Principles of Mathematical Analysis
Chapter 6
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(Note: no guarantee of correctness. Just an undergrad writing their answers down and archiving because notebooks have become unwieldy.)

Q1. Suppose α increases on $[a,b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

A: IDEA

x_0 may be covered with an interval of arbitrarily small width, and this segment will have an arbitrarily small weight in the sum due to α's continuity at x_0.

PROOF THAT $f \in \mathscr{R}(\alpha)$:

Let $\varepsilon > 0$.

Since α is continuous at x_0, choose δ such that $|x - x_0| \leq 2\delta$ implies $|\alpha(x) - \alpha(x_0)| \leq \varepsilon$.

Then let $P = [a, x_0 - \delta, x_0 + \delta, b]$. Then

$$\begin{aligned} U(P, f, \alpha) &= \sum_{i=0}^{n} M_i \Delta\alpha_i \\ &= M_0 \Delta\alpha_0 + M_1 \Delta\alpha_1 + M_2 \Delta\alpha_2 \end{aligned}$$

Since $f(x) = 0$ on $[a, x_0 - \delta]$ and $[x_0 + \delta, b]$, $M_0 = 0$ and $M_2 = 0$. Moreover, $M_1 = 1$

$$U(P, f, \alpha) = 0 + 1 \cdot \Delta\alpha_1 + 0\Delta = \alpha_1$$

As for L, the following holds for any partition at all, as any nonempty interval must contain a point that is not x_0

$$L(P, f, \alpha) = 0$$

So

$$U(P, f, \alpha) - L(P, f, \alpha) = \Delta\alpha_1 = \alpha(x + \delta) - \alpha(x - \delta) < \varepsilon$$

Then $f \in \mathscr{R}(\alpha)$.

PROOF THAT $\int f d\alpha = 0$:

Since $L(P, f, \alpha) = 0$ for all P,

$$\underline{\int} f d\alpha = 0$$

Which lets us conclude

$$\int f d\alpha = 0$$

Q2. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)

A: IDEA

We show a contradiction that a "bump" of nonzero area must exist if f is not identically zero, hence the integral must be nonzero.

PROOF

Choose any arbitrary point q in $[a,b]$. Suppose $f(q) > 0$. Let m be a number such that $0 < m < f(q)$, and use the continuity of f to create an interval $[s,t]$ containing q such that $x \in [s,t]$ implies $f(x) \geq m$.

By additivity of the integral,

$$\int_a^b f dx = \int_a^s f dx + \int_s^t f dx + \int_t^b f dx$$

Hence

$$\int_a^b f dx \geq \int_s^t f dx$$

But $\int_s^t f dx \geq m(t - s) > 0$, so

$$\int_a^b f dx > 0$$

A contradiction to the assertion that $\int_a^b f dx = 0$

Q3. Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0^+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

(b) State and prove a similar result for β_2

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0

(d) If f is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

A: Idea

With β_i, the only intervals with non-zero weight in any partition will be those which contain 0. Moreover, these segments will have fixed weights due to our definitions of β_i. Then, bounds on the variation of the upper and lower Riemann sums automatically pass through as bounds on f in the segment itself.

Hence we can convert convergence of Riemann sums into convergence of f.

(a) Proof that $f \in \mathscr{R}(\beta_1)$ implies $f(0+) = f(0)$

Let $f \in \mathscr{R}(\beta_1)$. Let $\varepsilon > 0$. Choose a partition P such that $0 \in P$, and

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

The only nonzero term in either of the sums $U(P, f, \beta_1)$ or $L(P, f, \beta_1)$ comes from the segment containing $x = 0$ as a left endpoint. Let this segment be $[0, x_i]$. Then

$$U(P, f, \beta_1) = M_i \cdot \Delta x_i = M_i \cdot (\beta(x_i) - \beta(0)) = M_i \cdot 1 = M_i$$

$$L(P, f, \beta_1) = M_i \cdot \Delta x_i = m_i \cdot (\beta(x_i) - \beta(0)) = m_i \cdot 1 = m_i$$

