Principles of Mathematical Analysis
Chapter 7
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1 Pointwise Convergence

Pointwise convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions. If f is a function such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then we say f_n converges *pointwise* to f.

This type of convergence is very *weak*. It guarantees very little in the way of actually *working with* the limit.
This definition is readily adapted to infinite sums of functions.

Infinite sums of functions
Definition
If f is a function such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in E$, then we say f is the *sum* of the series f_n.

An example of the weakness of pointwise convergence is

Continuity is not preserved under pointwise convergence
Example
Let $f_n : [0,1] \to [0,1]$ be defined by

$$f_n(x) := x^n$$

Then $f := \lim f_n$ is

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

by Theorem 3.20(e)

In this case, a sequence of continuous functions converges to a function that is eminently discontinuous. We use the preceding idea of "letting $f < 0$ sink and letting $f = 1$ float using the n^{th} power limit" to show the following.

Integrability is not preserved under pointwise convergence

Example

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

If we let

$$f_m(x) := \lim_{n\to\infty}(\cos m!\pi x)^{2n}$$

the above shows that a limit of integrable functions ($\int f_m dx = 0$ for all m) may fail to be integrable.

Proof

By a similar argument as in the previous example,

$$\lim_{n\to\infty}(\cos m!x)^{2n} = \begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases}$$

Let $x = p/q$ be rational. Then $m!x$ is rational for all $m \geq q$. Let x be irrational, $m!x$ cannot be an integer for any m, otherwise we can show a contradiction. Then

$$\lim_{m\to\infty}\begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

These two examples show that *properties* of f_n may not pass through the limit to f.
Next, we show that *operations* on f_n may not be passed through the limit to f.

A limit of differentiated functions may not be the differentiated limit of functions

Example

Let

$$f_n(x) := \frac{\sin nx}{\sqrt{n}}$$

Then,

$$0 = \frac{d}{dx}\left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\frac{d}{dx} f_n\right] = \sqrt{n}\cos nx$$

A limit of integrated functions may not be the integral of a limit of functions

Example

Let

$$f_n(x) := nx(1-x^2)^n$$

Then

$$0 = \int_0^1 \left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\int_0^1 f_n\right] = \frac{1}{2}$$

2 Uniform convergence

UNIFORM CONVERGENCE OF FUNCTIONS
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions.
If f is a function such that for all ε there exists N such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all x, we say that f converges *uniformly*.
This definition carries over to sums of functions (the partial sums must converge to the limit function uniformly).

This is a *much stronger* notion of convergence, as it, in a sense, "tethers" together convergence of of all points in the domain.

There are useful criteria for uniform convergence. These hint at the idea of being able to make sense of the idea of "distance" between two functions, which Rudin makes precise later. The first one tells us that uniform convergence of functions corresponds to convergence (via the Cauchy criterion) in a certain space of functions.

CAUCHY CRITERION FOR UNIFORM CONVERGENCE
Theorem
f_n converges uniformly on E if and only if there exists an integer N such that for all $m, n \geq N$,

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \varepsilon$$

Proof
For the forward implication, let $f_n \to f$ uniformly, then choose N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$$

Then use the triangle inequality.
For the converse, we note that the criterion is strong enough itself to guarantee $f_n \to f$ pointwise, hence if we take the inequality

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

and let $m \to \infty$, we recover the original condition for uniform convergence

The next criterion is just a rephrasing of the original definition in similar terms. It tells us that uniform convergence corresponds to the "distance" between two functions vanishing.

SUPREMUM CRITERION FOR UNIFORM CONVERGENCE
Theorem
$f_n \to f$ uniformly on E if and only if

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0 \;\text{ as }\; n \to \infty$$

Next, we show that uniform convergence does not share the same failures as pointwise convergence when it comes to passing through important properties and operations.

We first take continuity.

Continuity is preserved under uniform convergence
Theorem
Let f_n be a sequence of continuous functions and let $f_n \to f$ uniformly. Then f is continuous.
Proof
Let $\varepsilon > 0$. Let $x \in E$. Consider the inequality

$$|f(x) - f(t)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(t)| + |f_n(t) - f(t)|$$

First use uniform convergence to choose n such that $|f(x) - f_n(x)|$ and $|f(t) - f_n(t)|$ are arbitrarily small.
Then use continuity to choose δ such that $|f_n(x) - f_n(t)|$ can be made arbitrarily small by letting $|x - t| \leq \delta$.

This however is not an "if and only if". A continuous function can be the non-uniform limit of continuous functions. In one case, however, we can use compactness and monotonicity to take enough control such that a sequence of functions can *only* converge uniformly.

