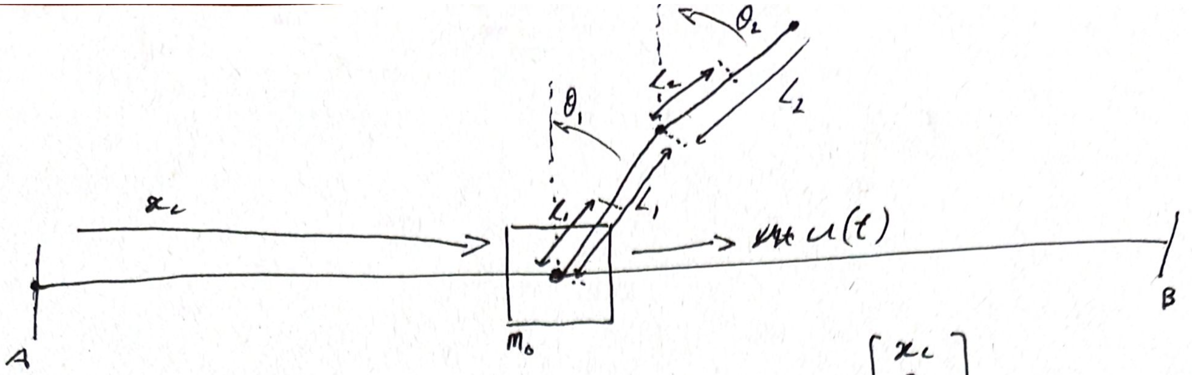

State vector: $x = \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \end{bmatrix}$

m_0 - Mass of cart
m_1 - Mass of pendulum 1
m_2 - Mass of pendulum 2

Cart:

$x_c = x_c$
$y_c = 0$

Pendulum 1: (Midpoint positions)

$x_1 = x_c + l_1 \sin(\theta_1)$ $\qquad \frac{dx_1}{dt} = \dot{x}_c + \dot{\theta}_1 l_1 \cos(\theta_1)$

$y_1 = 0 + l_1 \cos(\theta_1)$ $\qquad \frac{dy_1}{dt} = -l_1 \dot{\theta}_1 \sin(\theta_1)$

Pendulum 2:

$x_2 = x_c + L_1 \sin(\theta_1) + l_2 \sin(\theta_2)$
$y_2 = 0 + L_1 \cos(\theta_1) + l_2 \cos(\theta_2)$

Lagrange:

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{x}}\right] - \frac{\delta L}{\delta x} = q$$

q - Vector of generalised forces in terms of x.

$$q = \begin{bmatrix} u(t) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$L = E_k - E_p$

E_k - Kinetic energy.
E_p - Potential energy.

Cart:

$E_k = \frac{1}{2} m v^2$ $\qquad$ $E_p = 0$

$= \frac{1}{2} m_c \dot{x}_c^2$

Pendulum 1:

$E_k = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$ $\qquad$ $v^2 = \dot{x}_1^2 + \dot{y}_1^2$, $\omega^2 = \dot{\theta}_1^2$

$$E_k = \frac{m}{2}\left[\dot{x}_1^2 + \dot{y}_1^2\right] + \frac{1}{2} I \dot{\theta}_1^2$$

$$\cancel{= \frac{1}{2} m \left[(x_c + l_1 \sin(\theta_1))^2 + l_1^2 \cos^2(\theta_1)\right] + \frac{1}{2} I_1 \dot{\theta}_1^2}$$

$$\cancel{= \frac{1}{2} m \left[x_c^2 + 2 x_c l_1 \sin(\theta_1) + l_1^2 \sin^2(\theta_1) + l_1^2 \cos^2(\theta_1)\right] + \frac{1}{2} I_1 \dot{\theta}_1^2}$$

$$= \frac{1}{2} m_1 \left[(\dot{x}_c + \dot{\theta}_1 l_1 \cos(\theta_1))^2 + \dot{\theta}_1^2 l_1^2 \sin^2(\theta_1)\right] + \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$= \frac{1}{2} m_1 \left[\dot{x}_c^2 + 2 \dot{x}_c \dot{\theta}_1 l_1 \cos(\theta_1) + \dot{\theta}_1^2 l_1^2 \cos^2(\theta_1) + \theta_1^2 l_1^2 \sin^2(\theta_1)\right] + \frac{1}{2} I_1 \dot{\theta}_1^2$$

$$= \frac{1}{2} m_1 \dot{x}_c^2 + \frac{1}{2}(m_1 l_1^2 + I_1) \dot{\theta}_1^2 + m_1 l_1 \dot{x}_c \dot{\theta}_1 \cos(\theta_1) \quad \star$$

