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1 Pointwise Convergence

Pointwise convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions. If f is a function such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then we say f_n converges *pointwise* to f.

This type of convergence is very *weak*. It guarantees very little in the way of actually *working with* the limit.
This definition is readily adapted to infinite sums of functions.

Infinite sums of functions
Definition
If f is a function such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in E$, then we say f is the *sum* of the series f_n.

An example of the weakness of pointwise convergence is

Continuity is not preserved under pointwise convergence
Example
Let $f_n : [0,1] \to [0,1]$ be defined by

$$f_n(x) := x^n$$

Then $f := \lim f_n$ is

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

by Theorem 3.20(e)

In this case, a sequence of continuous functions converges to a function that is eminently discontinuous. We use the preceding idea of "letting $f < 0$ sink and letting $f = 1$ float using the n^{th} power limit" to show the following.

INTEGRABILITY IS NOT PRESERVED UNDER POINTWISE CONVERGENCE
Example

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

If we let

$$f_m(x) := \lim_{n\to\infty}(\cos m!\pi x)^{2n}$$

the above shows that a limit of integrable functions ($\int f_m dx = 0$ for all m) may fail to be integrable.
Proof
By a similar argument as in the previous example,

$$\lim_{n\to\infty}(\cos m!x)^{2n} = \begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases}$$

Let $x = p/q$ be rational. Then $m!x$ is rational for all $m \geq q$. Let x be irrational, $m!x$ cannot be an integer for any m, otherwise we can show a contradiction. Then

$$\lim_{m\to\infty}\begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

These two examples show that *properties* of f_n may not pass through the limit to f.
Next, we show that *operations* on f_n may not be passed through the limit to f.

A LIMIT OF DIFFERENTIATED FUNCTIONS MAY NOT BE THE DIFFERENTIATED LIMIT OF FUNCTIONS
Example
Let

$$f_n(x) := \frac{\sin nx}{\sqrt{n}}$$

Then,

$$0 = \frac{d}{dx}\left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\frac{d}{dx}f_n\right] = \sqrt{n}\cos nx$$

A LIMIT OF INTEGRATED FUNCTIONS MAY NOT BE THE INTEGRAL OF A LIMIT OF FUNCTIONS
Example
Let

$$f_n(x) := nx(1-x^2)^n$$

Then

$$0 = \int_0^1 \left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\int_0^1 f_n\right] = \frac{1}{2}$$

2 Uniform convergence

Uniform convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions.
If f is a function such that for all ε there exists N such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all x, we say that f converges *uniformly*.
This definition carries over to sums of functions (the partial sums must converge to the limit function uniformly).

This is a *much stronger* notion of convergence, as it, in a sense, "tethers" together convergence of of all points in the domain.

There are useful criteria for uniform convergence. These hint at the idea of being able to make sense of the idea of "distance" between two functions, which Rudin makes precise later. The first one tells us that uniform convergence of functions corresponds to convergence (via the Cauchy criterion) with respect to a certain kind of metric.

Cauchy criterion for uniform convergence
Theorem
f_n converges uniformly on E if and only if there exists an integer N such that for all $m, n \geq N$,

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \varepsilon$$

Proof
For the forward implication, let $f_n \to f$ uniformly, then choose N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$$

Then use the triangle inequality.
For the converse, we note that the criterion is strong enough itself to guarantee $f_n \to f$ pointwise, hence if we take the inequality

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

and let $m \to \infty$, we recover the original condition for uniform convergence

The next criterion is just a rephrasing of the first definition of uniform convergence in similar terms. It tells us that uniform convergence corresponds to the "distance" between two functions vanishing.

Supremum criterion for uniform convergence
Theorem
$f_n \to f$ uniformly on E if and only if

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty$$

Next, we show that uniform convergence does not share the same failures as pointwise convergence when it comes to passing through important properties and operations.

We first take continuity.

