Jasper Ty

A1.4.1: Let a be the side of length 4. Let the adjacent side be b and the hypotenuse c. The altitude perpendicular to the hypotenuse splits the triangle into two similar triangles (proof via the angle sum) that are similar to the original triangle. s is the hypotenuse of one of them, a is the hypotenuse of the other. Hence, if the length of the altitude is x, we have the relationship

$$x/a = b/c$$

Simplifiying

$$x = \frac{4\sqrt{c^2 - 16}}{c}$$

A1.4.2: The perimeter of a triangle is

$$P = a + b + c$$

Then

$$P^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Replacing $a^2 + b^2 = c^2$,

$$P^2 = c^2 + c^2 + 2ab + 2ac + 2bc = 2c^2 + 2ab + 2ac + 2bc$$

Collecting a common c,

$$P^2 = 2c(a + b + c) + 2ab$$

Replacing $a + b + c = P$

$$P^2 = 2Pc + 2ab$$

Using the previously found identity for the altitude perpendicular to the hypotenuse $A = ab/c$,

$$P^2 = 2Pc + 2(12c)$$

Then

$$P^2 = 2c \cdot P + 2c \cdot 12$$
$$P^2 = 2c(P + 12)$$

Solving for c,

$$\frac{P^2}{2(P + 12)} = c$$

A1.4.3: This gives us four possible equations, one of which is impossible and omitted.

$$(2x - 1) - (x + 5) = 3, \qquad \frac{1}{2} < x$$
$$-(2x - 1) - (x + 5) = 3, \qquad -5 < x \leq \frac{1}{2}$$
$$-(2x + 1) + (x + 5) = 3, \qquad x \leq -5$$

Solving,

$$x = 9$$
$$x = -\frac{7}{3}$$
$$x = 3$$

The third one is a contradiction, so the solutions are $x = 9$ and $x = -7/3$

A1.4.4: By the reverse triangle inequality,

$$2 = |x - x + 2| = |(x - 1) - (x - 3)| \geq \Big||x - 1| - |x - 3|\Big|$$

Hence $|x - 1| - |x - 3| \leq 2$, which implies that there are no solutions to $|x - 1| - |x - 3| \geq 5$

A1.4.5: Skip

A1.4.6: Skip

A1.4.7: Skip

A1.4.8: Skip

A1.4.9: Skip

A1.4.10: Skip

A1.4.11:

$$(\log_2 3)(\log_3 4)\cdots(\log_{31} 32)$$

is, by change of base, equal to

$$\frac{\ln 3}{\ln 2}\frac{\ln 4}{\ln 3}\cdots\frac{\ln 32}{\ln 31}$$

Which is, after cancellations,

$$\frac{\ln 32}{\ln 2} = \log_2 32 = 5$$

A1.4.12:

(a)

$$\begin{aligned}
f(-x) &= \ln(-x + \sqrt{(-x)^2 + 1}) \\
&= \ln(-x + \sqrt{x^2 + 1}) \\
&= \ln((x + \sqrt{x^2 + 1})^{-1}) \\
&= -\ln(x + \sqrt{x^2 + 1}) \\
&= -f(x)
\end{aligned}$$

(b)

$$\begin{aligned}
y &= \ln(x + \sqrt{x^2 + 1}) \\
e^y &= x + \sqrt{x^2 + 1} \\
e^y - e^{-y} &= (x + \sqrt{x^2 + 1}) - (-x + \sqrt{x^2 + 1}) \\
e^y - e^{-y} &= 2x \\
\frac{e^y - e^{-y}}{2} &= x
\end{aligned}$$

A1.4.13: By the monotonicity of ln, and the fact that $\ln 1 = 0$,

$$\ln(x^2 - 2x - 2) \leq 0$$

$$\ln(x^2 - 2x - 2) \leq \ln 1$$

$$x^2 - 2x - 2 \leq 1$$

$$x^2 - 2x - 3 \leq 0$$

$$(x - 3)(x + 1) \leq 0$$

Hence it must be that $-1 \leq x \leq 3$

A1.4.14: Suppose $\log_2 5$ is rational. Then $\log_2 5 = p/q$ for $p, q \in \mathbb{Z}$ and p, q coprime. Then

$$2^{p/q} = 5$$

Then,

$$2^p = 5^q$$

Which is impossible, as neither side share any prime factors.

A1.4.15: Let the distance travelled be D. Then the time taken in the first half T_1 and the second half T_2 is

$$T_1 = \frac{D/2}{30} \qquad T_2 = \frac{D/2}{60}$$

Hence $T_1/T_2 = 60/30 = 2$. Then the average speed is

$$(30 \cdot 2 + 60 \cdot 1)/3 = 40$$

A1.4.16: No. Let f be a function such that $f(1) := 0$, $f(2) = 1$. Then let g and h be the constant functions $g := 1$ and $h := 1$. Then $(f \circ (g + h))(x) = f(2) = 1$, but $(f \circ g + f \circ h)(x) = f(1) + f(1) = 0$

A1.4.17:

$$7 \equiv 1 \mod 6$$
$$7^n \equiv 1 \mod 6$$
$$7^n - 1 \equiv 0 \mod 6$$

A1.4.18:

$$(n+1)^2 - n^2 = (n+1-n)(n+1+n) = 2n+1 = 2(n+1) - 1$$

Hence if $n^2 = 1 + 3 + 5 + \cdots + (2n-1)$,

$$1 + 3 + 4 + \cdots + (2n-1) + (2(n+1)-1) = n^2 + (2(n+1)-1) = (n+1)^2$$

Since the identity is easily verified for $n = 1$, it follows by induction that it is true for all n.

A1.4.19:

$$f_{n+1}(x) = f_0(f_n(x)) = [f_n(x)]^2 = x^{2(n+2)}$$

Reindexing, $f_n(x) = x^{2n+2}$

A1.4.20: