Chapter 1

Binary words avoiding powers

A finite set is referred to as an **alphabet**; its elements are **letters**. Let Σ be an alphabet.

An **overlap** is a word of the form BBb where B is a word and b is the first letter of B. For example, $acdcacdca$ is an overlap, taking $B = acdc$, $b = a$. A word w is **overlap-free** if none of its factors are overlaps. Otherwise w **contains an overlap**.

If word w can be written $w = ps$, we call p a **prefix** of w, and s a **suffix** of w. We write $p \leq_p w$, $s \leq_s w$. We say that w is a **right-extension** of p, and a **left-extension** of s. If $w = pus$, we call u a **factor** of w, and w an **extension** of u. The **index** of factor u is $|p|$, the number of letters in p. Thus 123 is a prefix of 123456, while 56 is a suffix. Word 345 is a factor of 123456. On the other hand, 123456 is a right extension of 123, a left extension of 56 and an extension of 123, of 56, and of 345.

An ω**-word** over Σ is a sequence $w = \{a_n\}_{n=1}^{\infty}$ where $a_n \in A$ for $n \in \mathbb{N}$. The prefixes of w are the finite words $a_1a_2\cdots a_n$ for $n \in \mathbb{N}$. (When $n = 0$ we interpret $a_1a_2\cdots a_n$ to be ϵ, the empty word.) We say that finite word u is a factor of w if u is a factor of some finite prefix of w.

1. Fix a finite alphabet Σ. Let $L \subseteq \Sigma^*$. Suppose that L is infinite.

   (a) Show that if L is closed under taking prefixes, then there is an ω-word w over Σ such that every prefix of w is in L.

   (b) Give an example to show that if L is not closed under taking prefixes, then the claim may fail.

2. Show that if u is an overlap, then there exist words v, w, z such that $u = wv = zw$ and $|w| > |v|$.

3. Show that if u is an overlap, then either

   (a) There is a non-empty word x such that $u = xxx$

   or

   (b) There are non-empty words x and y such that $u = xyxyx$.

   Let $\mu : \{0,1\}^* \to \{0,1\}^*$ be the morphism generated by $\mu(0) = 01$, $\mu(1) = 10$.

4. Suppose that $v \in \{0,1\}^*$ contains an overlap; then $\mu(v)$ contains an overlap.

5. Let $w \in \{0,1\}^*$ be overlap-free. Show that there exist words $u, v, z \in \{0,1\}^*$ such that $w = u\mu(v)z$ where v is overlap-free and $u, z \in \{\epsilon, 0, 1, 00, 11\}$.

6. Let n be a non-negative integer, and write $\mu^n(0) = a_0 a_1 a_2 \cdots a_{2^n-1}$, where $a_i \in \{0,1\}$, $i = 0, 1, \ldots 2^n - 1$. For each non-negative integer i, denote by $b(i)$ the (mod 2) sum of the digits in the binary representation of i; for example, $b(13) = 1 + 1 + 0 + 1 = 1$. For $i = 0, 1, \ldots 2^n - 1$, show that $a_i = b_i$.

7. Define a sequence of words $\{w_n\}_{n=0}^{\infty}$ by $w_0 = 0$, $w_{n+1} = w_n \overline{w}_n$, $n > 0$. Show that for each n, $w_n = \mu^n(0)$.

8. Show that if $w \in \{0,1\}^*$ is overlap-free, then so is $\mu(w)$. Conclude that $\mu^\omega(0)$ is overlap-free.

9. Show that if BB is a non-empty factor of $\mu^\omega(0)$ then $|B|$ has the form 2^n or $3(2^n)$ for some integer n.

   Suppose that alphabet Σ is ordered by $\leq$. The **lexicographic order** $\preceq$ on Σ^* is given recursively by

   (a) $\epsilon \preceq \epsilon$

   (b) For $a, b \in \Sigma$, $u, v \in \Sigma^*$, we have $au \preceq bv$ if and only if

      i. $a \leq b$ or

      ii. $a = b$ and $u \preceq v$

10. Show that there is a lexicographically least binary ω-word over $\{0,1\}$.

11. Show that μ is order-preserving with respect to the lexicographic order on $\{0,1\}^*$; if $u \preceq v$, then $\mu(u) \preceq \mu(v)$.

