

(Note: no guarantee of correctness. Just an undergrad writing their answers down and archiving because notebooks have become unwieldy.)

**Q1.** Suppose  $\alpha$  increases on  $[a, b]$ ,  $a \leq x_0 \leq b$ ,  $\alpha$  is continuous at  $x_0$ ,  $f(x_0) = 1$ , and  $f(x) = 0$  if  $x \neq x_0$ . Prove that  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$ .

**A: IDEA**

$x_0$  may be covered with an interval of arbitrarily small width, and this segment will have an arbitrarily small weight in the sum due to  $\alpha$ 's continuity at  $x_0$ .

PROOF THAT  $f \in \mathcal{R}(\alpha)$ :

Let  $\varepsilon > 0$ .

Since  $\alpha$  is continuous at  $x_0$ , choose  $\delta$  such that  $|x - x_0| \leq 2\delta$  implies  $|\alpha(x) - \alpha(x_0)| \leq \varepsilon$ .

Then let  $P = [a, x_0 - \delta, x_0 + \delta, b]$ . Then

$$\begin{aligned} U(P, f, \alpha) &= \sum_{i=0}^n M_i \Delta\alpha_i \\ &= M_0 \Delta\alpha_0 + M_1 \Delta\alpha_1 + M_2 \Delta\alpha_2 \end{aligned}$$

Since  $f(x) = 0$  on  $[a, x_0 - \delta]$  and  $[x_0 + \delta, b]$ ,  $M_0 = 0$  and  $M_2 = 0$ . Moreover,  $M_1 = 1$

$$U(P, f, \alpha) = 0 + 1 \cdot \Delta\alpha_1 + 0\Delta\alpha_2 = \Delta\alpha_1$$

As for  $L$ , the following holds for any partition at all, as any nonempty interval must contain a point that is not  $x_0$

$$L(P, f, \alpha) = 0$$

So

$$U(P, f, \alpha) - L(P, f, \alpha) = \Delta\alpha_1 = \alpha(x_0 + \delta) - \alpha(x_0 - \delta) < \varepsilon$$

Then  $f \in \mathcal{R}(\alpha)$ .

PROOF THAT  $\int f d\alpha = 0$ :

Since  $L(P, f, \alpha) = 0$  for all  $P$ ,

$$\underline{\int} f d\alpha = 0$$

Which lets us conclude

$$\int f d\alpha = 0$$

**Q2.** Suppose  $f \geq 0$ ,  $f$  is continuous on  $[a, b]$ , and  $\int_a^b f(x) dx = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ . (Compare this with Exercise 1.)

**A: IDEA**

We show a contradiction that a “bump” of nonzero area must exist if  $f$  is not identically zero, hence the integral must be nonzero.

PROOF

Choose any arbitrary point  $q$  in  $[a, b]$ . Suppose  $f(q) > 0$ . Let  $m$  be a number such that  $0 < m < f(q)$ , and use the continuity of  $f$  to create an interval  $[s, t]$  containing  $q$  such that  $x \in [s, t]$  implies  $f(x) \geq m$ .

By additivity of the integral,

$$\int_a^b f dx = \int_a^s f dx + \int_s^t f dx + \int_t^b f dx$$

Hence

$$\int_a^b f dx \geq \int_s^t f dx$$

But  $\int_s^t f dx \geq m(t - s) > 0$ , so

$$\int_a^b f dx > 0$$

A contradiction to the assertion that  $\int_a^b f dx = 0$

**Q3.** Define three functions  $\beta_1, \beta_2, \beta_3$  as follows:  $\beta_j(x) = 0$  if  $x < 0$ ,  $\beta_j(x) = 1$  if  $x > 0$  for  $j = 1, 2, 3$ ; and  $\beta_1(0) = 0$ ,  $\beta_2(0) = 1$ ,  $\beta_3(0) = \frac{1}{2}$ . Let  $f$  be a bounded function on  $[-1, 1]$ .

