Principles of Mathematical Analysis
Chapter 7
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1 Pointwise Convergence

Pointwise convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions. If f is a function such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then we say f_n converges *pointwise* to f.

This type of convergence is very *weak*. It guarantees very little in the way of actually *working with* the limit.
This definition is readily adapted to infinite sums of functions.

Infinite sums of functions
Definition
If f is a function such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in E$, then we say f is the *sum* of the series f_n.

An example of the weakness of pointwise convergence is

Continuity is not preserved
Example
Let $f_n : [0,1] \to [0,1]$ be defined by

$$f_n(x) := x^n$$

Then $f := \lim f_n$ is

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

by Theorem 3.20(e)

In this case, a sequence of continuous functions converges to a function that is eminently discontinuous. We use the preceding idea of "letting $f < 0$ sink and letting $f = 1$ float using the n^{th} power limit" to show the following.

Integrability is not preserved

Example

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

If we let

$$f_m(x) := \lim_{n\to\infty}(\cos m!\pi x)^{2n}$$

the above shows that a limit of integrable functions ($\int f_m dx = 0$ for all m) may fail to be integrable.

Proof

By a similar argument as in the previous example,

$$\lim_{n\to\infty}(\cos m!x)^{2n} = \begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases}$$

Let $x = p/q$ be rational. Then $m!x$ is rational for all $m \geq q$. Let x be irrational, $m!x$ cannot be an integer for any m, otherwise we can show a contradiction. Then

$$\lim_{m\to\infty}\begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

These two examples show that *properties* of f_n may not be pass through the limit to f.

Next, we show that *operations* on f_n may not be passed through the limit to f.

A limit of differentiated functions may not be the differentiated limit of functions

Example

Let

$$f_n(x) := \frac{\sin nx}{\sqrt{n}}$$

Then,

$$0 = \frac{d}{dx}\left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\frac{d}{dx} f_n\right] = \sqrt{n}\cos nx$$

A limit of integrated functions may not be the integral of a limit of functions

Example

Let

$$f_n(x) := nx(1-x^2)^n$$

Then

$$0 = \int_0^1 \left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\int_0^1 f_n\right] = \frac{1}{2}$$

2 Uniform convergence

Uniform convergence of functions

Definition

Let $f_n : E \to \mathbb{R}$ be a sequence of functions.
If f is a function such that for all ε there exists N such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all x, we say that f converges *uniformly*.