12. Show that $001001\mu^\omega(1)$ is overlap-free.

13. Show that $\mu^\omega(1)$ is the lexicographically least overlap-free ω-word over $\{0,1\}$ that starts with 1.

14. Show that $001001\mu^\omega(1)$ is the lexicographically least overlap-free ω-word over $\{0,1\}$.

15. For each non-negative integer n let c_n be the number of distinct overlap-free words of length n in $\{0,1\}^*$. Show that there is a constant k such that $c_n \leq n^k$ for every non-negative integer n.

16. Let $h : \{0,1\}^* \to \{0,1\}^*$ be the morphism generated by $h(0) = 001$, $h(1) = 011$. Show that the word $h^\omega(0)$ has no factor xxx with x non-empty.

17. The 'Mephisto waltz' morphism $g : \{0,1\}^* \to \{0,1\}^*$ is generated by $g(0) = 001$, $g(1) = 110$. Show that the word $g^\omega(0)$ has no factor $xxxx$ with x non-empty.

18. The Fibonacci word $f^\omega(0)$ is the fixed point of the morphism $f : \{0,1\}^* \to \{0,1\}^*$ generated by $f(0) = 01$, $f(1) = 0$. Show that the Fibonacci word has no factor xxy where y is a prefix of x with $|y| > \frac{1+\sqrt{5}}{2}|x|$.

19. Show that if uu is a non-empty factor of the Fibonacci word, then u is a conjugate of $f^n(0)$ for some positive integer n.

20. Show that there uncountably many overlap-free ω-words in $\{0,1\}^*$.

    Define the **distance** between ω-words u and v to be

$$d(u,v) = \begin{cases} 0 & u = v \\ 1/k & \text{the longest common prefix of } u \text{ and } v \text{ has length } k \end{cases}$$

21. Show that if u is an overlap-free ω-word over $\{0,1\}$, there are overlap-free ω-words over $\{0,1\}$other than u which are arbitrarily close to u.

    A **bi-infinite word** over Σ is a function $w : BbbZ \to \Sigma$. Let $u = u_1 u_2 \cdots u_n \in \Sigma^*$. Word u is a factor of w if for some integer k, $u_i = w(k+i)$, $i = 1, 2, \cdots n$.

22. Show that there is a bi-infinite word w over $\{0,1\}$ such that no factor of w is an overlap.

23. For each non-negative integer i, denote by $b(i)$ the sum of the digits in the binary representation of i. Show that $\sum_{i=0}^{\infty} b_i 2^{-i}$ is a transcendental number.

    Word w is **a circular overlap-free word** if ww contains no overlaps of length $|w|$ or less. Word v is a **conjugate** of w if there are words u and z such that $w = uz$ and $v = zu$.

24. Show that the relation v *is a conjugate of* w is an equivalence relation.

25. Show that w is a circular overlap-free word if and only if every conjugate of w is overlap-free.

26. Show that for each non-negative integer n, word $\mu^n(0)$ is circular overlap-free.

27. Show that if $w \in \{0,1\}^*$ circular overlap-free, then $|w|$ has the form 2^n or $3(2^n)$ for some integer n.

    An word w over $\{0,1\}$ is a **maximal overlap-free word** if w is overlap-free, but none of its proper extensions is overlap-free.

28. Show that every overlap-free word over $\{0,1\}$ has an extension which is a maximal overlap-free word.

    The **paper-folding word** $a_1 a_2 a_3 a_4 \cdots$ is the ω-word over $\{0,1\}$ given by

$$a_i = \begin{cases} (1 + (-1)^{(i+1)/2})/2, & i \text{ odd} \\ a_{i/2}, & i \text{ even} \end{cases}$$

29. Show that the paper-folding word contains no nonempty factors of the form $xxxx$.

30. Show that the only factors of the form xxx in the paper-folding word are 000 and 111.

    Let u, $v \in \{0,1\}^*$. We say that u **encounters** v if there is a morphism $\phi : \{0,1\}^* \to \{0,1\}^*$ $\phi(v)$ is a factor of u. If u doesn't encounter v, then u **avoids** v. If u avoids v and v avoids u we say that u and v are **mutually incomparable**.

