CO 351 : Network Flow Theory A6

Term: Spring 2019

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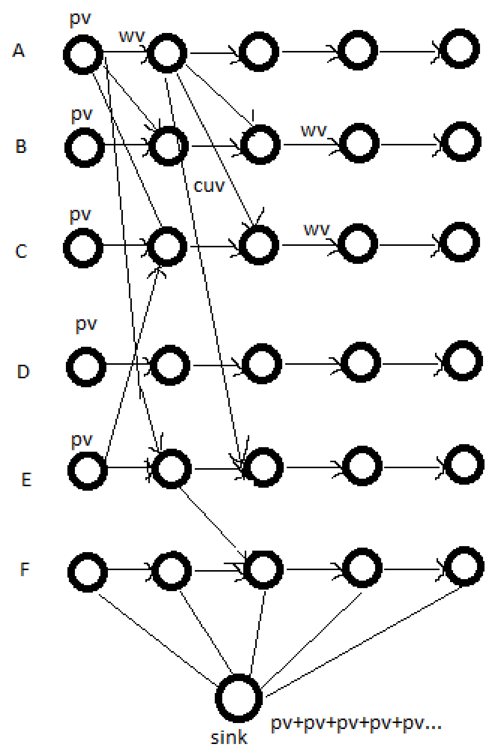
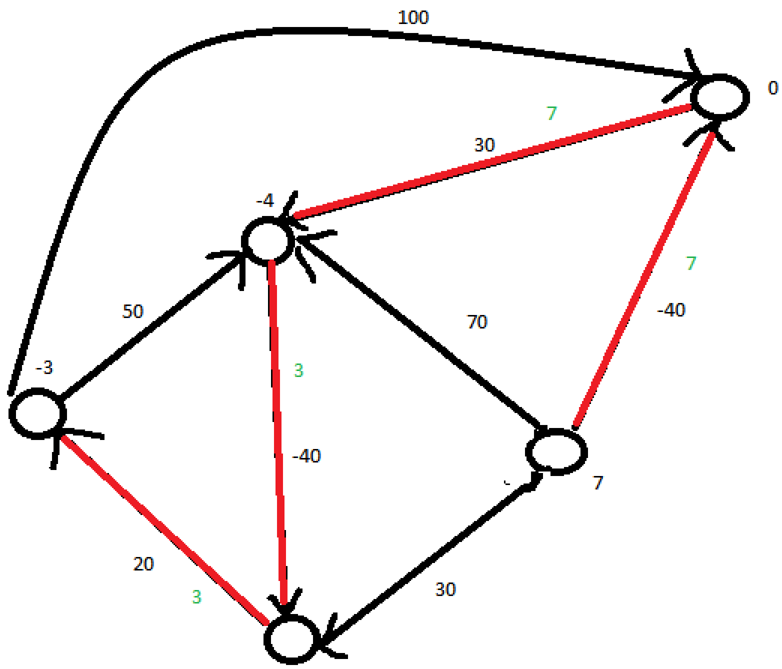
1. Factory Evacuation
   As the hint suggests we model the problem using $|N| * (M + 1) + 1$ nodes.
   So for this example we will have 31 nodes. 30 of which will be representing each minute we have in each room, and one will be a "sink" node at the end. For the 30 nodes they will be divided into 6 groups of 5. Each set represents a room and each node within the set represents a minute in the room starting at the 0th minute to the 4th minute. We see from the diagram that each room's 0th minute node has p_v as the supply meaning that at the 0th min there are p_v number of people in the room all other minute node have no demand or supply. For all nodes that are not part of the exit room F. The arcs connecting the minute nodes (t_{uv}) in each group is showing the fact that there are people choosing to remain in the room for an additional minute per node travelled. The cost of t_{uv} will be the total harm a person takes while staying in the room for an additional minute and have a capacity of ∞ unless each room has a capacity then we use that instead. We then introduce arcs r_{uv} for inter-room travel. We have one arc going from every minute t in room u to minute $t + 1$ in room v. Arcs r_{uv} will have no cost but a capacity of the original capacities introduced for travelling between rooms. For example the 0th minute node in room$A$ will have an r_{uv} arc to the 1st minute node of room B, C and E with capacities 15, 10, 5 respectively. This is repeated until we reach the 4th minute node where unfortunately the person has ran out of time :(.
   As we reach the last set of minute nodes which is the exit room F. We do not need arcs connecting the minute nodes since logically a person reaching the exit is safe and there is no harm. Instead each node will have an arc s_{uv} to our sink described earlier, with ∞ capacity and 0 cost. The sink will have a demand that is the sum of all the people in the factory. The graph structure forces the problem to be solved under 4 minutes as all 5th minute nodes are "dead" nodes with no demand so people can't just stay there. Example diagram below:



The optimal solution to the MCFP corresponds to the optimal solution to the original problem is that since the MCFP's obj func will be the sum of the costs of the arcs going from minute to minute for each room we are minimizing the harm done. The 0 demand and supply on the minute nodes ensures that this is done in 4 min as explained above so in order to satisfy the contraint of $x(\delta(\bar{v})) - x(\delta(v)) = 0$ they need to leave by the 4th minute. Satisfying the demand on the sink shows that all the people has escaped. The capacities contraints are also held in the MCFP since moving from room to room for every minute is restricted by r_{uv} arc's capacity.

2. transshipment



We see that the red lines show the optimal path with the green numbers showing the flow on each arc. We sub those into the obj func and get a value of -130. Then we calculate our potentials. Since $\bar{x}$ is optimal our cs conditions for those arcs are tight. So we see that
$z_2 = 0 + 30 = 30$
$z_4 = 0 - (-40) = 40$
$z_3 = 30 + (-40) = -10$
$z_1 = -10 + 20 = 10$
subbing these values of z into the obj func of our original lp we get a value of -130 as well.

3. shortest path and rooted trees

   (a) The algorithm will involve finding an st dipath from the node s to every $t \in N$ for the graph D. And the resulting arcs will form a spanning tree rooted at S. Then for the certificate we can say that if there exists a node $v \in N$ such that there is no sv dipath then the spanning tree rooted at s does not exist.
   We prove this by showing that D has a spanning tree that is rooted at s if and only if there exists an st dipath from s to every $t \in N$.
   $(\rightarrow)$ So suppose that there exists a spanning tree T rooted at s for D. Then by the definition of a tree there exists a path from the root node to every $t \in T$. Since T is a spanning tree then by the def of a spanning tree we can say that there exists an st dipath for every node in D
   $(\leftarrow)$ So lets suppose that every $t \in N$ has a dipath from s, and we a spanning tree at rooted at s for D doesn't exist. Then we can still produce a tree T. Since a spanning tree does not exist we can say that the nodes of T is strictly a subset of the nodes of D. So then let a node v be the difference between the two sets. By our hypothesis we can see that an sv dipath exists for this node v so then $T + sv$ is a tree with more nodes than our previous tree. So if we continue to find add nodes from N that are not in T then we can produce a spanning tree since sv dipath exists for all $v \in N$. So contradiction.

   (b) $(\rightarrow)$ Let T be a spanning tree of D rooted at s. In order of the node s to be considered to be a root it must have no "parent" nodes i.e nodes that give it an input so the node s has an in-degree of 0. We also see by definition that a tree T is rooted at S if there is an unique path from s to all $v \in N$. So lets assume that there exists $v \in T$ where the in degree for v is not one. So we have two cases, if the in-degree is 0 then we are done as that shows there is no dipath from s to v which contradicts the fact that T is a spanning tree. The second case is that the in-degree is > 1 This means there are multiple arcs uv going into v so that signifies multiple sv dipaths violating the definition given and the definition of a tree. So contradiction.
   $(\leftarrow)$ Lets suppose that the in-degree of s is 0 and the in-degree of all other nodes is 1 in T. Since s has an in-degree of zero we know that it is the root by definition. Since all $v \in T$ only has a in degree of 1 we know each v has only one uv arc in T so there is a unique $sv - path$ for all $v \in N$ meaning that by the definition the spanning tree T is a tree rooted at s.