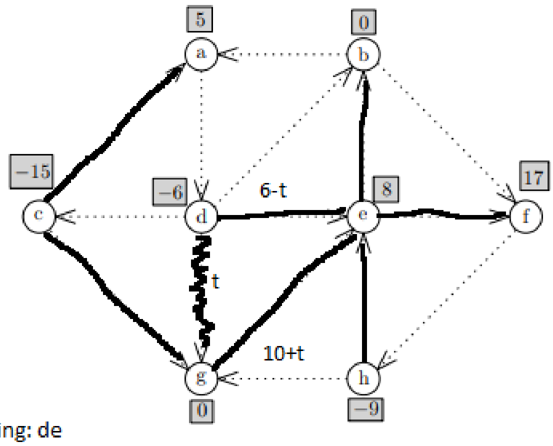
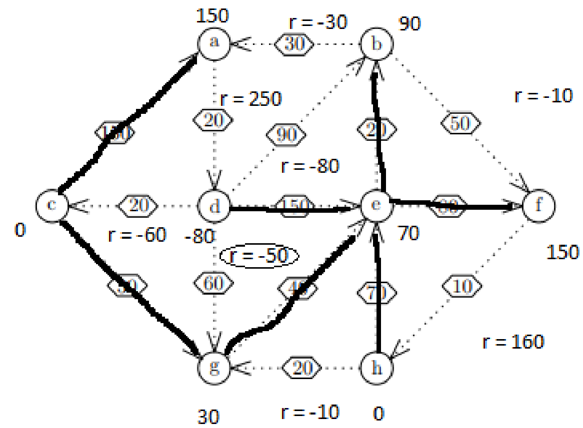
CO 351 : Network Flow Theory A3

Term: Spring 2019

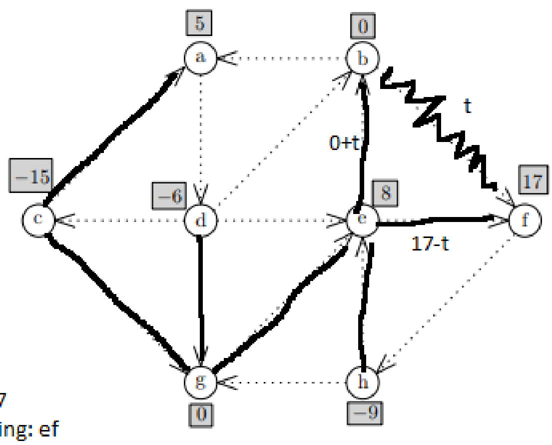
Instructor: M. Pei

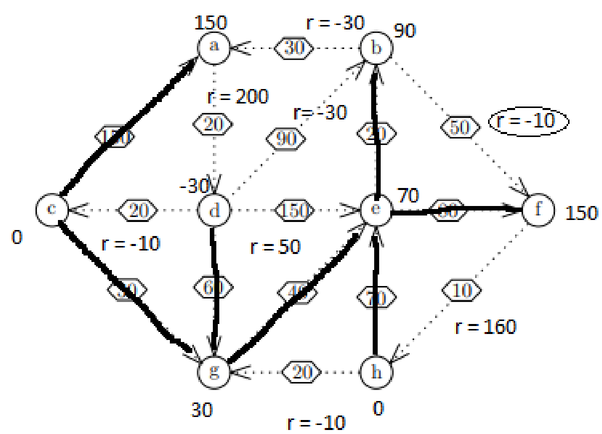
1. Let B be the set of columns in incidence matrix M that contains a the set of linearly dependent columns. Given D and M we know that each row of M represents a vertex and each column represents an arc of D corresponding to the vertices. By the definition of linear independence we know that the solution to the linear system of equations has to be the 0 vector. So we make a claim that since B is linearly dependent, there a is a none zero solutions to $Bx = 0$. So each row and column sum to 0 from some linear combination with a non-zero vector. Since this is an incidence matrix every row has at least one non-zero element. For each row $M_{v,e}$ that has less than 2 non-zero elements we remove them from our matrix B and form another submatrix B' we also remove the corresponding arc column, M_e from the set. This is the equivalent of removing nodes with only one incident arc in the subgraph formed by B. Then this submatrix B' is also linearly dependent as the variable we removed has a value of 0 and does not affect the solution. We do so until we are left with a matrix that only has rows with 2 or more non-zero elements. Then by the lemma we are given the matrix contains a cycle, which is what remains of our graph.



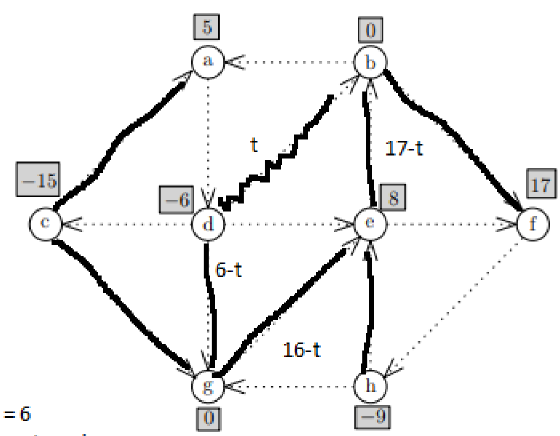


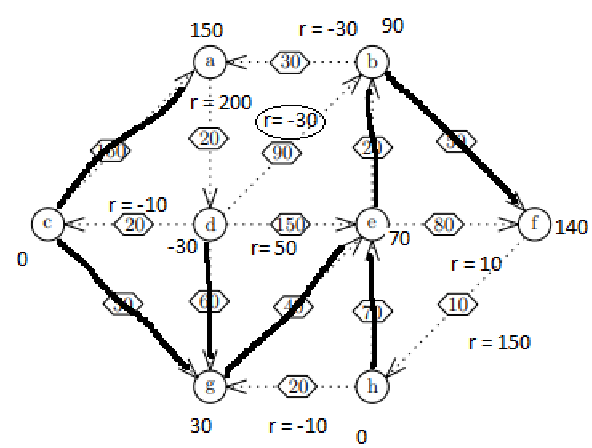
t = 6
leaving: de
entering: dg





t = 17
leaving: ef
entering: bf





t = 6
leaving: dg
entering: db

2.

3. (a) the reduced cost of uv is the cost of all forward arcs subtract the cost of all backward arcs in C. Proof:
   we know that $\bar{w}_{uv} = y_u + w_{uv} - y_v$. We know that the sum of costs on forward arcs minus sum of costs on backward arcs is the total cost of the direct cycle. Let the cycle be formed by $y_1, y_2, .., y_k$ Let $y_v = y_k$ and $y_u = y_1$. We see that $y_v = y_u + w_{u,2} + w_{2,3} + ... + w_{i,v}$. Any $w_{u,v}$ can be positive or negative to represent their directions. So we sub this equation back into our equation for $\bar{w}_{uv}$ we cancel out the y_u and are just left with arc costs. So it is the sum of costs on forward arcs minus sum of costs on backward arcs. Also since the reduced cost of all other arcs in C are zero then we know $\bar{w}_{uv} =$ sum of costs on forward arcs minus sum of costs on backward arcs

   (b) Since pq was the previous leaving arc we know it was a backward arc w.r.t uv so then by part a we know that sum of backward arc costs $>$ sum of forward arcs in C so that $\bar{w}_{uv} < 0$ So if pq is the next entering arc we would have to use the same C from before and this time pq is the forward arc. But we know that sum of backward arc costs $>$ sum of forward arcs in C which means in this iteration sum of backward arc costs $<$ sum of forward arcs in C so $\bar{w}_{pg} > 0$ which is a contradiction.