31. Show that there is an infinite collection $\mathcal{C}$ of overlap-free words over $\{0,1\}$ such that if $u, v \in \mathcal{C}$, then u and v are mutually incomparable.

32. Let $\alpha > 2$ be a rational number. Let w be a binary word, and suppose that $\mu(w)$ contains an α power z of period p, $|z| = \alpha p$. Then w contains a word u of period $p/2$, with $|u| \geq |z|/2$.

33. Let k be a rational number. Let w be a binary circular k^+ power-free word. Then $\mu(w)$ is circular k^+ power-free.

Chapter 2

Hints

2.1 Binary words avoiding powers

1.

2. Write $u = BBb$.

3. Consider $\mu(BBb)$.

4. The cases correspond to $|B| = 1, |B| > 1$.

5. Consider the longest word v such that we can write $w = u\mu(v)z$, with $u, v, z \in \{0,1\}^*$.

6. Show that for $1 \leq i \leq 2^n - 1$,

$$a_i = \begin{cases} a_{i/2}, & i \text{ even} \\ 1 - a_{(i-1)/2}, & i \text{ odd} \end{cases}$$

7. Consider the most significant binary digit of i for $2^{n-1} \leq i < 2^n$..

Chapter 3

Solutions

3.1 Binary words avoiding powers

1. (a) We define a sequence of words $\epsilon = w_0 \leq_p w_1 \leq_p w_2 \leq_p w_3 \leq_p \cdots$ and a sequence of sets $L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots$ so that for each non-negative integer n

   i. The set L_n is infinite,

   ii. $|w_n| = n$,

   iii. w_n is a prefix of every word of L_n.

   Let $w_0 = \epsilon$, $L_0 = L$. Certainly, (a), (b), (c) hold here with $n = 0$. For some non-negative integer n, suppose that L_n, w_n have been defined in such a way that (a), (b), (c) hold. Let $u_1, u_2, \ldots, u_{|\Sigma|}$ be all the extensions of w_n of length $n+1$. For $i = 1, 2, \ldots |\Sigma|$, let $U_i = \{u \in L_n : u_i \leq_p u\}$. Since $L_n = \{w_n\} \cup \bigcup_{i=1}^{|\Sigma|} U_i$ is infinite, set U_{i_0} is infinite for some $i_0 \in \{1, 2, \ldots |\Sigma|\}$. We let $w_{n+1} = u_{i_0}$, $L_{n+1} = U_{i_0}$. Then $w_n \leq_p w_{n+1}$, $L_n \supseteq L_{n+1}$ and (a), (b), (c) hold with $n+1$ in place of n.

   We define w to be the unique ω-word having all the w_n as prefixes. Any prefix of w will be w_n for some n, and thus a prefix of a word of L.

   (b) Let $L = \bigcup_{m=0}^{\infty} \{0^{2m}, 1^{2m+1}\}$.

2. Suppose that u is an overlap, $u = BBb$, where b is the first letter of B. If $|B| = 1$, then $B = b$, and we can let $w = v = z = b$.

Otherwise, $|B| \geq 2$, and we can write $B = bC$, some word C. Then $u = BBb = bCbCb$, and the result is true with $w = bCb$, $v = Cb$, $z = bC$.

3. Let $u = BBb$ be an overlap, where b is the first letter of B. If $|B| = 1$, then let $x = B = b$, so that $u = xxx$. If $|B| > 1$, let $b = x$ and write $B = xy$ where y is a non-empty word. Then $BBb = xyxyx$ where $x = b$.

4. Suppose that $bCbCb$ is a factor of v, some $b \in \{0, 1\}$. It follows that $\mu(v)$ contains factor $\mu(bC)\mu(bC)\mu(b)$. From the definition of μ, letter b is the first letter of $\mu(b)$, and hence of $\mu(bC)$. Thus $\mu(bC)\mu(bC)b$ is an overlap contained in $\mu(v)$.