$E_p = m_1 g y$

$= m g l_1 \cos(\theta_1) \quad \star$

Pendulum 2:

$$\frac{dx_2}{dt} = \dot{x}_c + \dot{\theta}_1 L_1 \cos(\theta_1) + \dot{\theta}_2 l_2 \cos(\theta_2)$$

$$\frac{dy_2}{dt} = -\dot{\theta}_1 L_1 \sin(\theta_1) - \dot{\theta}_2 l_2 \sin(\theta_2)$$

$$E_k = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 \longrightarrow \quad v^2 = \dot{x}_2^2 + \dot{y}_2^2 \quad , \omega^2 = \dot{\theta}_2^2$$

$$(\dot{x}_2)^2 = \left[\dot{x}_c + \dot{\theta}_1 L_1 \cos(\theta_1) + \dot{\theta}_2 l_2 \cos(\theta_2)\right]^2$$

$$= \dot{x}_c^2 + \dot{\theta}_1^2 L_1^2 \cos^2(\theta_1) + \dot{\theta}_2^2 l_2^2 \cos^2(\theta_2) + 2\dot{x}_c \dot{\theta}_1 L_1 \cos(\theta_1) + 2\dot{x}_c \dot{\theta}_2 l_2 \cos(\theta_2)$$
$$+ 2\dot{\theta}_1 L_1 \cos(\theta_1) \dot{\theta}_2 l_2 \cos(\theta_2)$$

$$(\dot{y}_2)^2 = \left[-\dot{\theta}_1 L_1 \sin(\theta_1) - \dot{\theta}_2 l_2 \sin(\theta_2)\right]^2$$

$$= \dot{\theta}_1^2 L_1^2 \sin^2(\theta_1) + \dot{\theta}_2^2 l_2^2 \sin^2(\theta_2) + 2\dot{\theta}_1 L_1 \sin(\theta_1) \dot{\theta}_2 l_2 \sin(\theta_2)$$

$$v^2 = \dot{x}_c^2 + \dot{\theta}_1^2 L_1^2 \cos^2(\theta_1) + \dot{\theta}_2^2 l_2^2 \cos^2(\theta_2) + 2\dot{x}_c \dot{\theta}_1 L_1 \cos(\theta_1) + 2\dot{x}_c \dot{\theta}_2 l_2 \cos(\theta_2) + 2\dot{\theta}_1 L_1 \cos(\theta_1) \dot{\theta}_2 l_2 \cos(\theta_2)$$
$$+ \dot{\theta}_1^2 L_1 \sin^2(\theta_1) + \dot{\theta}_2^2 l_2^2 \sin^2(\theta_2) + 2\dot{\theta}_1 L_1 \sin(\theta_1) \dot{\theta}_2 l_2 \sin(\theta_2)$$

$$= \dot{x}_c^2 + \dot{\theta}_1^2 L_1^2 + \dot{\theta}_2^2 l_2^2 + 2\dot{\theta}_1 L_1 \dot{\theta}_2 l_2 \overbrace{\left[\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right]}^{\cos(\theta_1 - \theta_2)}$$
$$+ 2\dot{x}_c \dot{\theta}_1 L_1 \cos(\theta_1) + 2\dot{x}_c \dot{\theta}_2 l_2 \cos(\theta_2)$$