So $U(P, f, \beta_1) - L(P, f, \beta_1) = M_i - m_i < \varepsilon$. This tells us that on $[0, x_i]$, f stays within ε of $f(0)$. Since ε was arbitrary, we conclude that $f(0+) = f(0)$

Proof that $f(0+) = f(0)$ implies $f \in \mathscr{R}(\beta_1)$

Let $f(0+) = f(0)$. Let $\varepsilon > 0$ Choose δ such that $x \in (0, \delta)$ implies $f(x) \in (f(0) - \varepsilon, f(0) + \varepsilon)$. Let P be the partition $\{a, 0, \delta, b\}$. Then for the same reasons as above,

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

So $f \in \mathscr{R}(\beta_1)$.

Proof that $\int f d\beta_1 = 0$

By Theorem 6.7(b), letting $t_i = 0$, we have that

$$\left| f(0) - \int f d\beta_1 \right| < \varepsilon$$

Hence we conclude that

$$\int f d\beta_1 = f(0)$$

(b) Statement

$f \in \mathscr{R}(\beta_1)$ if and only if $f(0-) = f(0)$, and if it exists,

$$\int f d\beta_1 = f(0)$$

Proof

Extremely similar as before, but with $[x_{i-1}, 0]$ instead of $[0, x_i]$

(c) Proof that $f \in \mathscr{R}(\beta_3)$ implies f is continuous at 0

Let $f \in \mathscr{R}(\beta_3)$. Let $\varepsilon > 0$. Let P be a partition that such that $0 \in P$ and $U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$

Then, consider the intervals $[x_{i-1}, 0]$, $[0, x_i]$. We have that

$$\beta_3(0) - \beta_3(x_{i-1}) = \frac{1}{2}$$

$$\beta_3(x_i) - \beta_3(0) = \frac{1}{2}$$

Then

$$U(P, f, \beta_3) = \frac{M_i + M_{i-1}}{2}$$

$$L(P, f, \beta_3) = \frac{m_i + m_{i-1}}{2}$$

Let $M = \min\{M_i, M_{i-1}\}$ and $m = \max\{m_i, m_{i-1}\}$, then

$$U(P, f, \beta_3) \geq M$$

$$L(P, f, \beta_3) \leq m$$

Then

$$M - m \leq U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence $f(x)$ is within ε of $f(0)$ if $f(x)$ is in $[x_{i-1}, x_i]$. Hence f is continuous.

Proof that f continuous at 0 implies $f \in \mathscr{R}(\beta_3)$

Let f be continuous at 0. Let $\varepsilon > 0$. Choose δ such that $|x| \leq 2\delta$ implies $|f(x) - f(0)| \leq \varepsilon/2$ then take the partition $P = \{a, -\delta, \delta, b\}$. Then

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence $f \in \mathscr{R}(\beta_3)$. Also, with a similar argument as in (a),

$$\int f d\beta_3 = f(0)$$

(d) f being continuous at zero is a strong enough condition for all three previous proofs above to apply. Since all of them show equality of the integral with respect to β_i and $f(0)$, the result follows.

Q4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$

A: Let P be a partition. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for all x_i we can find two rational numbers p and q such that $x_{i-1} < p < q < x_i$. This tells us that $M_i = 1$ for all x_i. Furthermore, the number $s = p + (\sqrt{2}/2)(q - p)$ is irrational and between p and q. This tells us that $m_i = 1$ for all x_i. Hence it must be that

$$U(P, f) = 1$$

$$L(P, f) = 0$$

for any partition P. Then f is not Riemann-integrable.

Q5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

A: No. Consider the function

$$f(x) := \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases}$$

We have that $(f^2)(x) = 1$, so $\int_a^b f^2 dx = (b - a)$. But f itself is not Riemann-integrable, as the exercise above shows.