5. Write $w = u\mu(v)z$, with $u, v, z \in \{0, 1\}^*$, and v as long as possible. By the previous exercise, v is overlap-free. Suppose that $u \notin \{\epsilon, 0, 1, 00, 11\}$. (The case where $z \notin \{\epsilon, 0, 1, 00, 11\}$ is similar.) By the maximality of $|v|$, u cannot end in 01 or 10. Thus u ends in 00 or 11. Replacing w by its binary complement if necessary, we suppose that u ends in 00. Since $u \neq 00$, word u ends in 100. Since $\mu(1) = 10$ is a factor of u, the maximality of v indicates that $|\mu(v)| \geq 2$. Since w is overlap free, v starts with 1, lest w contain 100.01, (the dot is for clarity, and indicates the break between u and $\mu(v)$) and hence the overlap 000. Now $\mu(10) = 1001$ is a factor of 100.10, and hence of w. The maximality of v therefore implies that $|\mu(v)| \geq 4$. If v starts 11, then w contains 100.1010, and the overlap 01010. If v starts 10, then w contains 100.1001, which is an overlap. Either case gives a contradiction.

6. For $x \in \{0, 1\}$, $|\mu(x)| = 2$, the first letter of $\mu(x)$ is x, and the second letter of $\mu(x)$ is $1 - x$. Suppose that i is odd, $1 \leq i \leq 2^n - 1$; then $|a_0 a_1 \cdots a_i|$ is even, and $a_0 a_1 \cdots a_i = \mu(a_0 a_1 \cdots a_{(i-1)/2})$. Thus $a_{i-1} a_i = \mu(a_{(i-1)/2})$, and $a_i = 1 - a_{(i-1)/2}$. Suppose that i is even, $1 \leq i \leq 2^n - 1$; then $a_0 a_1 \cdots a_i a_{i+1} = \mu(a_0 a_1 \cdots a_{i/2})$. Thus $a_i a_{i+1} = \mu(a_{i/2})$, and $a_i = a_{i/2}$.

   If i is even, then the binary representation of i ends in a 0, so that $b_i = b_{i/2}$. If i is odd, the binary representation of i ends in a 1, so that $b_i = 1 - b_{(i-1)/2}$, and the b_i obey the same recursion as the a_i. Since, $a_0 = 0 = b_0$, it follows that $a_i = b_i$, $0 \leq i \leq 2^n - 1$.

7. For a non-negative integer n, write $w_n = a_0a_1a_2\cdots a_{2^n-1}$, where $a_i \in \{0,1\}$, $i = 0,1,\ldots 2^n-1$. For each non-negative integer i, denote by $b(i)$ the (mod 2) sum of the digits in the binary representation of i. By the definition of the w_n, for $2^{n-1} \leq i < 2^n$, we have $a_i = 1 - a_{i-2^{n-1}}$. Now consider the binary representation of i; this will consist of a 1, followed by the binary representation of $i-2^{n-1}$. It follows that $b_i = 1-b_{i-2^{n-1}}$. Since $a_0 = b_0 = 0$ and the a_i and b_i satisfy the same recursion, the result follows by induction and the previous exercise.

8. Suppose that $w \in \{0,1\}^*$ is overlap-free, but $\mu(w) = xBBby$ for some words x, y, B with b the first letter of B. We form cases based on whether $|x|$ is even or odd.

   (a) If $|x|$ is even, then $x = \mu(z)$ for some prefix z of w, and $xBB = \mu(zz')$ for some prefix zz' of w. Thus $BB = \mu(z')$, and $|B| = |z'| = |\mu(z')|_1 = |BB|_1 = 2|B|_1$, which is even. Write $B = b_1b_2b_3\cdots b_m$, where $b_1 = b$ and m is even. Then $\mu(z') = BB = b_1b_2b_3\cdots b_mb_1b_2b_3\cdots b_m$, so that $z' = b_1b_3\cdots b_{m-1}b_1b_3\cdots b_{m-1}$. Further, since $xBBb$ is a prefix of $\mu(w)$, and $|xBB|$ is even, we must have $xBBb\overline{b}$ as a prefix of $\mu(w)$, where $\overline{b} = 1-b$. It follows that $zz'b$ is a prefix of w, and w contains the overlap $z'b = b_1b_3\cdots b_{m-1}b_1b_3\cdots b_{m-1}b_1$.

