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1 Pointwise Convergence

Pointwise convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions. If f is a function such that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in E$, then we say f_n converges *pointwise* to f.

This type of convergence is very *weak*. It guarantees very little in the way of actually *working with* the limit.
This definition is readily adapted to infinite sums of functions.

Infinite sums of functions
Definition
If f is a function such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

for all $x \in E$, then we say f is the *sum* of the series f_n.

An example of the weakness of pointwise convergence is

Continuity is not preserved under pointwise convergence
Example
Let $f_n : [0,1] \to [0,1]$ be defined by

$$f_n(x) := x^n$$

Then $f := \lim f_n$ is

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

by Theorem 3.20(e)

In this case, a sequence of continuous functions converges to a function that is eminently discontinuous. We use the preceding idea of "letting $f < 0$ sink and letting $f = 1$ float using the n^{th} power limit" to show the following.

Integrability is not preserved under pointwise convergence

Example

$$\lim_{m\to\infty}\lim_{n\to\infty}(\cos m!\pi x)^{2n} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

If we let

$$f_m(x) := \lim_{n\to\infty}(\cos m!\pi x)^{2n}$$

the above shows that a limit of integrable functions ($\int f_m dx = 0$ for all m) may fail to be integrable.

Proof

By a similar argument as in the previous example,

$$\lim_{n\to\infty}(\cos m!x)^{2n} = \begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases}$$

Let $x = p/q$ be rational. Then $m!x$ is rational for all $m \geq q$. Let x be irrational, $m!x$ cannot be an integer for any m, otherwise we can show a contradiction. Then

$$\lim_{m\to\infty}\begin{cases} 0 & m!x \text{ is not an integer} \\ 1 & m!x \text{ is an integer} \end{cases} = \begin{cases} 0 & x \text{ irrational} \\ 1 & x \text{ rational} \end{cases}$$

These two examples show that *properties* of f_n may not pass through the limit to f.

Next, we show that *operations* on f_n may not be passed through the limit to f.

A limit of differentiated functions may not be the differentiated limit of functions

Example

Let

$$f_n(x) := \frac{\sin nx}{\sqrt{n}}$$

Then,

$$0 = \frac{d}{dx}\left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\frac{d}{dx} f_n\right] = \sqrt{n}\cos nx$$

A limit of integrated functions may not be the integral of a limit of functions

Example

Let

$$f_n(x) := nx(1-x^2)^n$$

Then

$$0 = \int_0^1 \left[\lim_{n\to\infty} f_n\right] \neq \lim_{n\to\infty}\left[\int_0^1 f_n\right] = \frac{1}{2}$$

2 Uniform convergence

Uniform convergence of functions
Definition
Let $f_n : E \to \mathbb{R}$ be a sequence of functions.
If f is a function such that for all ε there exists N such that

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all x, we say that f converges *uniformly*.
This definition carries over to sums of functions (the partial sums must converge to the limit function uniformly).

This is a *much stronger* notion of convergence, as it, in a sense, "tethers" together convergence of of all points in the domain.

There are useful criteria for uniform convergence. These hint at the idea of being able to make sense of the idea of "distance" between two functions, which will be made precise later. The first one tells us that uniform convergence of functions corresponds to convergence (via the Cauchy criterion) in a certain space of functions.

Cauchy criterion for uniform convergence
Theorem
f_n converges uniformly on E if and only if there exists an integer N such that for all $m, n \geq N$,

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \varepsilon$$

Proof
For the forward implication, let $f_n \to f$ uniformly, then choose N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$$

Then use the triangle inequality.
For the converse, we note that the criterion is strong enough itself to guarantee $f_n \to f$ pointwise, hence if we take the inequality

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

and let $m \to \infty$, we recover the original condition for uniform convergence

The next criterion is just a rephrasing of the original definition in similar terms. It tells us that uniform convergence corresponds to the "distance" between two functions vanishing.

Supremum criterion for uniform convergence
Theorem
$f_n \to f$ uniformly on E if and only if

$$\sup_{x \in E} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty$$

Next, we show that uniform convergence does not share the same failures as pointwise convergence when it comes to passing through important properties and operations.

Continuity is preserved under uniform convergence

Theorem

Let f_n be a sequence of continuous functions and let $f_n \to f$ uniformly. Then f is continuous.

Proof

We want to show that, for all x,

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = f(x)$$

If we were allowed to interchange this limit, we can use the continuity of f_n to achieve what we want

$$\lim_{n \to \infty} \lim_{t \to x} f_n(x) = \lim_{n \to \infty} f_n(x) = f(x)$$