MATH-531 Homework Solutions

Jasper Ty

1 Before we start...

1.1 Binomial coefficients and elementary counting

Exercise 1.1.1. $\boxed{3}$ Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n} \binom{-2}{k} = (-1)^n \left\lfloor \frac{n+2}{2} \right\rfloor$$

2 Generating functions

3 Integer partitions and q-binomial coefficients

4 Permutations

5 Alternating sums, signed counting and determinants

Exercise 5.0.1. (3 points)
Prove a generalization of Lemma 6.1.4 in which f is only required to be a bijection, not an involution, but the assumption "sign $I = 0$ for all $I \in \mathcal{X}$ satisfying $f(I) = I$" is replaced by the stronger assumption "sign $I = 0$ for all $I \in \mathcal{X}$ and all **odd** $k \in \mathbb{N}$ satisfying $f^k(I) = I$"

Since $\mathcal{X}$ is finite, there exists a smallest k.

Exercise 5.0.2. Recall the concepts of Dyck words and Dyck paths defined in Example 2 in Section 3.1. Let $n \in \mathbb{N}$. If $w \in 0, 1^{2n}$ is a $2n$-tuple, and if $k \in \{0, 1, \ldots, 2n\}$, then we define *k-height* $h_k(w)$ of w to be the number

$$\begin{aligned}&(\text{\# of 1's among the first } k \text{ entries of } w)\\&\qquad - (\text{\# of 0's among the first } k \text{ entries of } w)\end{aligned}$$

If w is a Dyck word, then this k-height $h_k(w)$ is a nonnegative integer. [For example, if $n = 4$ and $w = (1, 0, 0, 1)$ then $h_3(w)$ is a nonnegative integer.

6 Symmetric functions