   (b) If $|x|$ is odd, write $B = bB'$. Then $xb = \mu(z)$ for some prefix z of w, and $xBBb = \mu(zz')$ for some prefix zz' of w. Thus $B'bB'b = \mu(z')$, and $|B| = |B'b| = |z'| = |\mu(z')|_1 = |B'bB'b|_1 = 2|B'b|_1$, which is even. Write $B'b = b_1b_2b_3\cdots b_m$, where $b_m = b$ and m is even. Then $\mu(z') = B'bB'b = b_1b_2b_3\cdots b_mb_1b_2b_3\cdots b_m$, so that $z' = b_1b_3\cdots b_{m-1}b_1b_3\cdots b_{m-1}$, and $b_m = 1 - m_{m-1}$. Further, since xb is a prefix of $\mu(w)$, and $|xb|$ is even, we must have $x'b_{m-1}bB'bB'b$ as a prefix of $\mu(w)$, where $x = x'\overline{b}$. Write $x' = \mu(z")$ where $z"$ is a prefix of z. Then $z"b_{m-1}z'$ is a prefix of w, and w contains the overlap $b_{m-1}z- = b_{m-1}b_1b_3\cdots b_{m-1}b_1b_3\cdots b_{m-1}b_1$.

9. Let r be least such that BB is a factor of $\mu^r(0)$, and write $\mu^r(0) = xBBy$. If $r \leq 2$ then BB is a factor of $\mu^2(0) = 0110$, and $|B| = 1 = 2^0$. We prove the result by induction on r. Suppose that whenever uu is a non-empty factor of $\mu^{r-1}(0)$ then $|u|$ has the form 2^s or $3(2^s)$ for some integer s. We make cases depending on whether $|x|$ is odd or even.

(a) If $|x|$ is even, then $x = \mu(z)$ and $xBB = \mu(zz')$ for some prefix zz' of $\mu^{r-1}(0)$. Thus $BB = \mu(z')$, and $|B| = |z'| = |\mu(z')|_1 = |BB|_1 = 2|B|_1$, which is even. Write $B = b_1b_2b_3\cdots b_m$, where m is even. Then $\mu(z') = BB = b_1b_2b_3\cdots b_m b_1b_2b_3\cdots b_m$, so that $z' = b_1b_3\cdots b_{m-1}b_1b_3\cdots b_{m-1} = uu$ where $u = b_1b_3\cdots b_{m-1}$. By the induction hypothesis, $|u|$ has the form 2^s or $3(2^s)$ for some integer s, and $|B| = 2|u|$ has the form 2^n or $3(2^n)$ where $n = s+1$.

(b) Suppose that $|x|$ is odd. If $|B| = 1 = 2^0$ we are done. Suppose then that $B = aB'b$ where $a, b \in \{0,1\}$. Then $xa = \mu(z)$ and $xaB'baB' = \mu(zz')$ where zz' is some prefix of $\mu^{r-1}(0)$. Since $xa = \mu(z)$ we can write $x = x'\overline{a}$ where $\overline{a} = 1-a$. If $b \neq a$, then $b = \overline{a}$, and $xaB'baB' = x'baB'baB'$ where $|x'|$ is even. We then replace B by baB', and by the previous case $|B| = |baB'|$ has the form 2^n or $3(2^n)$, as desired. Otherwise, $b = a$, and $xaB'baB' = xaB'aaB'$. Since aa cannot be $\mu(0) = 01$ or $\mu(1) = 10$, we conclude that $|xaB'aa|$ is odd. As $|x|$ is odd, we conclude that $|B| = |B'aa| = |xaB'aa| - |x| - 1$ is odd. Let $|B| = 2t+1$, and write $B = b_1b_2\cdots b_{2t}b_{2t+1}$. Since $|xb_1|$ is even, we find that each of b_2b_3, $b_4b_5, \ldots, b_{2t}b_{2t+1}$ is either 01 or 10, and $b_{2i} = 1 - b_{2i+1}$, $i = 1, 2, \ldots t$. On the other hand, since $|xB|$ is even, we find that each of b_1b_2, $b_3b_4, \ldots, b_{2t-1}b_{2t}$ is either 01 or 10. In conclusion, B is an alternating string of 0's and 1's. Since BB must be overlap-free, we must have $|B| < 5$, to avoid the overlaps 01010 and 10101. Since $|B|$ is an odd number greater than 1, we have $|B| = 3 = 3(2^0)$.

