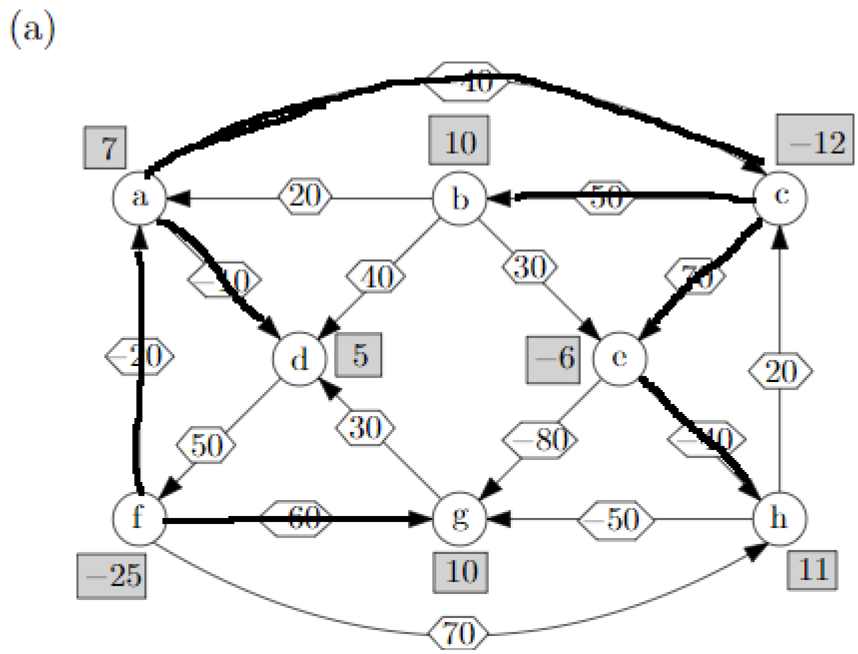
CO 351 : Network Flow Theory A3

Term: Spring 2019

Instructor: M. Pei

1. Optimal, unbounded or infeasible

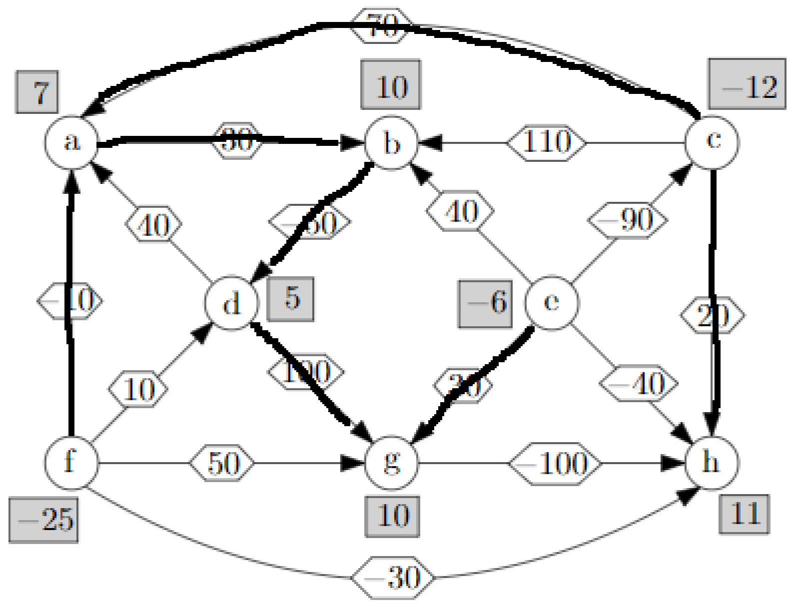
   (a) We see that this TP has an optimal solution, since a feasible flow exists

   we also see that there are no negative dicycles with G for this TP so we know it is not unbounded by theorem from class. Then by the fundamental theorem of LP since this LP isn't unbounded and it is feasible then it must have an optimal solution

   (b) We see that this TP is infeasible as $\{f, a, d, h\} \subseteq N$ such that $b(S) < 0$ and $\delta(S) = \emptyset$ since $b(s) = -2$ f, a, d, h's leaving arcs are not in the cut $\delta(S)$ we see that the only three outward arcs are fa, ad, fd and their end nodes are all contained within the set. So by theorem from class this TP is infeasible.