(a) Prove that  $f \in \mathcal{R}(\beta_1)$  if and only if  $f(0^+) = f(0)$  and that then

$$\int f d\beta_1 = f(0)$$

(b) State and prove a similar result for  $\beta_2$

(c) Prove that  $f \in \mathcal{R}(\beta_3)$  if and only if  $f$  is continuous at 0

(d) If  $f$  is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

**A: IDEA**

With  $\beta_i$ , the only intervals with non-zero weight in any partition will be those which contain 0. Moreover, these segments will have fixed weights due to our definitions of  $\beta_i$ . Then, bounds on the variation of the upper and lower Riemann sums automatically pass through as bounds on  $f$  in the segment itself.

Hence we can convert convergence of Riemann sums into convergence of  $f$ .

(a) PROOF THAT  $f \in \mathcal{R}(\beta_1)$  IMPLIES  $f(0^+) = f(0)$

Let  $f \in \mathcal{R}(\beta_1)$ . Let  $\varepsilon > 0$ . Choose a partition  $P$  such that  $0 \in P$ , and

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

The only nonzero term in either of the sums  $U(P, f, \beta_1)$  or  $L(P, f, \beta_1)$  comes from the segment containing  $x = 0$  as a left endpoint. Let this segment be  $[0, x_i]$ . Then

$$U(P, f, \beta_1) = M_i \cdot \Delta x_i = M_i \cdot (\beta(x_i) - \beta(0)) = M_i \cdot 1 = M_i$$

$$L(P, f, \beta_1) = m_i \cdot \Delta x_i = m_i \cdot (\beta(x_i) - \beta(0)) = m_i \cdot 1 = m_i$$

So  $U(P, f, \beta_1) - L(P, f, \beta_1) = M_i - m_i < \varepsilon$ . This tells us that on  $[0, x_i]$ ,  $f$  stays within  $\varepsilon$  of  $f(0)$ . Since  $\varepsilon$  was arbitrary, we conclude that  $f(0^+) = f(0)$

PROOF THAT  $f(0^+) = f(0)$  IMPLIES  $f \in \mathcal{R}(\beta_1)$

Let  $f(0^+) = f(0)$ . Let  $\varepsilon > 0$  Choose  $\delta$  such that  $x \in (0, \delta)$  implies  $f(x) \in (f(0) - \varepsilon, f(0) + \varepsilon)$ . Let  $P$  be the partition  $\{a, 0, \delta, b\}$ . Then for the same reasons as above,

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

So  $f \in \mathcal{R}(\beta_1)$ .

PROOF THAT  $\int f d\beta_1 = 0$

By Theorem 6.7(b), letting  $t_i = 0$ , we have that

$$\left| f(0) - \int f d\beta_1 \right| < \varepsilon$$

Hence we conclude that

$$\int f d\beta_1 = f(0)$$

(b) STATEMENT

$f \in \mathcal{R}(\beta_1)$  if and only if  $f(0^-) = f(0)$ , and if it exists,

$$\int f d\beta_1 = f(0)$$

PROOF

Extremely similar as before, but with  $[x_{i-1}, 0]$  instead of  $[0, x_i]$

(c) PROOF THAT  $f \in \mathcal{R}(\beta_3)$  IMPLIES  $f$  IS CONTINUOUS AT 0

Let  $f \in \mathcal{R}(\beta_3)$ . Let  $\varepsilon > 0$ . Let  $P$  be a partition that such that  $0 \in P$  and  $U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$

Then, consider the intervals  $[x_{i-1}, 0]$ ,  $[0, x_i]$ . We have that

$$\beta_3(0) - \beta_3(x_{i-1}) = \frac{1}{2}$$

$$\beta_3(x_i) - \beta_3(0) = \frac{1}{2}$$

Then

$$U(P, f, \beta_3) = \frac{M_i + M_{i-1}}{2}$$

$$L(P, f, \beta_3) = \frac{m_i + m_{i-1}}{2}$$

Let  $M = \min\{M_i, M_{i-1}\}$  and  $m = \max\{m_i, m_{i-1}\}$ , then

$$U(P, f, \beta_3) \geq M$$

$$L(P, f, \beta_3) \leq m$$

Then

$$M - m \leq U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence  $f(x)$  is within  $\varepsilon$  of  $f(0)$  if  $f(x)$  is in  $[x_{i-1}, x_i]$ . Hence  $f$  is continuous.