10. For each positive integer n, let

$$F_n = \{u \in \{0,1\}^n : u \text{ is a prefix of an overlap-free } \omega\text{-word over } \{0,1\}\}\,.$$

Let w_n be the lexicographically least word in the finite set F_n. We claim that $w_n \leq_p w_{n+1}$. Write $w_{n+1} = ua$ where $u \in \{0,1\}^n$, $a \in \{0,1\}$. By the choice of w_n, $w_n \preceq u$. Let an overlap-free ω-word extending w_n be w_nbv, where $b \in \{0,1\}$. By the choice of w_{n+1}, we have $w_{n+1} = ua \preceq w_nb$. This implies that $u \preceq w_n$, so that $w_n = u \leq_p w_{n+1}$. Finally, let w be the unique ω-word having all the w_n as prefixes.

We claim that w is the lexicographically least overlap-free ω-word over $\{0,1\}$. Suppose not. Let $y \neq w$ be an overlap-free ω-word over $\{0,1\}$

so that $y \preceq w$. There are then finite words $y' \leq_p y$ and $w_n \leq_p w$ with $|y'| = |w_n|$, $y \neq w_n$, $y \preceq w_n$. This contradicts the choice of the w_n.

11. Suppose that $u, v \in \{0,1\}^*$ and $u \preceq v$. Let the longest common prefix of u and v be w. It follows that $u = w0u'$ and $v = w1v'$ for some $u, v \in \{0,1\}^*$. Then $\mu(u) = \mu(w)01\mu(u') \prec \mu(w)10\mu(v')$.

12. For each positive n, $\mu^n(1)$ is a prefix of $\mu^{2n+1}(1)$. Also, 01001 is a suffix of $\mu^{2n+1}(0)$. It follows that $01001\mu^n(1)$ is a factor of $\mu^{2n+1}(01)$, and is thus overlap-free.

    Suppose now that $001001\mu^n(1) = xbB'bB'by$ for x, y, $B' \in \{0,1\}^*$, $b \in \{0,1\}$. Since $01001\mu^n(1)$ is overlap-free, $x = \epsilon$ and $001001\mu^n(1) = bB'bB'by$.

    If w is a factor of $\mu^\omega(0)$, then $w = a\mu(v)b$ for some $v \in \{0,1\}^*$, $a, b \in \{\epsilon, 0, 1\}$. Since $|\mu(v)|_0 = |\mu(v)|_1$, we see that $||w|_0 - |w|_1| \leq 2$. On the other hand, $|00100|_0 - |00100|_1 = 4 - 1 = 3$. It follows that 00100 is not a factor of $1001\mu^n(1)$. Word bB' is a prefix of $001001\mu^n(1)$, and also a factor of $01001\mu^n(1)$. If $|bB'| \geq 5$, then 00100 is a prefix of bB', which is impossible, since 00100 is not a factor of $1001\mu^n(1)$. We conclude then that $|bB'| \leq 4$, and overlap $bB'bB'b$ is a prefix of $001001\mu^n(1)$ of length at most 9. Then overlap $bB'bB'b$ is a prefix of overlap-free word 001001100. This is impossible.□

13. Word $\mu^\omega(1)$ is an overlap-free ω-word beginning with 100101. Suppose that y is the lexicographically least overlap-free ω-word over $\{0,1\}$ which starts with 1. Over $\{0,1\}$, the lexicographically least overlap-free word of length 6 starting with 1 is 100100. However, the two right extensions 1001000 and 1001001 of this word contain overlaps. Therefore, the lexicographically least overlap-free word of length 6 which extends to an ω-word is 100101, which extends to $\mu^\omega(1)$. It follows that y has 100101 as a prefix. By Exercise 4, $y = \mu(t)$, some ω-word t over $\{0,1\}$ where t starts with a 1. Now $y \preceq t$, and since μ is order-preserving, $\mu(y) \preceq \mu(t) = y$. However, since y is lexicographically least, we also have $y \preceq \mu(y)$, whence $\mu(y) = y$. It follows that y is the fixed point of μ starting with 1, namely $\mu^\omega(1)$.

14. Suppose that x is the lexicographically least overlap-free ω-word over $\{0,1\}$. Let $y = \mu^\omega(1)$ as in the previous exercise. Over $\{0,1\}$, the lexicographically least overlap-free word of length 7 is 0010011. Since this

extends to an overlap-free ω-word $001001y$, word x must have the form $x = 0010011u$ for some u. By Exercise 4 then, $x = 001001\mu(t)$, some ω-word t over $\{0,1\}$ where t starts with a 1. Now $y \preceq t$ by the previous exercise, and since μ is order-preserving, $001001y = 001001\mu(y) \preceq 001001\mu(t) = x$. However, since x is lexicographically least, we also have $x \preceq 001001y$, whence $x = 001001y$.

15. Let C_n be the set of overlap-free binary words of length n. By Exercise 5, we can write any overlap-free binary word w of length n in the form $u\mu(v)z$ where $u, v \in \{\epsilon, 0, 1, 00, 11\}$. If n is even, we thus have $C_n \subseteq \mu(C_{n/2}) \cup \{0,1\}\mu(C_{(n-2)/2})\{0,1\} \cup \{00,11\}\mu(C_{(n-2)/2}) \cup \mu(C_{(n-2)/2})\{00,11\} \cup \{00,11\}\mu(C_{(n-4)/2})\{00,11\}$.

    If n is odd, we have $C_n \subseteq \{0,1\}\mu(C_{(n-1)/2}) \cup \mu(C_{(n-1)/2})\{0,1\} \cup \{00,11\}\mu(C_{(n-3)/2})\{0,1$
    $\{0,1\}\mu(C_{(n-3)/2})\{00,11\}$.

Proof: Note that $\alpha > 2$ is necessary, since 01 is 2 power-free, but $\mu(01)$ contains the square 11.

Write $z = (z_1 z_2 \cdots z_p)^n z_1 z_2 \cdots z_m$ where the z_i are letters, n, m are integers, $n \geq 2$ and $m < p$. Write $\mu(w) = xzy$. If $|x|$ is even, then for some $\underline{z}$ we can write the even length prefix $(z_1 z_2 \cdots z_p)^2$ of z as $\mu(\underline{z})$. We see that

$$\begin{aligned} p &= |\underline{z}| \\ &= |\mu(\underline{z})|_1 \\ &= |(z_1 z_2 \cdots z_p)^2|_1 \\ &= 2|(z_1 z_2 \cdots z_p)|_1 \end{aligned}$$

so that p is even. If $|x|$ is odd, then $|xz_1|$ is even, and we can write $(z_2 \cdots z_p z_1)^2 = \mu(\underline{z})$ for some $\underline{z}$. Again we find that p is even.

Without loss of generality, assume that z is the longest factor of $\mu(w)$ having period p. We will show that $|x|$ is even. Suppose that $|x|$ is odd. Write $x = \mu(\underline{x})x_0$, where x_0 is a letter, $\underline{x}$ some word. Since p is even, write $xz_1 z_2 \cdots z_p z_1$ as $\mu(\underline{x})x_0 z_1 \mu(\underline{z}) z_p z_1$ for some $\underline{z}$. It follows that $x_0 = \bar{z}_1 = z_p$. Now, however, $x_0 z$ has period p, but is longer than z. This is a contradiction. We conclude that $|x|$ must be even. Symmetrically, $|y|$ must be even, so that $|z|$ is even also. This implies that m is even and $z = \mu(u)$ where $u = (z_1 z_3 \cdots z_{p-1})^n z_1 z_3 \cdots z_{m-1}$. We see that u has period $p/2$, while $|u| = |z|/2$.$\Box$

Proof: Suppose that $\mu(w)$ is not circular k^+ power-free. This means that $\mu(w)\mu(w) = \mu(ww)$ contains some α power z, $\alpha > k$, $|z| \leq |\mu(w)|$. Word z has period $p = |z|/\alpha$. By the previous lemma, ww contains a word u of period $p/2$, with $|u| = \lceil |z|/2 \rceil \leq |w|$. Moreover, u is a β power, where $\beta = |u|/(p/2) = \lceil |z|/2 \rceil/(p/2) \geq |z|/p = \alpha$.

Now ww contains a k^+ power u, with $|u| \leq |w|$. This means that w is not circular k^+ power-free.$\square$

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