Jasper Ty

RIEMANN INTEGRAL OF $f\alpha'$ IS EQUAL TO RIEMANN-STIELTJES INTEGRAL OF f WITH INTEGRATOR α

Theorem

Let $\alpha : [a, b] \to \mathbb{R}$ be a differentiable monotonic function such that $\alpha' \in \mathscr{R}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. And

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Proof

Let $\varepsilon > 0$ and choose a partition P such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

By the mean value theorem, there exist points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i$$

Hence, if we pick any point $s_i \in [x_{i-1}, x_i]$ at all

CHANGE OF VARIABLE