$$E_k = \frac{1}{2} m_2 \left[x_c^2 + \dot{\theta}_1^2 L_1^2 + \dot{\theta}_2^2 l_2^2 + 2\dot{\theta}_1 L_1 \dot{\theta}_2 l_2 \cos(\theta_1 - \theta_2) + 2\dot{x}_c \dot{\theta}_1 L_1 \cos(\theta_1) + 2\dot{x}_c \dot{\theta}_2 l_2 \cos(\theta_2)\right]$$
$$+ \frac{1}{2} I_2 \dot{\theta}_2^2$$

$$= \frac{1}{2} m_2 x_c^2 + \frac{m_2}{2} \dot{\theta}_1^2 L_1^2 + \frac{1}{2}(m_2 l_2^2 + I_2)\dot{\theta}_2^2 + m_2 L_1 \dot{\theta}_1 \dot{x}_c \cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \dot{x}_c \cos(\theta_2)$$
$$+ m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$E_p = m_2 g y_2$$

$$= m_2 g \left[L_1 \cos(\theta_1) + l_2 \cos(\theta_2) \right]$$

Total kinetic energy is then:

$$E_K = \frac{1}{2} m_c \dot{x}_c^2 + \frac{1}{2} m_1 \dot{x}_c^2 + \frac{1}{2}\left[m_1 l_1^2 + I_1\right] \dot{\theta}_1^2 + m_1 l_1 \dot{x}_c \dot{\theta}_1 \cos(\theta_1)$$
$$+ \frac{1}{2} m_2 x_c^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 L_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right) \dot{\theta}_2^2 + m_2 L_1 \dot{\theta}_1 \dot{x}_c \cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \dot{x}_c \cos(\theta_2)$$
$$+ m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

Total Potential energy is:

$$E_p = m_1 g l_1 \cos(\theta_1) + m_2 g \left[L_1 \cos(\theta_1) + l_2 \cos(\theta_2) \right]$$
$$= g(m_1 l_1 + m_2 L_1) \cos(\theta_1) + m_2 g l_2 \cos(\theta_2)$$

$$L = \frac{1}{2} m_c \dot{x}_c^2 + \frac{1}{2} m_1 \dot{x}_c^2 + \frac{1}{2}\left[m_1 l_1^2 + I_1\right] \dot{\theta}_1^2 + m_1 l_1 \dot{x}_c \dot{\theta}_1 \cos(\theta_1) + \frac{1}{2} m_2 x_c^2 + \frac{1}{2} m_2 \dot{\theta}_1^2 L_1^2$$
$$+ \frac{1}{2}\left(m_2 l_2^2 + I_2\right) \dot{\theta}_2^2 + m_2 L_1 \dot{\theta}_1 \dot{x}_c \cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \dot{x}_c \cos(\theta_2) + m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$- g(m_1 l_1 + m_2 L_1) \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)$$

$$= \frac{1}{2}(m_c + m_1 + m_2) \dot{x}_c^2 + \frac{1}{2}\left[m_1 l_1^2 + I_1 + m_2 L_1^2\right] \dot{\theta}_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right) \dot{\theta}_2^2$$
$$+ (m_1 l_1 + m_2 L_1) \dot{x}_c \dot{\theta}_1 \cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \dot{x}_c \cos(\theta_2) + m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$- g(m_1 l_1 + m_2 L_1) \cos(\theta_1) + m_2 g l_2 \cos(\theta_2)$$

Now we need:

$$x = \begin{bmatrix} x_c \\ \theta_1 \\ \theta_2 \end{bmatrix}, \quad q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{x}}\right] - \frac{\delta L}{\delta x} = q$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{x}_c}\right] - \frac{\delta L}{\delta x_c} = u(t) \quad (1)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{\theta}_1}\right] - \frac{\delta L}{\delta \theta_1} = 0 \quad (2)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{\theta}_2}\right] - \frac{\delta L}{\delta \theta_2} = 0 \quad (3)$$

$$\frac{\delta L}{\delta \dot{x}_c} = (m_c + m_1 + m_2)\dot{x}_c + (m_1 l_1 + m_2 L_1)\dot{\theta}_1 \cos(\theta_1) + m_2 l_2 \dot{\theta}_2 \cos(\theta_2)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{x}_c}\right] = (m_c + m_1 + m_2)\ddot{x}_c + (m_1 l_1 + m_2 L_1)\ddot{\theta}_1 \cos(\theta_1) + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2)$$
$$- (m_1 l_1 + m_2 L_1)\dot{\theta}_1^2 \sin(\theta) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$