(c) We see that this TP has an optimal solution, since a feasible flow exists



we also see that there are no negative dicycles with G for this TP so we know it is not unbounded by theorem from class. Then by the fundamental theorem of LP since this LP isn't unbounded and it is feasible then it must have an optimal solution

(d) We see that this TP is infeasible as $\{c, e, h\} \subseteq N$ such that $b(S) < 0$ and $\delta(S) = \emptyset$ since $b(s) = -7$ and all of e, c, h's leaving arcs are not in the cut $\delta(S)$ as eh, ce, hc are all arcs with ending nodes in S then by theorems from class since this TP is infeasible.

2. Negative cycle and unboundedness

   (a) By contradiction let's assume that y is not feasible for D'. This means $\bar{w_{uv}} \leq 0$ and so $y_v - y_u \geq w_u v$. Let S be a dipath from z to u. So then we have two cases one where $v \in S$ and $v \notin S$. Consider the $v \in S$ case So from z to v the cost should be y_v at least which means that the cost from v to u is $y_u - y_v$. Then let cycle C be constructed from path of v to u and the uv arc. So the cost of this cycle is $y_u - y_v + w_u v$ we see that this cost is negative since our original assumption is that $y_v - y_u \geq w_{uv}$ so this creates a negative dicycle which is a contradiction. Now we assume that $v \notin S$ Then $S + v$ is a dipath connecting $z - v$ where the cost is $y_u + w_{uv}$ which by our original assumption is less than y_v since $y_v - y_u \geq w_{uv}$ meaning that it is not the least cost z-v dipath. So contradiction

   (b) Assume that D does not contain a negative dicycle. Then by part a we know that y is a feasible potential for D' which by extension is also feasible for D as the newly added arcs cost 0. So since y are feasible for D then we know that $\bar{w_{uv}} \geq 0$ so it has an optimal solution and is bounded.

3. Lower bound arcs on TP
   (=>) We assume that the TP is infeasible with the lower bound restrictions on the arcs. Since the TP is infeasible, either $x(\delta(\bar{v})) - x(\delta(v)) = b_v$ doesn't hold or $x_e \geq l_e$ doesn't hold. Let $v \in S$ and v is a supply node, if $x(\delta(\bar{v})) - x(\delta(v)) < b_v$, then since the flow is trying to meet the $x_e \geq l_e$. Then v is sending out more flow than it has for supply or v so $b(v) < l(\delta(\bar{v}))$ and since $v \in S$, $b(S) > l(\delta(\bar{S}))$ and S is non-empty. The case $x(\delta(\bar{v})) - x(\delta(v)) > b_v$ is not considered as if the TP has extra supply a new sink node can be added, shown in A2. If $x_e \geq l_e$ doesn't hold then that means the flow was unable to meet the lower bound, so let $v \in S$ and $b(v) < l(\delta(\bar{v}))$ as the total supply in v cannot meet the minimum requirements. So since $v \in S$, $b(S) > l(\delta(\bar{S}))$ and S is non-empty.
   (<=) Assume that $b(S) > l(\delta(\bar{S}))$ and S non-empty. Let $v \in S$, since $b(v) > l(\delta(\bar{v}))$ and the we try to satisy both $x(\delta(\bar{v})) - x(\delta(v)) = b_v$ and $x_e \geq l_e$. If $x(\delta(\bar{v})) - x(\delta(v)) = b_v$ holds then it is easy to see that $x_e \geq l_e$ since the supply is less than the minimium requirements. If $x_e \geq l_e$. holds then that means the flow must be sending out more units than it has to meet the lower bound as $b(v) > l(\delta(\bar{v}))$. So one of the constraints is always not fulfilled and thus the TP is infeasible.