PROOF THAT  $f$  CONTINUOUS AT 0 IMPLIES  $f \in \mathcal{R}(\beta_3)$

Let  $f$  be continuous at 0. Let  $\varepsilon > 0$ . Choose  $\delta$  such that  $|x| \leq 2\delta$  implies  $|f(x) - f(0)| \leq \varepsilon/2$  then take the partition  $P = \{a, -\delta, \delta, b\}$ . Then

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence  $f \in \mathcal{R}(\beta_3)$ . Also, with a similar argument as in (a),

$$\int f d\beta_3 = f(0)$$

(d)  $f$  being continuous at zero is a strong enough condition for all three previous proofs above to apply. Since all of them show equality of the integral with respect to  $\beta_i$  and  $f(0)$ , the result follows.

**Q4.** If  $f(x) = 0$  for all irrational  $x$ ,  $f(x) = 1$  for all rational  $x$ , prove that  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$

**A:** IDEA

Any nonempty interval will contain both irrational and rational numbers, so the “gap” between the lower and upper sums never closes.

PROOF

Let  $P$  be a partition. Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , for all  $x_i$  we can find two rational numbers  $p$  and  $q$  such that  $x_{i-1} < p < q < x_i$ . This tells us that  $M_i = 1$  for all  $x_i$ . Furthermore, the number  $s = p + (\sqrt{2}/2)(q - p)$  is irrational and between  $p$  and  $q$ . This tells us that  $m_i = 0$  for all  $x_i$ . Hence it must be that

$$U(P, f) = 1$$

$$L(P, f) = 0$$

for any partition  $P$ . Then  $f$  is not Riemann-integrable.

**Q5.** Suppose  $f$  is a bounded real function on  $[a, b]$ , and  $f^2 \in \mathcal{R}$  on  $[a, b]$ . Does it follow that  $f \in \mathcal{R}$ ? Does the answer change if we assume that  $f^3 \in \mathcal{R}$ ?

**A:** IDEA

Because of squaring's non-injectivity, we can construct "terrible" discontinuities which can be undone via squaring. On the other hand, cubing is much nicer (it is a continuous bijection of the real line to itself).

PROOF

No. Consider the function

$$f(x) := \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases}$$

We have that  $(f^2)(x) = 1$ , so  $\int_a^b f^2 dx = (b - a)$ . But  $f$  itself is not Riemann-integrable, as the exercise above shows.

For  $f^3$ , use Theorem 6.11, which tells us that the composition of continuous functions with integrable functions is integrable. Let  $\phi = \sqrt[3]{x}$ . Then  $\phi \circ (f^3) \in \mathcal{R}$ . Since  $\phi \circ f^3 = f$ ,  $f \in \mathcal{R}$ .

**Q6.** Let  $P$  be the Cantor set constructed in Sec. 2.44. Let  $f$  be a bounded real function on  $[0, 1]$  which is continuous at every point outside  $P$ . Prove that  $f \in \mathcal{R}$  on  $[0, 1]$ . *Hint:*  $P$  can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

**A:** Each set of the sequence that constructs the Cantor set has  $2^n$  intervals, each of width  $1/3^n$ . Let  $E_n$  be the  $n$ th set in the sequence. Cover each interval in  $E_n$  with an open interval whose endpoints have a distance of  $1.1 \cdot (1/3^n)$ . Then, the sum of all lengths of each cover will be  $1.1 \cdot (2/3)^n$ . This also covers the Cantor set itself, since the Cantor set  $E$  is a subset of  $E_n$  for all  $n$ . Hence, we have made a cover of the Cantor set with arbitrarily small area.

Take the endpoints of each interval and construct a partition, and proceed as in Theorem 6.10. This sequence of partitions eventually.

**Q7.** Suppose  $f$  is a real function on  $(0, 1]$  and  $f \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Define

$$\int_0^1 f(x)dx = \lim_{c \rightarrow 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

- (a) If  $f \in \mathcal{R}$  on  $[0, 1]$ , show that this definition of the integral agrees with the old one.
- (b) Construct a function  $F$  such that the above limit exists, although it fails to exist with  $|f|$  in place of  $f$

**A:**

- (a) Lemma: If  $f \in \mathcal{R}$  on  $[a, b]$ , then

$$\lim_{h \rightarrow 0} \int_a^{a+h} f(x)dx = 0$$

Proof: Let  $|f(x)| \leq M$ . Let  $\varepsilon > 0$ . Choose  $\delta = \varepsilon/M$ . Then, letting  $h \leq \delta$

$$\left| \int_a^{a+h} f(x)dx \right| \leq M((a+h) - a) = Mh \leq M\delta = \varepsilon$$

Since the  $\varepsilon$  was arbitrary, the lemma holds. Then, by Theorem 6.12, if  $f \in \mathcal{R}$  on  $[0, 1]$ ,

$$\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx$$

Then, taking limits

$$\begin{aligned} \lim_{c \rightarrow 0} \int_0^1 f(x)dx &= \lim_{c \rightarrow 0} \int_0^c f(x)dx + \lim_{c \rightarrow 0} \int_c^1 f(x)dx \\ \int_0^1 f(x)dx &= 0 + \lim_{c \rightarrow 0} \int_c^1 f(x)dx \\ \int_0^1 f(x)dx &= \lim_{c \rightarrow 0} \int_c^1 f(x)dx \end{aligned}$$

Hence showing that the definition agrees.

- (b) Let  $n = 1, 2, \dots$ . Consider the function (with infinitely many cases)

$$f(x) = \begin{cases} (-1)^{n+1}n + 1 & \text{if } \frac{1}{n+1} \leq x < \frac{1}{n} \end{cases}$$

Then the graph of  $f$  consists of boxes of width  $1/n - 1/(n+1)$  and height  $n+1$ . The area of each box is then  $1/n$ . Then as the lower bound of the integral approaches 0, the integral approximates the alternating sum  $S = 1 - 1/2 + 1/3 - 1/4 + \dots$ , which converges to  $\ln 2$ . However, in the case of  $|f|$ , the integral approximates the sum  $S = 1 + 1/2 + 1/3 + 1/4 + \dots$ , which diverges.

**Q8.** Suppose  $f \in \mathcal{R}$  on  $[a, b]$  for every  $b > a$  where  $a$  is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after  $f$  has been replaced by  $|f|$ , it is said to converge *absolutely*. Assume that  $f(x) \geq 0$  and that  $f$  decreases monotonically on  $[1, \infty)$ . Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called “integral test” for convergence of series.)

**A:** Suppose  $\int_1^\infty f(x)dx$  converges. Then if we show that, for any positive integer  $b$ ,

$$\sum_{n=1}^b f(n) \leq \int_1^b f(x)dx$$

we can prove convergence of the sum, as the partial sums form a monotonic sequence ( $f$  is non-negative). Let  $P$  be the partition consisting of the points  $0, 1, 2, \dots, b-1, b$ . Then, since  $f$  is monotonically decreasing,

$$\inf_{x \in [n-1, n]} f(x) = f(n)$$

Hence

$$\sum_{n=1}^b f(n) = L(P, f) \leq \int_1^b f(x)dx$$

which was the inequality we wanted. Then the sum converges.

Suppose  $\int_1^\infty f(x)dx$  converges. Then we make a similar argument, with  $U(P, f)$  bounding  $\int_1^b f(x)dx$  from above. Take the same partition of  $[a, b]$ , then

$$\sup_{x \in [n, n+1]} f(x) = f(n)$$

Hence

$$U(P, f) = \sum_{n=0}^{b-1} f(n)$$

Which proves that the integral converges.

**Q9.** Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx$$

**Q10.** Suppose  $f$  is a real function on  $(0, 1]$  and  $f \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Define

$$\int_0^1 f(x)dx = \lim_{c \rightarrow 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite)