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RIEMANN INTEGRAL OF $f\alpha'$ IS EQUAL TO RIEMANN-STIELTJES INTEGRAL OF f WITH INTEGRATOR α

Theorem

Let $\alpha : [a, b] \to \mathbb{R}$ be a differentiable monotonic function such that $\alpha' \in \mathscr{R}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. And

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Proof

Let $\varepsilon > 0$ and choose a partition P such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

If we pick any points $s_i, t_i \in [x_{i-1}, x_i]$, we have that

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \le \varepsilon$$

Let $|f(x)| \le M$, then

$$\sum_{i=1}^{n} |\alpha'(s_i)\Delta x_i - \alpha'(t_i)\Delta x_i||f(s_i)| \le M\varepsilon$$

Pushing the $f(s_i)$ in,

$$\sum_{i=1}^{n} |f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\alpha'(t_i)\Delta x_i| \le M\varepsilon$$

By the mean value theorem, we can choose t_i such that

$$\Delta\alpha_i = \alpha'(t_i)\Delta x_i$$

Then

$$\sum_{i=1}^{n} |f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\Delta\alpha_i| \le M\varepsilon$$

By the triangle inequality,

$$\left|\sum_{i=1}^{n}[f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\Delta\alpha_i]\right| \le \sum_{i=1}^{n} |f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\Delta\alpha_i| \le M\varepsilon$$

So

$$\left|\sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\Delta\alpha_i\right| \le \sum_{i=1}^{n} |f(s_i)\alpha'(s_i)\Delta x_i - f(s_i)\Delta\alpha_i| \le M\varepsilon$$

In particular, letting $f(s_i)$ be M_i and m_i respectively,

$$|U(P, f\alpha') - U(P, f, \alpha)| \le M\varepsilon$$

$$|L(P, f\alpha') - L(P, f, \alpha)| \le M\varepsilon$$

Since the triangle inequality holds for sup, consider the reverse triangle inequality (for sup)

$$\left| \sup_P |U(P, f\alpha')| - \sup_P |U(P, f\alpha')| \right| \leq \sup_P |U(P, f\alpha') - U(P, f, \alpha)| \leq M\varepsilon$$

Then

$$\left| \overline{\int_a^b} f\alpha' dx - \overline{\int_a^b} f d\alpha \right| \leq M\varepsilon$$

Similarly,

$$\left| \underline{\int_a^b} f\alpha' dx - \underline{\int_a^b} f d\alpha \right| \leq M\varepsilon$$

Hence, if one integral exists, the other does. Moreover, their integrals are equal.

Change of variable
Theorem

Integration by parts
Theorem