Reflection groups and Coxeter groups

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What is this?

These are notes based on my reading of Humphreys's "Reflection groups and Coxeter groups".

Contents

1 Finite reflection groups

We define *reflections*.

Definition 1.0.1. Let V be a real Euclidean space, equipped with an inner product $\langle \_, \_ \rangle$. Let $\alpha \in V$ be some nonzero vector. The *reflection* s_α is defined to be.

$$s_\alpha \lambda := \lambda - \frac{2 \langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

This has the effect of sending α to $-\alpha$, and fixing the hyperplane orthogonal to α. We do some paperwork to show that this is all defined correctly.

Proposition 1.0.2. Let V be a real vector space, and let $\alpha \in V$. The reflection s_α is an orthogonal linear operator, which sends α to $-\alpha$, and fixes the hyperplane

$$H_\alpha := \{w \in V : \langle w, \alpha \rangle = 0\}.$$

Proof. Linearity follows from the bilinearity of the inner product. Let $a, b \in \mathbb{R}$. Then

$$\begin{aligned}
\mathrm{s}_\alpha(a\lambda + b\beta) &= (a\lambda + b\beta) - \frac{2\left\langle a\lambda + b\beta, \alpha \right\rangle}{\langle \alpha, \alpha \rangle}\alpha \\
&= (a\lambda + b\beta) - \frac{2\Big(a\langle \lambda, \alpha\rangle + b\left\langle \beta, \alpha \right\rangle\Big)}{\langle \alpha, \alpha \rangle}\alpha \\
&= (a\lambda + b\beta) - \left(\frac{2a\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha + \frac{2b\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha\right) \\
&= \left(a\lambda - \frac{2a\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha\right) + \left(b\beta - \frac{2b\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha\right) \\
&= a\left(\lambda - \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha\right) + b\left(\beta - \frac{2\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha\right) \\
&= a\,\mathrm{s}_\alpha\,\lambda + b\,\mathrm{s}_\alpha\,\beta.
\end{aligned}$$

Next, we compute that it is orthogonal.

$$\begin{aligned}
&\left\langle \mathrm{s}_\alpha\,\lambda, \mathrm{s}_\alpha\,\beta \right\rangle \\
&= \left\langle \lambda - \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha, \beta - \frac{2\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle \\
&= \left\langle \lambda, \beta \right\rangle + \left\langle \lambda, -\frac{2\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle + \left\langle -\frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha, \beta \right\rangle + \left\langle -\frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha, -\frac{2\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\alpha \right\rangle \\
&= \left\langle \lambda, \beta \right\rangle - \frac{2\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle}\langle \lambda, \alpha\rangle - \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha \rangle}\left\langle \alpha, \beta \right\rangle + \frac{4\langle \lambda, \alpha\rangle\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle^2}\langle \alpha, \alpha\rangle \\
&= \left\langle \lambda, \beta \right\rangle - \frac{4\langle \lambda, \alpha\rangle\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle} + \frac{4\langle \lambda, \alpha\rangle\left\langle \beta, \alpha\right\rangle}{\langle \alpha, \alpha \rangle} \\
&= \left\langle \lambda, \beta \right\rangle.
\end{aligned}$$

Finally, we verify its "reflection" properties. First,

$$\mathrm{s}_\alpha\, \alpha = \alpha - \frac{2\langle \alpha, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \alpha - 2\alpha = -\alpha.$$

And, if $\langle \lambda, \alpha \rangle = 0$

$$\mathrm{s}_\alpha\, \lambda = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha = \lambda - \frac{2 \cdot 0}{\langle \alpha, \alpha \rangle} = \lambda.$$

□

And finally,

Remark 1.0.3. Let s_α be a reflection in V. Then it is involutory, i.e s_α^2 is the identity operator on V.

Proof. We know that $V = \mathbb{R}\alpha \oplus H_\alpha$. Then if we write down a vector $w \in V$ as $p\alpha + q\lambda$, where $p, q \in \mathbb{R}$ and $\lambda \in H_\alpha$,

$$\mathrm{s}_\alpha\, \mathrm{s}_\alpha\, w = \mathrm{s}_\alpha\, \mathrm{s}_\alpha(p\alpha) + \mathrm{s}_\alpha\, \mathrm{s}_\alpha(q\lambda) = \mathrm{s}_\alpha(-p\alpha) + \mathrm{s}_\alpha(q\lambda) = p\alpha + q\beta = w.$$

□

Definition 1.0.4. A *finite reflection group* is a finite subgroup of $\mathrm{O}(V)$ generated by reflections.

1.1 Some examples

Definition 1.1.1. Let $V = \mathbb{R}^2$, and fix some integer $m \geq 3$. The *dihedral group* Dih_m is the finite reflection group consisting of all orthogonal transformations which fix a regular m-sided polygon centered at the origin.

Definition 1.1.2. The *symmetric group* Sym_n is the finite reflection group consisting of all orthogonal transformations which swap two basis vectors.