Inferring uniform convergence from pointwise convergence
Example
Let $f_n : K \to \mathbb{R}$ be a sequence of continuous functions, and let $f_n \to f$ pointwise. Let f be continuous. Impose the two conditions

- Let K be compact
- Let $f_n(x) \geq f_{n+1}(x)$ for all x and all n

Then $f_n \to f$ uniformly.
Proof
We must show that, as $n \to \infty$,

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0$$

To do this, we choose ε and argue that the set of points for which $|f_n(x) - f(x)| \geq \varepsilon$ can be made empty if we take n large enough. Let this set of exceptional points be K_n.
K's compactness tells us that K_n is compact:

- Because $f_n - f$ is a continuous function and $(\infty, \varepsilon]$ is closed, K_n is closed. (Theorem 4.8; inverse images of closed sets under continuous functions are closed)
- Because K_n is closed, and K is compact, K_n is compact. (Theorem 2.35; closed subsets of compact spaces are compact).

Moreover, because of the monotonicity of f_n, $K_n \supseteq K_{n+1}$. Hence K_n is a sequence of nested compact sets.
It must be that $\bigcap K_n$ is empty, as f_n converges pointwise to f, and so for each x, the fact that $|f_n(x) - f| < \varepsilon$ for some n means there must be some K_n it is not a member of.
But an intersection of nonempty nested compact sets must be nonempty (Theorem 2.36), so there has to be an empty set somewhere in the sequence. Let this set be K_N. Then for all $n \geq N$, K_n is empty. This is precisely the N we need to keep the distance between f_n and f below ε.

Restricting our study to continuous, bounded functions (the functions above were bounded because K was compact) gives rise to a very nice structure, one we're already very familiar with: a metric space. (More importantly, a vector space!)
The concept of distance here corresponds with how it has been used in the previous theorems and examples.

The space of continuous and bounded functions on X
Definition
Let X be a metric space. Let $\mathscr{C}(X)$ denote the set of all continuous, bounded, complex-valued functions on X. Let $f \in \mathscr{C}(X)$. Define the *supremum norm* as follows.

$$\|f\| = \sup_{x \in X} |f(x)|$$

Let $d(f, g)$ for $f, g \in \mathscr{C}(X)$ be induced by the norm; i.e

$$d(f, g) = \|f - g\|$$

This turns $\mathscr{C}(X)$ into a metric space.

This space is complete, and the fact that it is so can be bootstrapped with the previous theorems.

$\mathscr{C}(X)$ is complete
Proof
Let f_n be a Cauchy sequence in $\mathscr{C}(X)$. Then f_n converges uniformly to a function f. Moreover, since each f_n is continuous, f is continuous. That f is bounded follows from the fact that $f = f_n + (f - f_n)$, so $\|f\| \leq \|f_n\| + \|f - f_n\| = \|f_n\| + \varepsilon$. Hence $f \in \mathscr{C}(X)$. Since $f_n \to f$ uniformly, $\|f - f_n\| \to 0$.

Next, we tackle integration. Uniform convergence allows us to bound f in a, well, uniform way. This, along with a version of the squeeze theorem for integrals, shows us that uniform convergence plays nicely with integration.

The integral of a uniformly convergent limit is a uniformly convergent limit of integrals
Theorem Let $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ for all n. Let $f_n \to f$ uniformly. Then $f \in \mathscr{R}(\alpha)$, and

$$\int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f_n d\alpha$$

Proof
The idea is that you can "squeeze" f with two f_n shaped calipers, and that the gap decreases as n goes to infinity. Let this gap be

$$\varepsilon_n = \sup_{x \in [a,b]} |f_n(x) - f(x)|$$

Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n$$

Then this bounds f's lower and upper integrals.

$$\int_a^b (f_n - \varepsilon_n) d\alpha \leq \underline{\int} f d\alpha \leq \overline{\int} f d\alpha \leq \int_a^b (f_n + \varepsilon_n) d\alpha$$

Then

$$0 \leq \overline{\int} f d\alpha - \underline{\int} f d\alpha \leq 2\varepsilon(\alpha(b) - \alpha(a))$$

So $f \in \mathscr{R}(\alpha)$. From the inequality we can also read off

$$\left| \int_a^b f d\alpha - \int_a^b f_n d\alpha \right| \leq \varepsilon_n (\alpha(b) - \alpha(a))$$

Which proves the equality of the integral and the limit.

Finally, we look at differentiation. This is not as nice as integration.

3 Equicontinuous families of functions