Schubert polynomials

Jasper Ty

These are notes based on my study of Schubert polynomials. My main references are [KnutsonSP] and [MacdonaldSP].

Contents

1 Notation and conventions

1.1 Sets

We take $\mathbb{N}$ to be the set of natural numbers *including* zero,

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

We take $\mathbb{P}$ to be the set of *positive integers*,

$$\mathbb{P} := \{1, 2, \ldots\}.$$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are defined as usual.

1.2 Partitions and compositions

A *weak composition* α of $n \in \mathbb{N}$ is an infinite tuple of nonnegative integers

$$(\alpha_1, \alpha_2, \ldots)$$

such that $\sum_i \alpha_i = n$. We define $|\alpha| = \sum_i \alpha_i$ to have notation for recovering n given α.

A *partition* λ of n is a weak composition whose entries are *weakly decreasing*. That a particular partition λ is a partition of a particular n is denoted $\lambda \vdash n$. We define $|\lambda|$ the exact same way.

I use English notation when drawing diagrams and tableaux, meaning, row index increases *north to south*, and column index increases *west to east*.

1.3 Rings, polynomials, and formal power series

The following notation is (mostly) in accordance with the notation in [GrinbergAC], with a few additions.

All rings considered are commutative and unital. An arbitrary ring will be denoted $\mathbb{K}$.

$\mathbb{K}[[t]]$ will denote the formal power series ring over $\mathbb{K}$ in the indeterminate t.

We will fix notation for the following sets of indeterminates, which we will use when convenient:

(a) $X_N := (x_1, x_2, \ldots x_N)$ for a set of N indeterminates.

(b) $X := (x_1, x_2, \ldots)$ for a set of countably many indeterminates.

(c) Y, Y_N, Z, Z_N, Q, Q_N and so on are defined similarly.

With compositions, partitions, or otherwise any finitely supported tuple of nonnegative integers α, we define *multi-index notation* for compactly writing down monomials.

$$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots .$$

We will let $[x^\alpha]f$ denote the coefficient of $[x^\alpha]$ in the polynomial or formal power series f.

1.4 Permutations and the symmetric group

S_n will denote the symmetric group on n letters.

I use cycle notation, so e.g the cycle that sends 1 to 7, 7 to 4, and 4 to 1 will be written as (174).

The simple transpositions $(i\ i+1)$ will be denoted s_i.

The length of a permutation w will be denoted $\ell(w)$.

Permutations will act on polynomials or power series by permuting *places*, meaning that if $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in \mathbb{K}[X_n]$, we define

$$\sigma f(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots x_{\sigma(n)}).$$

2 Schubert Polynomials

Apparently these are "Schubert cycles in flag varieties".

2.1 Divided difference operators

These strike me as a tool to measure how "unsymmetric" a polynomial is in a local sense, in two variables at a time.

2.1.1 Definition

Definition 2.1.1. Let f be a polynomial in N indeterminates. We define the *divided difference operators* ∂_i by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}} \tag{1}$$

Example 2.1.2. If $f(x_1, x_2, x_3) = x_1 x_2$, then

$$\begin{aligned} \partial_2 f(x_1, x_2, x_3) &= \frac{x_1 x_2 - x_1 x_3}{x_2 - x_3} \\ &= x_1 \left(\frac{x_2 - x_3}{x_2 - x_3} \right) \\ &= x_1. \end{aligned}$$

2.1.2 Basic facts

We have the following characterization of ∂_i that does not invoke division.

Lemma 2.1.3. Fix i. Consider some monomial $f = \cdots x_i^a x_{i+1}^b \cdots$. Then

$$\partial_i(\cdots x_i^a x_{i+1}^b \cdots) = \varepsilon_{ba} \sum_{\substack{u,v \geq \min\{a,b\} \\ u+v=a+b-1}} \cdots x_i^u x_{i+1}^v \cdots,$$

where ε is defined to be

$$\varepsilon_{rs} := \begin{cases} 0 & \text{if } r = s \\ 1 & \text{if } r < s \\ -1 & \text{if } r > s \end{cases}.$$

Proof. The proof is not hard but it's a slog. We compute

$$\begin{aligned} \partial_i(\cdots x_i^a x_{i+1}^b \cdots) &= \frac{(\cdots x_i^a x_{i+1}^b \cdots) - (\cdots x_i^b x_{i+1}^a \cdots)}{x_i - x_{i+1}} \\ &= (\cdots) \frac{x_i^a x_{i+1}^b - x_i^b x_{i+1}^a}{x_i - x_{i+1}}. \end{aligned}$$

We recall our (well, mine) favorite high-school algebra identity

$$\frac{x^n - y^n}{x - y} = x^{n-1} y^0 + x^{n-2} y^1 + \cdots + x^1 y^{n-2} + x^0 y^{n-1},$$

which we will modify a little

$$\frac{x^{n+m} y^m - x^m y^{n+m}}{x - y} = x^{m+n-1} y^m + x^{m+n-2} y^{m+1} + \cdots + x^{m+1} y^{m+n-2} x^m y^{m+n-1},$$

and we note that the pairs $(u, v) \in \{(m+n-1, m), \ldots, (m, m+n-1)\}$ are precisely those such that $u, v \geq \min\{a, b\}$ and $u + v = 2m + n - 1$. We then put $a = m + n$ and $b = m$, to get that

$$\frac{x^a y^b - x^a y^b}{x - y} = \sum_{\substack{u,v \geq \min\{a,b\} \\ u+v=a+b-1}} x^u y^v, \qquad \text{given } a \geq b.$$

Then, to forget $a \geq b$, we pick up a ε_{ba} term to keep track of sign. Applying this identity now to our computation, we finish the lemma. □

Then the following properties of the operator ∂_i can be read off

Corollary 2.1.4. Let f be a polynomial.

(a) $\partial_i f$ is a polynomial.

(b) If f is homogeneous of degree d, then $\partial_i f$ is homogeneous of degree $d-1$.

Proof. Left to reader. □

Theorem 2.1.5. The divided difference operators satsify the following relations

(a) The braid relation
$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \tag{2}$$

(b) Far commutativity
$$\partial_i \partial_j = \partial_j \partial_i \qquad \text{whenever} \qquad |i-j| > 1$$

(c) Reflection by a simple
$$\partial_i s_i = -\partial_i$$

(d) Chain condition
$$\partial_i^2 = 0$$

Proof. We have that
$$\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i).$$
Then □

I wonder if $\partial_i^2 = 0$ has to do with the Schuberts arising from a cohomology theory.

2.2 The definition of a Schubert polynomial

Definition 2.2.1. The *Schubert polynomials* $\mathfrak{S}_w$ are defined by the rules
$$\begin{cases} \mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1, \\ \partial_i \mathfrak{S}_w := \mathfrak{S}_{w s_i} \end{cases}$$

Actually, this definition is a theorem if we start with the "Representatives of cohomology classes of Schubert cycles in flag varieties" definition, but I don't understand that unfortunately.

3 The ring of coinvariants of

Theorem 3.0.1. The Schuberts form a basis for the coinvariant ring

3.1 Definition

References

[StanleyEC2] Richard P. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge University Press 2023.

[GrinbergAC] Darij Grinberg, *An Introduction to Algebraic Combinatorics*, http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf

[KnutsonSP] Allen Knutson, *Schubert Polynomials and Symmetric Functions*, https://pi.math.cornell.edu/~allenk/schubnotes.pdf

[MacdonaldSP] Ian Macdonald, *Notes on Schubert Polynomials*