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Riemann integral of $f\alpha'$ is equal to Riemann-Stieltjes integral of $f$ with integrator $\alpha$

**Theorem**

Let $\alpha : [a, b] \to \mathbb{R}$ be a differentiable monotonic function such that $\alpha' \in \mathscr{R}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. And

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

**Proof**

Let $\varepsilon > 0$ and choose a partition $P$ such that

$$U(P, \alpha') - L(P, \alpha') < \varepsilon$$

Change of variable