Continuity is preserved under uniform convergence
Theorem
Let f_n be a sequence of continuous functions and let $f_n \to f$ uniformly. Then f is continuous.
Proof
Let $\varepsilon > 0$. Let $x \in E$. Consider the inequality

$$|f(x) - f(t)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(t)| + |f_n(t) - f(t)|$$

First use uniform convergence to choose n such that $|f(x) - f_n(x)|$ and $|f(t) - f_n(t)|$ are arbitrarily small.
Then use continuity to choose δ such that $|f_n(x) - f_n(t)|$ can be made arbitrarily small by letting $|x - t| \leq \delta$.

This however is not an "if and only if". A continuous function can be the non-uniform limit of continuous functions. In one case, however, we can use compactness and monotonicity to take enough control such that a sequence of functions can *only* converge uniformly.

Inferring uniform convergence from pointwise convergence
Example
Let $f_n : K \to \mathbb{R}$ be a sequence of continuous functions, and let $f_n \to f$ pointwise. Let f be continuous. Impose the two conditions

- Let K be compact
- Let $f_n(x) \geq f_{n+1}(x)$ for all x and all n

Then $f_n \to f$ uniformly.
Proof
We must show that, as $n \to \infty$,

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0$$

To do this, we choose ε and argue that the set of points for which $|f_n(x) - f(x)| \geq \varepsilon$ can be made empty if we take n large enough. Let this set of exceptional points be K_n.
K's compactness tells us that K_n is compact:

- Because $f_n - f$ is a continuous function and $(\infty, \varepsilon]$ is closed, K_n is closed. (Theorem 4.8; inverse images of closed sets under continuous functions are closed)
- Because K_n is closed, and K is compact, K_n is compact. (Theorem 2.35; closed subsets of compact spaces are compact).

Moreover, because of the monotonicity of f_n, $K_n \supseteq K_{n+1}$. Hence K_n is a sequence of nested compact sets.
It must be that $\bigcap K_n$ is empty, as f_n converges pointwise to f, and so for each x, the fact that $|f_n(x) - f| < \varepsilon$ for some n means there must be some K_n it is not a member of.
But an intersection of nonempty nested compact sets must be nonempty (Theorem 2.36), so there has to be an empty set somewhere in the sequence. Let this set be K_N. Then for all $n \geq N$, K_n is empty. This is precisely the N we need to keep the distance between f_n and f below ε.

Restricting our study to continuous, bounded functions (the functions above were bounded because K was compact) gives rise to a very nice structure, one we're already very familiar with: a metric space. (More importantly, a vector space!)
The concept of distance here corresponds with how it has been used in the previous theorems and examples.

The space of continuous and bounded functions on X

Definition

Let X be a metric space. Let $\mathscr{C}(X)$ denote the set of all continuous, bounded, complex-valued functions on X. Let $f \in \mathscr{C}(X)$. Define the *supremum norm* as follows.

$$\|f\| = \sup_{x \in X} |f(x)|$$

Let $d(f, g)$ for $f, g \in \mathscr{C}(X)$ be induced by the norm; i.e

$$d(f, g) = \|f - g\|$$

This turns $\mathscr{C}(X)$ into a metric space.

This space is complete, and the fact that it is so can be bootstrapped with the previous theorems.

$\mathscr{C}(X)$ is complete

Proof

Let f_n be a Cauchy sequence in $\mathscr{C}(X)$. Then f_n converges uniformly to a function f. Moreover, since each f_n is continuous, f is continuous. That f is bounded follows from the fact that $f = f_n + (f - f_n)$, so $\|f\| \leq \|f_n\| + \|f - f_n\| = \|f_n\| + \varepsilon$. Hence $f \in \mathscr{C}(X)$. Since $f_n \to f$ uniformly, $\|f - f_n\| \to 0$.

Next, we tackle integration. Uniform convergence allows us to bound f in a, well, uniform way. This, along with a version of the squeeze theorem for integrals, shows us that uniform convergence plays nicely with integration.