For f^3, use Theorem 6.11, which tells us that the composition of continuous functions with integrable functions is integrable. Let $\phi = \sqrt[3]{x}$. Then $\phi \circ (f^3) \in \mathscr{R}$ Since $\phi \circ f^3 = f$, $f \in \mathscr{R}$.

Q6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0, 1]$. *Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

A: Each set of the sequence that constructs the Cantor set has 2^n intervals, each of width $1/3^n$. Let E_n be the nth set in the sequence. Cover each interval in E_n with an open interval whose endpoints have a distance of $1.1 \cdot (1/3^n)$. Then, the sum of all lengths of each cover will be $1.1 \cdot (2/3)^n$. This also covers the Cantor set itself, since the Cantor set E is a subset of E_n for all n. Hence, we have made a cover of the Cantor set with arbitrarily small area.

Take the endpoints of each interval and construct a partition, and proceed as in Theorem 6.10. This sequence of partitions eventually.

Q7. Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function F such that the above limit exists, although it fails to exist with $|f|$ in place of f

A:

(a) Lemma: If $f \in \mathscr{R}$ on $[a,b]$, then

$$\lim_{h\to a} \int_a^h f(x)dx = 0$$

Proof: Let $|f(x)| \leq M$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then, letting $h \leq \delta$

$$\left|\int_a^h f(x)dx\right| \leq M((a+h)-a) = Mh \leq M\delta = \varepsilon$$

Since the ε was arbitrary, the lemma holds. Then, by Theorem 6.12, if $f \in \mathscr{R}$ on $[0,1]$,

$$\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx$$

Then, taking limits

$$\begin{aligned}
\lim_{c\to 0} \int_0^1 f(x)dx &= \lim_{c\to 0} \int_0^c f(x)dx + \lim_{c\to 0} \int_c^1 f(x)dx \\
\int_0^1 f(x)dx &= 0 + \lim_{c\to 0} \int_c^1 f(x)dx \\
\int_0^1 f(x)dx &= \lim_{c\to 0} \int_c^1 f(x)dx
\end{aligned}$$

Hence showing that the definition agrees.

(b) Let $n = 1, 2, \ldots$. Consider the function (with infinitely many cases)

$$f(x) = \left\{(-1)^{n+1}n + 1 \quad \text{if } \frac{1}{n+1} \leq x < \frac{1}{n}\right.$$

Then the graph of f consists of boxes of width $1/n - 1/(n+1)$ and height $n+1$. The area of each box is then $1/n$. Then as the lower bound of the integral approaches 0, the integral approximates the alternating sum $S = 1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges to $\ln 2$. However, in the case of $|f|$, the integral apporximates the sum $S = 1 + 1/2 + 1/3 + 1/4 + \cdots$, which diverges.

Q8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*. Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

A: Suppose $\int_1^\infty f(x)dx$ converges. Then if we show that, for any positive integer b,

$$\sum_{n=1}^b f(n) \leq \int_1^b f(x)dx$$

we can prove convergence of the sum, as the partial sums form a monotonic sequence (f is non-negative). Let P be the partition consisting of the points $0, 1, 2, \ldots, b-1, b$. Then, since f is monotonically decreasing,

$$\inf_{x\in[n-1,n]} f(x) = f(n)$$

Hence

$$\sum_{n=1}^b f(n) = L(P, f) \leq \int_1^b f(x)dx$$

which was the inequality we wanted. Then the sum converges.

Suppose $\int_1^\infty f(x)dx$ converges. Then we make a similar argument, with $U(P, f)$ bounding $\int_1^b f(x)dx$ from above. Take the same partition of $[a, b]$, then

$$\sup_{x\in[n,n+1]} f(x) = f(n)$$

Hence

$$U(P, f) = \sum_{n=0}^{b-1} f(n)$$

Which proves that the integral converges.

Q9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}$$