Schubert polynomials

Jasper Ty

These are notes based on my study of Schubert polynomials. My main references are [KnutsonSP] and [MacdonaldSP].

Contents

1 Notation and conventions

1.1 Sets

We take $\mathbb{N}$ to be the set of natural numbers *including* zero,

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

We take $\mathbb{P}$ to be the set of *positive integers*,

$$\mathbb{P} := \{1, 2, \ldots\}.$$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are defined as usual.

1.2 Partitions and compositions

A *weak composition* α of $n \in \mathbb{N}$ is an infinite tuple of nonnegative integers

$$(\alpha_1, \alpha_2, \ldots)$$

such that $\sum_i \alpha_i = n$. We define $|\alpha| = \sum_i \alpha_i$ to have notation for recovering n given α.

A *partition* λ of n is a weak composition whose entries are *weakly decreasing*. That a particular partition λ is a partition of a particular n is denoted $\lambda \vdash n$. We define $|\lambda|$ the exact same way.

I use English notation when drawing diagrams and tableaux, meaning, rows are drawn further downward as the index increases.

1.3 Rings, polynomials, and formal power series

The following notation is (mostly) in accordance with the notation in [GrinbergAC], with a few additions.

All rings considered are commutative and unital. An arbitrary ring will be denoted $\mathbb{K}$.

$\mathbb{K}[[t]]$ will denote the formal power series ring over $\mathbb{K}$ in the indeterminate t.

We will fix notation for the following sets of indeterminates, which we will use when convenient:

(a) $X_N := (x_1, x_2, \ldots x_N)$ for a set of N indeterminates.

(b) $X := (x_1, x_2, \ldots)$ for a set of countably many indeterminates.

(c) Y, Y_N, Z, Z_N, Q, Q_N and so on are defined similarly.

Let $\mathbb{K}[[X]]$ be a formal power series ring. With compositions, partitions, or otherwise any finitely supported tuple of nonnegative integers α, we define *multi-index notation* for compactly writing down monomials in $\mathbb{K}[[X]]$.

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots .$$

We will let $[x^\alpha]f$ denote the coefficient of $[x^\alpha]$ in the formal power series f.

1.4 Permutations and the symmetric group

S_n will denote the symmetric group on n letters.

I use cycle notation, so e.g the cycle that sends 1 to 7, 7 to 4, and 4 to 1 will be written as (174).

The simple transpositions $(i\ i+1)$ will be denoted s_i.

The length of a permutation w will be denoted $\ell(w)$.

Permutations will act on polynomials or power series by permuting *places*, meaning that if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in \mathbb{K}[X_n]$, we define

$$\sigma f(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots x_{\sigma(n)}).$$

2 Schubert Polynomials

Apparently these are "Schubert cycles in flag varieties".

2.1 Divided difference operators

These strike me as a tool to measure how "unsymmetric" a polynomial is in a local sense, in two variables at a time.

2.1.1 Definition

Definition 2.1.1. Let f be a polynomial in N indeterminates. We define the *divided difference operators* ∂_i by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}} \tag{1}$$

Example 2.1.2. If $f(x_1, x_2, x_3) = x_1 x_2$, then

$$\begin{aligned} \partial_2 f(x_1, x_2, x_3) &= \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} \\ &= x_1 \left(\frac{x_2 - x_3}{x_2 - x_3} \right) \\ &= x_1. \end{aligned}$$

2.1.2 Basic facts

Theorem 2.1.3. Let f be a polynomial.

(a) $\partial_i f$ is a polynomial.

(b) If f is homogeneous of degree d, then $\partial_i f$ is homogeneous of degree $d - 1$.

Proof. The proof is not hard but it's a slog. We'll take some liberty with notation so it's a little less of a slog. Consider some monomial $\cdots x_i^a x_{i+1}^b \cdots$. We compute

$$\begin{aligned} \partial_i(\cdots x_i^a x_{i+1}^b \cdots) &= \frac{(\cdots x_i^a x_{i+1}^b \cdots) - (\cdots x_i^b x_{i+1}^a \cdots)}{x_i - x_{i+1}} \\ &= (\cdots) \frac{x_i^a x_{i+1}^b - x_i^b x_{i+1}^a}{x_i - x_{i+1}}. \end{aligned}$$

Let $k = |a - b|$, and let $[]_\pm$ denote the sign of a number. Then

$$(\cdots) \frac{x_i^a x_{i+1}^b - x_i^b x_{i+1}^a}{x_i - x_{i+1}} = (\cdots x_i^a x_{i+1}^a \cdots)[k]_\pm \frac{x_{i+1}^k - x_i^k}{x_i - x_{i+1}}.$$

We recall a favorite high-school algebra identity

$$\frac{a^n - b^n}{a - b} = a^{n-1}b^0 + a^{n-2}b^1 + \cdots + a^1 b^{n-2} a^0 b^{n-1},$$

and continue computing to finally get

$$(\cdots x_i^a x_{i+1}^a \cdots)[k]_\pm \frac{x_i^k - x_{i+1}^k}{x_i - x_{i+1}} = [k]_\pm (\cdots x_i^a x_{i+1}^a \cdots)(x_i^{k-1} x_{i+1}^0 + \cdots + x_i^0 x_{i+1}^{k-1}),$$

which is evidently a polynomial. We also note that in the above computation we had a monomial of, say, degree $\_ + a + b$, and ∂_i gave us a monomial with degree $\_ + a + a$ multiplied by a homogeneous polynomial of degree $b - a - 1$, hence the resulting polynomial is homogeneous of degree $(\_ + a + a) + (b - a - 1) = \_ + a + b - 1$.

Since ∂_i is linear, the proof is complete, as we've proven it for an arbitrary monomial. □

Theorem 2.1.4. The divided difference operators satsify the following relations

(a) The braid relation

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \tag{2}$$

(b) Far commutativity

$$\partial_i \partial_j = \partial_j \partial_i \qquad \text{whenever} \qquad |i - j| > 1$$

(c) Reflection by a simple

$$\partial_i s_i = -\partial_i$$

(d) Chain condition

$$\partial_i^2 = 0$$

Proof. □

I wonder if $\partial_i^2 = 0$ has to do with the Schuberts arising from a cohomology theory.

2.2 The definition of a Schubert polynomial

Definition 2.2.1. The *Schubert polynomials* $\mathfrak{S}_w$ are defined by the rules

$$\begin{cases} \mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1, \\ \partial_i \mathfrak{S}_w := \mathfrak{S}_{ws_i} \end{cases}$$

3 The ring of coinvariants of

Theorem 3.0.1. The Schuberts form a basis for the coinvariant ring

3.1 Definition

References

[StanleyEC2] Richard P. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge University Press 2023.

[GrinbergAC] Darij Grinberg, *An Introduction to Algebraic Combinatorics*, http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf

[KnutsonSP] Allen Knutson, *Schubert Polynomials and Symmetric Functions*, https://pi.math.cornell.edu/~allenk/schubnotes.pdf

[MacdonaldSP] Ian Macdonald, *Notes on Schubert Polynomials*