The integral of a uniformly convergent limit is a uniformly convergent limit of integrals

Theorem Let $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ for all n. Let $f_n \to f$ uniformly. Then $f \in \mathscr{R}(\alpha)$, and

$$\int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f_n d\alpha$$

Proof

The idea is that you can "squeeze" f with two f_n shaped calipers, and that the gap decreases as n goes to infinity. Let this gap be

$$\varepsilon_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$$

Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$$

Then this bounds f's lower and upper integrals.

$$\int_a^b (f_n - \varepsilon_n) d\alpha \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq \int_a^b (f_n + \varepsilon_n) d\alpha$$

Then

$$0 \leq \overline{\int} f d\alpha - \underline{\int} f d\alpha \leq 2\varepsilon(\alpha(b) - \alpha(a))$$

So $f \in \mathscr{R}(\alpha)$. From the inequality we can also read off

$$\left| \int_a^b f d\alpha - \int_a^b f_n d\alpha \right| \leq \varepsilon_n(\alpha(b) - \alpha(a))$$

Which proves the equality of the integral and the limit.

Finally, we look at differentiation.

A SEQUENCE OF FUNCTIONS WHOSE DERIVATIVES CONVERGE UNIFORMLY CONVERGES UNIFORMLY TO A LIMIT WITH THE CORRECT DERIVATIVE

Theorem

Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of differentiable functions that converge pointwise for at least *some* point x_0. Then if f_n' converge uniformly, there is a function f such that $f_n \to f$ uniformly and such that $f_n' \to f'$.

Proof

Let $\varepsilon > 0$. Use the Cauchy criterion for both ordinary sequences in $\mathbb{R}$ and for uniformly convergent sequences of functions to produce N such that $n, m \geq N$ implies

$$|f_n(x_0) - f_m(x_0)| \leq \frac{\varepsilon}{2}$$

$$\sup_{t \in [a,b]} |f_n'(t) - f_m'(t)| \leq \frac{\varepsilon}{2(b-a)}$$

Consider the function $f_n - f_m$, whose derivative is $f_n' - f_m'$. The mean value theorem gives us a c with which we can show

$$\left| \frac{[f_n(x) - f_m(x)] - [f_n(y) - f_m(y)]}{x - y} \right| = |f_n'(c) - f_m'(c)| \leq \sup_{t \in [a,b]} |f_n'(t) - f_m'(t)| \leq \frac{\varepsilon}{2(b-a)}$$

Then, since $a \leq x < y \leq b$

$$|[f_n(x) - f_m(x)] - [f_n(y) - f_m(y)]| \leq \frac{\varepsilon |x-y|}{2(b-a)} \leq \frac{\varepsilon}{2}$$

for any $x, y \in [a,b]$. Then, by the triangle inequality,

$$|f_n(x) - f_m(x)| \leq |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| + |f_n(x_0) - f_m(x_0)| \leq \varepsilon$$

Then f_n converges uniformly by the Cauchy criterion.

Next we prove that the derivative of the limit is the limit of the derivatives. Let $f_n \to f$. Define the difference quotients

$$\phi_n(t) := \frac{f_n(t) - f_n(x)}{t - x}$$

$$\phi(t) := \frac{f(t) - f(x)}{t - x}$$

Reusing the same inequality, we know that for $n, m \geq N$

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)}$$

So that $\phi_n \to \phi$ uniformly. Then, by Theorem 7.11, the limits we are interested in commute:

$$\lim_{t \to x} \lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \lim_{t \to x} \phi_n(t)$$

The left hand side is

$$\lim_{t \to x} \lim_{n \to \infty} \phi_n(t) = \lim_{t \to x} \phi(t) = f'(x)$$

The right hand side is

$$\lim_{n \to \infty} \lim_{t \to x} \phi_n(t) = \lim_{n \to \infty} f_n'(t)$$

Hence $f_n' \to f_n$.

Now with the machinery of limits of function sequences, we can explore a wider variety of functions. As an example.

A NOWHERE DIFFERENTIABLE CONTINUOUS FUNCTION

Example

Define the periodic triangle function φ to be

$$\varphi(x) := |x|$$

for $0 < x \leq 1$, and let its periodicity be defined by

$$\varphi(x + 2) := \varphi(x)$$

Then φ is continuous. Define f as

$$f(x) := \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

By Theorem 7.10, the series converges uniformly. Then f is continuous, as it is a uniformly convergent sequence of continuous functions. To prove it is not differentiable, we construct a sequence $x_n \to x$ such that the difference quotient diverges.

3 Equicontinuous families of functions

Now we go up another level. How do we *produce* these uniformly convergent sequences of functions out of existing sequences? The same way we have conditions (boundedness in $\mathbb{R}^n$) for squeezing out convergent sequences of numbers? First consider two different kinds of boundedness.

POINTWISE AND UNIFORM BOUNDEDNESS
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions.
f_n is *pointwise bounded* if for some $\varphi : E \to \mathbb{R}$,

$$|f_n(x)| < \varphi(x)$$

f_n is *uniformly bounded* if for some $M \geq 0$,

$$|f_n(x)| < M$$

A sequence of functions being pointwise bounded has an important consequence

Every pointwise bounded sequence has a subsequence that converges pointwise on a countable set

Theorem

Let f_n be defined on a countable set E. (Or define it on an arbitrary set, and let f_n be the restriction on a countable set E).

Then there exists a subsequence f_{n_k} of f_n such that f_{n_k} converges pointwise on E.

Proof

We will construct a table of f_n of which taking the diagonal will give us the subsequence we want. To do so, we introduce a lemma

Lemma

Let s_n be a bounded sequence. Then it contains a convergent subsequence s_{n_k} that may be obtained by simply deleting entries in s_n.

Proof

s_n as a set has a limit point L (Theorem 2.42). Let $\varepsilon_1 > 0$. Choose $s_{n_1} \neq L$ such that $d(L, s_{n_1}) < \varepsilon_1$. Since L is a limit point, we can find this point.

Choose ε_2 such that $d(L, s_{n_1}) > \varepsilon_2 > 0$ and that $\frac{\varepsilon_1}{2} > \varepsilon_2$. Choose s_{n_2} such that $n_2 > n_1$ and $d(L, s_{n_2}) < \varepsilon_2$. This is possible because again L is a limit point of s_n, and there are only finitely many entries before n_1 to be excluded.

Continue recursively to construct the two sequences ε_k and s_{n_k}. $\varepsilon_k \to 0$ and $d(L, s_{n_k}) < \varepsilon_k$, so $s_{n_k} \to L$. Moreover, $i > j$ implies $n_i > n_j$. Which implies s_{n_k} is constructed by simply skipping entries in s_n.

This table will be defined row-by-row Let x_i be an enumeration of E. Since f_n is pointwise bounded, we can produce a subsequence f_{n_k} such that $f_{n_k}(x_1)$ converges as $k \to \infty$. Denote this sequence by f_k^1.

Then in a similar fashion, we recursively construct the following sequences f_k^{i+1} out of f_k^i using the above lemma. This results in a table of functions

$$\begin{matrix} f_1^1 & f_2^1 & f_3^1 & \cdots \\ f_1^2 & f_2^2 & f_3^2 & \cdots \\ f_1^3 & f_2^3 & f_3^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

which satisfy the properties

(a) f_k^{i+1} is a subsequence of f_k^i

(b) $f_k^i(x_i)$ converges as $k \to \infty$

(c) f_k^{i+1}, as a subsequence of f_k^i, is simply f_k^i with entries deleted.

Then, if we take the diagonal

$$f_1^1, f_2^2, f_3^3, \dots$$

(c) implies that this sequence is a subsequence of all f_k^i, thus (b) implies that this sequence converges for all x_i.