CO 351 : Network Flow Theory A2

Term: Spring 2019

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1. Digraph proofs

(a)

$$x(\delta(\bar{S})) - x(\delta(S)) = \sum_{v \in S}[x(\delta(\bar{v}))) - x(\delta(v))]$$

$$\sum_{v \in S}[x(\delta(\bar{v}))) - x(\delta(v))] = \sum_{v \in S}[\sum_{e \in \delta(\bar{v})} x(e) - \sum_{e \in \delta(v)} x(e)]$$

From here we can see that except the supply and demand nodes

$$\sum_{e \in \delta(\bar{v})} x(e) - \sum_{e \in \delta(v)} x(e) = 0$$

and the nodes with supply and demand is just b_v so we get

$$\sum_{v \in S} \sum_{e \in \delta(\bar{v})} x(e) - \sum_{v \in S} \sum_{e \in \delta(v)} x(e)$$

expanding

$$[(b_v + \sum_{e \in \delta(\bar{S})} x(e)] - [(b_v + \sum_{e \in \delta(S)} x(e)]$$

cancel out the b_v and we get our LHS as $x(\delta(S)) = \sum_{e \in \delta(S)} x(e)$ so the equality is proven.

(b) We prove the hypothesis with induction.The first case is when the s, t-dipath leaves the s-t cut set and never re-enters. By theorem from class we know every s,t-dipath must use at least one edge from any s-t cut. So in this case P will only ever use one arc from $\delta(S)$ which is intuitively true. Let this one arc be uv since $uv \in A(P)$ and $uv \in \delta(S)$, $u \in S$, $v \notin S$ so any further arc from v are not part of $\delta(S)$. So as in this case P never re-enters S, $\delta(\bar{S}) = 0$ so $|A(P) \bigcap \delta(\bar{S})| = 0$ and so $|A(P) \bigcap \delta(S)| = 1$ so the equality holds.
The other case is P uses more than one edge from $\delta(S)$ meaning it re-enters S as we shown above, in order to use more than one arc from $\delta(S)$, P would have to re-enter S. Then by the definition of a s-t dipath and s-t cut. In order for P to re-enter S it would have to leave S again so for every re-entry arc $\in \delta(\bar{S})$ in to S a leaving arc $\in \delta(S)$ is also used. As P needs to reach t and $t \notin S$, for every n arcs it shares with $\delta(\bar{S})$ it shares $n + 1$ arcs with $\delta(S)$. So our equality always holds.

2. Transshipment LP and Dual

(a) min $120x_{fc} + 10x_{cb} + 70x_{ba} + 20x_{ac} + 50x_{da} + 110x_{de} + 60x_{be} + 10x_{ef} + 40x_{fd}$

s.t $-x_{cb} + x_{ac} + x_{fc} = 15$

$-x_{fc} - x_{fd} + x_{ef} = -10$

$-x_{ef} + x_{be} + x_{de} = 5$

$-x_{da} - x_{de} + x_{fd} = -8$

$-x_{ac} + x_{da} + x_{ba} = 5$

$-x_{ba} + x_{cb} - x_{be} = -7$

$x \geq 0$

(b) max $5y_a - 7y_b + 15y + c - 10y_f + 5y_e - 8y_d$

s.t $y_a - y_b \leq 70$

$y_a - y_d \leq 50$

$y_b - y_c \leq 10$

$y_c - y_a \leq 20$

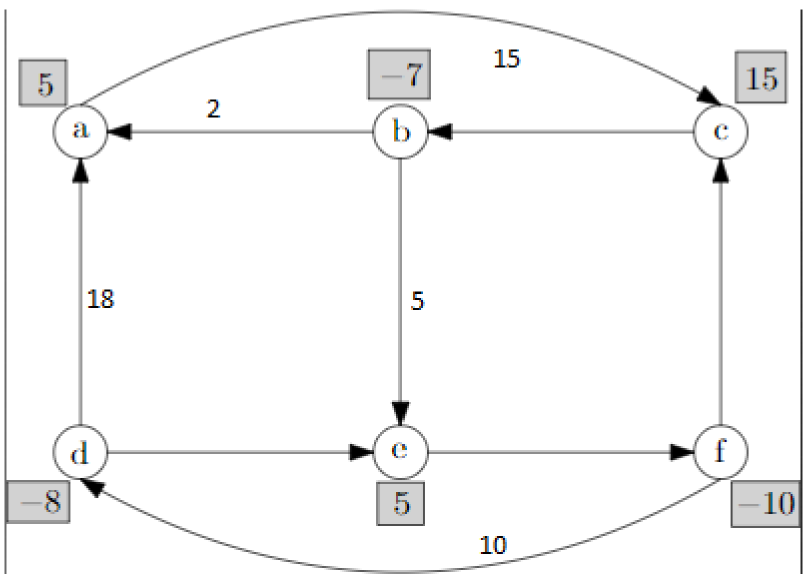
$y_c - y_f \leq 120$

$y_d - y_f \leq 40$

$y_e - y_d \leq 110$

$y_e - y_b \leq 60$

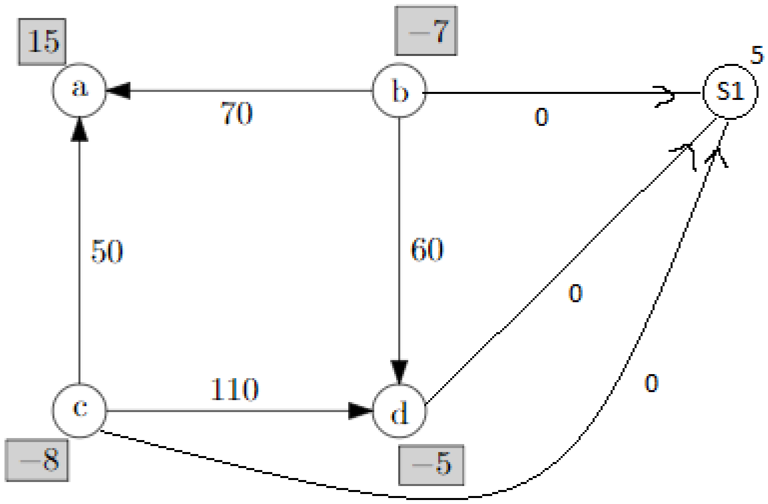
$y_f - y_e \leq 10$



(c) so from this flow we get the optimal solution value of 2040
from $2 * 70 + 4 * 60 + 10 * 40 + 18 * 50 + 15 * 20$
we used $x_{ba} = 2, x_{ac} = 15, x_{da} = 18, x_{be} = 5, x_{fd} = 10$

(d) so from part c) we see that $y_a - yb \leq 70$, $y_c - ya \leq 20$, $y_a - yd \leq 50$, $y_e - yb \leq 60$, $y_d - yf \leq 40$ are tight constraints as their corresponding $x \geq 0$. with $y_a = 0$ we do some simple algebra and get $y_b = -70, y_e = -10, y_d = -50, y_f = -90, y_c = 20$. Subbing into the obj func we get 2040 which is the same answer as our primal. Then we check that this is feasible by subbing the y values back into the constraints we will do this for y_d so $0 - y_d = 50 \leq 70$ and $-10 - (-50) = 40 \leq 60$ we repeat this for every y and find that the solution is feasible. So we conclude the above is the optimal sol for our dual

3. Modified transshipment problem
Given LP1 we introduce some slack variables to convert the inequality to equality constraints. We use this to simulate an extra "demand" node that will take the extra supply. In order to keep the obj function consistant between the two LPs we make the cost of arcs that are directed to this new demand node 0. Example below



this allows us to intoduce a new constraint for the new demand node so the LP2 will be
min $70x_{ba} + 60x_{ab} + 110x_{cd} + 50x_{ca} + 0x_{cs1} + 0x_{bs1} + 0x_{ds1}$
s.t $-x_{bd} - x_{ba} - x_{bs1} = -7$
$-x_{cd} - x_{ca} - x_{cs1} = -8$
$x_{bd} + x_{cd} - x_{ds1} = -5$
$x_{ca} + x_{ba} = 15$
$x_{bs1} + x_{cs1} + x_{ds1} = 5$
$x \geq 0$

Now lets assume $\bar{x}^{'}$ to be the optimal solution for LP2 with the format $(x_1, x_2, ..., x_k, s_i, ..., s_k)^T$ we can then construct our optimal solution $\bar{x}$ for LP1 by removing the s variables and leaving the values assigned to the x vars the same. We retain the same optimal objective value as the coefficients for s are 0 and the values for x are unchanged. $\bar{x}$ is still feasible as we are changing from $=$ to $\geq$ for the supply constraints.