisfies

$$(5.22) \quad \text{MA}_g(\varphi_\infty) = e^{-t_\infty \varphi_\infty + \rho_0} \omega_0^n.$$

Since the right-hand side has L^p -density for some $p > 1$, one can apply [9], Theorem 1.2 (or more directly Kołodziej's a priori estimate [31] for the Monge-Ampère equation) so that φ_∞ is continuous. The regularity argument [25], Proposition 3.8 (by using the log concavity assumption for g) shows φ_∞ is smooth. One can also apply the argument of [19, 26]. At any rate it implies $t_\infty \in \mathcal{T}$. \square

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