$$\frac{\delta L}{\delta x_c} = 0$$

$\therefore$ (1):

$$(m_c + m_1 + m_2)\ddot{x}_c + (m_1 l_1 + m_2 L_1)\ddot{\theta}_1 \cos(\theta_1) + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - (m_1 l_1 + m_2 \overset{L_1}{\cancel{l_2}})\dot{\theta}_1^2 \sin(\theta_1)$$
$$- m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2) = u(t)$$

$$\frac{\delta L}{\delta \dot{\theta}_1} = (m_1 l_1^2 + I_1 + m_2 L_1^2)\dot{\theta}_1 + (m_1 l_1 + m_2 L_1)\dot{x}_c \cos(\theta_1) + m_2 L_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{\theta}_1}\right] = (m_1 l_1^2 + I_1 + m_2 L_1^2)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos(\theta_1)\ddot{x}_c - (m_1 l_1 + m_2 L_1)\dot{x}_c \overset{\sin}{\cancel{\cos}}(\theta_1)\dot{\theta}_1$$

$$+ m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 L_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 - m_2 L_{\cancel{m_1}} l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(-\dot{\theta}_2)$$

$$= (m_1 l_1^2 + I_1 + m_2 L_1^2)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos(\theta_1)\ddot{x}_c - (m_1 l_1 + m_2 L_1)\dot{x}_c \sin(\theta_1)\dot{\theta}_1$$

$$+ m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 L_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$

$$\cancel{\frac{\delta L}{\delta \theta_1} = (m_1 l_1 + m_2 L_1)\ddot{x}_c \dot{\theta}_1}$$

$$\frac{\delta L}{\delta \theta_1} = -(m_1 l_1 + m_2 L_1)\dot{x}_c \dot{\theta}_1 \sin(\theta_1) - m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + g(m_1 l_1 + m_2 L_1)\sin(\theta_1)$$

$\therefore$ 2 =

$$(m_1 l_1^2 + I_1 + m_2 L_1^2)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos(\theta_1)\ddot{x}_c \cancel{- (m_1 l_1 + m_2 L_1)\dot{x}_c \sin(\theta_1)\dot{\theta}_1}$$

$$+ m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 - m_2 L_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\cancel{\dot{\theta}_1} - \dot{\theta}_2) \cancel{+ (m_1 l_1 + m_2 L_1)\dot{x}_c \dot{\theta}_1 \sin(\theta_1)}$$

$$\cancel{+ m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} - g(m_1 l_1 + m_2 L_1)\sin(\theta_1) = 0$$

$$(m_1 l_1^2 + I_1 + m_2 L_1^2)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos(\theta_1)\ddot{x}_c + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2$$

$$\cancel{+ (m_1 l_1 + m_2 L_1)\sin(\theta_1)\ddot{x}_c \ddot{\theta}_1} + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 - g(m_1 l_1 + m_2 L_1)\sin(\theta_1)$$

$$= 0 \qquad (2)$$

$$\frac{\delta L}{\delta \dot{\theta}_2} = \frac{1}{2}(m_2 l_2^2 + I_2)\dot{\theta}_2 - m_2 l_2 \ddot{x}_c \dot{\theta} \cos(\theta_2) + m_2 L_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{\theta}_2}\right] =$$

$$\frac{\delta L}{\delta \dot{\theta}_2} = (m_2 l_2^2 + I_2)\dot{\theta}_2 + m_2 l_2 \dot{x}_c \cos(\theta_2) + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1$$

$$\frac{d}{dt}\left[\frac{\delta L}{\delta \dot{\theta}_2}\right] = (m_2 l_2^2 + I_2)\ddot{\theta}_2 + m_2 l_2 \ddot{x}_c \cos(\theta_2) - m_2 l_2 \dot{x}_c \dot{\theta}_2 \sin(\theta_2) + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1$$
$$- m_2 L_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)\dot{\theta}_1 - m_2 L_1 l_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2)(-\dot{\theta}_2)$$

$$= m_2 l_2 \cos(\theta_2)\ddot{x}_c + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2$$
$$- m_2 l_2 \dot{x}_c \dot{\theta}_2 \sin(\theta_2) - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 \quad *$$

$$\frac{\delta L}{\delta \theta_2} = - m_2 l_2 \dot{\theta}_2 \dot{x}_c \sin(\theta_2) + m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin(\theta_2) \quad *$$

