CO 351 : Network Flow Theory A7

Term: Spring 2019

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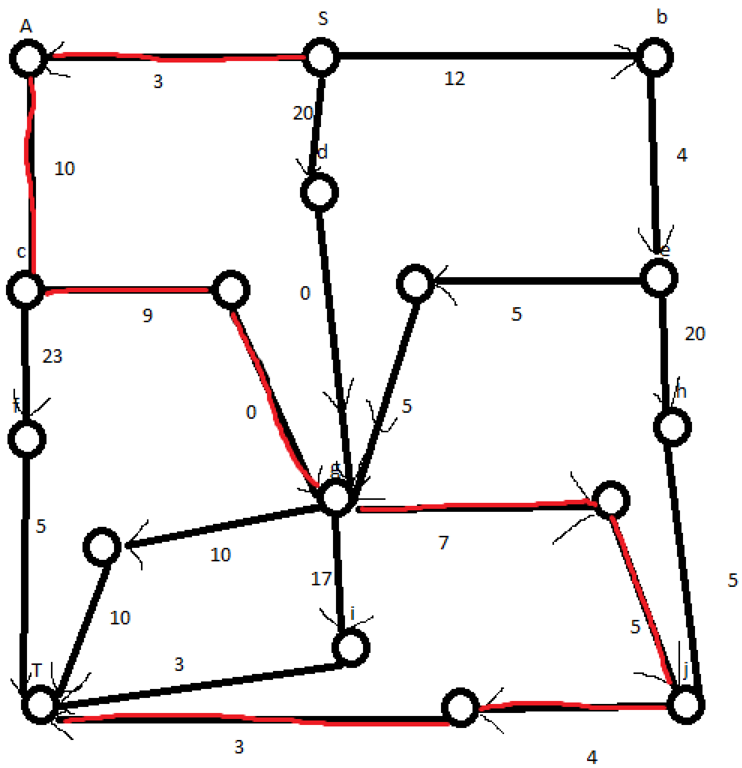
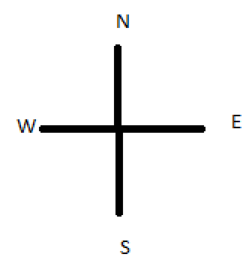
1. UPS BTW
   we assume that the starting direction of the car is south for the example. To formulate this as a min flow problem we create a node for each incoming arc a node has. For example d is represented by 3 nodes as it has three different directions to reach it and a will have one node as it has on direction trying to reach it. We do this for all nodes that are not s or t they will remain as one node. Then we add α to any arcs that represents a left turn. So for example the sequence s, a, c is a left turn and we will add α to 5. The arcs will connect the to the appropriate directions the nodes are representing. Example below. The red is the optimal solution from Dijkstra's algorithm.





2. Dual

(a) Min

By the max flow and min cut theorem we see that everytime we find a max flow x we find a minimum cut $\delta(S)$ with minimum capacity $c(\delta(S))$ So we can make a conclusion that the dual of the max flow problem can be formulated as a minimization problem for finding a min cut in a directed graph G. This result is from the proof of the strong duality theorem from CO250 which states that primal program has an optimal solution, x, then the dual program also has an optimal solution, y, such that the optimal values formed by the two solutions are equal . So the dual of the max flow problem is equivalent to the LP of the min-cut problem. So we will formulate the LP for finding a minimum cut in G. So we can show that the given LP is the LP for minimizing an st-cut in G. We define y to be variable fo each node in G that is not the sink or source. and z to be the variable for every arc in G. Then we assign 1 to z_{uv} if uv is in an s,t cut and 0 otherwise. Then we assign 1 to y_v if v is part of the set that is producing the s,t cut. Then set constraints to reinforces the idea of $\delta(S)$ to be an s,t cut of G. In order of $\delta(S)$ to be an s,t cut then for all nodes other than the since and source if u is in S and v is not in S then the arc uv has to be in $\delta(S)$ so we set $z_{uv} - y_u + y_v \geq 0$ as a constraint to make sure that if u is in S then z_{uv} has to be 1. Then by the definiton of the s,t cut s must always be in S and t must never be in S so we set $y_s = 1$ and $y_t = 0$ as two more constraints. And by definition $z_e \geq 0$ so we are done. We see that we have fomulated the LP for minimizing an s,t cut in G and it is equivalent to the given dual. And by strong duality and the max flow min cut theorem we can say that this dual is valid.

(b) Feasible Sol

Assume that y is a characteristic vector of S and z is a characteristic vector of $\delta(S)$ We see that by the definition of the characteristic vector $z_e \geq 0 \forall e \in A$ is satisfied we also see that $y_s = 1$ and $y_t = 0$ is satisfied because since y is the characteristic vector of S and $\delta(S)$ is an s,t cut $s \in S$ and $t \notin S$. We then see that $z_{uv} = 1$ if $uv \in \delta(S)$ else $z_{uv} = 0$. Then if $z_{uv} = 1$ then $y_v = 0$ and $y_u = 1$ and vice versa if $z_{uv} = 0$ then $y_v = 1$ and $y_u = 0$ or both y_v and y_u are 0 if neither v or u are a part of S. This comes from the definition of an s,t cut and the characteristic vector. So we see that in all situations. So the possible values for $-y_u + y_v + z_{uv}$ are 0 or 1. Which are both ≥ 0 So since we satisfy all of the constraints to the LP our solution is feasible, so (y, z) is a feasible solution to the dual in part (a).

(c) Optimal Sol

Let (y, z) be an optimal solution to the dual in part (a). Since z is the characteristic vector of $\delta(S)$ we see that by the obj func we are minimizing the capacities of an s,t cut. And since y is the characteristic vector of S we see that by the proof above this representation is feasible. So, since we have the optimal minimization of the capacities of an s,t cut we can conclude that $\delta(S)$ is a minimum cut of G.

3. Critical

(a) Downward Critical

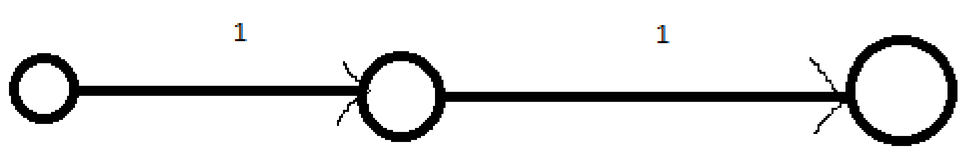
We say that every edge that is a part of a minimum cut in a graph G is a downward critical arc.
Lemma: If arc e belongs to the minimum s-t cut of a graph G then e is a downward critical arc.
Proof: $\rightarrow$ By the max flow min cut theorem we see that the maximum value of a flow x is equal to the minimum capacity of an s,t cut. Let $\delta(S)$ be an minimum st-cut. and let $e \in \delta(S)$ subtracting 1 from c_e means subtracting 1 from $c(\delta(S))$ so the value of x also decreases as $x(\delta(S)) - x(\delta(\bar{S}) = c(\delta(S))$. So e is a downward critical arc
$\leftarrow$ Suppose e is a downward critical arc then by subracting 1 from c_e means subtracting 1 from the max flow x. Since the value of the max flow is equal to the minimum capacity of a s,t cut that means the minimum capacity of the s,t cut has also been reduced by 1 and that is only possible if $e \in \delta(S)$ so $e \in \delta(S)$

(b) Upward critical

We say that an arc uv is upward critical if and only if it is in a minimum s,t cut and there are flow augmenting path that connect s to u and v to t in G
Proof: $\rightarrow$ Suppose that uv is upward critical then we can increase c_{uv} by 1 and thus increasing x by 1. This is only possible if the flow for the path from s to u is not at capacity and the flow for the path from v to t is not at capacity. So by definition those are flow augmenting pathes. Then again by the max flow min cut theorem, since we changed the flow number x then the capacity of the min cut must have also changed so uv is in a minimum s,t cut
$\leftarrow$ Suppose that uv is in a minimum s,t cut and there are flow augmenting path that connect s to u and v to t in G. Then we see that if we increased the capacity of uv by 1 then we would be able to send 1 more flow through the flow augmenting path that connects s to u and v to t as in the residual graph there is still a forward and backwards arc. Then since uv is in a minimum s,t cut of G then by the max flow min cut theorem we see that increasing c_{uv} also increases x so uv is an upward critical arc

(c) examples

Every network contains a downward critical arc, since it is impossible for a network to not have a min s,t cut.
Every network may not contain a upward critical arc, for example consider the following diagram



increasing the capacity of either arcs won't increase the max flow.