$\therefore$ (3) =

$$m_2 l_2 \cos(\theta_2)\ddot{x}_c + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (m_2 l_2^2 + I_2)\ddot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2$$
$$- m_2 g l_2 \sin(\theta_2) = 0 \qquad (3)$$

Now we write the system in matrix form.
Something like:

$$D(x)\ddot{x} + C(x, \dot{x})\dot{x} + G(x) = Hu$$

...

$$D(x) = \begin{bmatrix} m_c + m_1 + m_2 & (m_1 l_1 + m_2 L_1)\cos(\theta_1) & m_2 l_2 \cos(\theta_2) \\ (m_1 l_1 + m_2 L_1)\cos(\theta_1) & m_1 l_1^2 + I_1 + m_2 L_1^2 & m_2 L_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 l_2 \cos(\theta_2) & m_2 L_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 + I_2 \end{bmatrix}$$

$$C(x, \dot{x}) = \begin{bmatrix} 0 & -(m_1 l_1 + m_2 L_1)\sin(\theta_1)\dot{\theta}_1 & -m_2 l_2 \sin(\theta_2)\dot{\theta}_2 \\ 0 & 0 & m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ 0 & -m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ -g(m_1 l_1 + m_2 L_1)\sin(\theta_1) \\ -g m_2 l_2 \sin(\theta_2) \end{bmatrix}$$

$$H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$l_1 = \frac{1}{2}L_1, \quad l_2 = \frac{1}{2}L_2, \quad I_1 = \frac{1}{12}m_1 L_1^2, \quad I_2 = \frac{1}{12}m_2 L_2^2$$

$$D(x) = \begin{bmatrix} m_c + m_1 + m_2 & (\frac{1}{2}m_1 + m_2)L_1 \cos(\theta_1) & \frac{1}{2}m_2 L_2 \cos(\theta_2) \\ (\frac{1}{2}m_1 + m_2)L_1 \cos(\theta_1) & (\frac{1}{3}m_1 + m_2)L_1^2 & \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2}m_2 L_2 \cos(\theta_2) & \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1 - \theta_2) & \frac{1}{3}m_2 L_2^2 \end{bmatrix}$$

$$C(x,\dot{x}) = \begin{bmatrix} 0 & -(\frac{1}{2}m_1 + m_2)L_1 \sin(\theta_1)\dot{\theta}_1 & -\frac{1}{2}m_2 L_2 \sin(\theta_2)\dot{\theta}_2 \\ 0 & 0 & \frac{1}{2}m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2 \\ 0 & -\frac{1}{2}m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1 & 0 \end{bmatrix}$$

$$G(\theta) = \begin{bmatrix} 0 \\ -\frac{1}{2}g(m_1 + m_2)L_1 \sin(\theta_1) \\ -\frac{1}{2}g m_2 L_2 \sin(\theta_2) \end{bmatrix}$$

$$H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We want a first order system....

$$D\ddot{x} + C\dot{x} + G = Hu$$

$$D\ddot{x} = -C\dot{x} - G + Hu$$

$$\ddot{x} = -D^{-1}C\dot{x} - D^{-1}G + D^{-1}Hu \quad *$$

let $\theta = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \dot{\theta} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

$$\therefore \quad \dot{\theta} = \begin{bmatrix} 0^{3\times3} & I^{3\times3} \\ 0^{3\times3} & -D^{-1}C \end{bmatrix}\theta + \begin{bmatrix} 0^{3\times1} \\ -D^{-1}G \end{bmatrix} + \begin{bmatrix} 0^{3\times1} \\ D^{-1}H \end{bmatrix}u$$