Positive Polynomials and Sums

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What is this?

These are notes I am taking for the class "Positive Polynomials and Sums" at Drexel University, taught by Hugo Woerdeman.

Contents

1 Cones of Hermitian matrices and trigonometric polynomials

1.1 Cones

Definition 1.1.1. Let $\mathscr{H}$ be a real Hilbert space.

A **cone** $\mathscr{C} \subseteq \mathscr{H}$ is a nonempty subset of $\mathscr{H}$ such that

(i) $\mathscr{C} + \mathscr{C} \subseteq \mathscr{C}$,

(ii) $\alpha\mathscr{C} \subseteq \mathscr{C}$ for all $\alpha > 0$.

Namely, a cone $\mathscr{C}$ is a set closed under taking **conical combinations**.

Proposition 1.1.2. The set of all cones of a Hilbert space $\mathscr{H}$ forms a lattice, with join and meet given by sum and intersection respectively.

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be cones in $\mathscr{H}$.

Let $x, y \in \mathscr{C}_1 \cap \mathscr{C}_2$. Then $x + y \in \mathscr{C}_1 + \mathscr{C}_1$ □

Definition 1.1.3. Let $\mathscr{C}$ be a cone.

The **dual cone** $\mathscr{C}^*$ of $\mathscr{C}$ is

$$\mathscr{C}^* := \left\{ L \in \mathscr{H} : \langle L, K \rangle \geq 0 \text{ for all } K \in \mathscr{C} \right\}.$$

We say that $\mathscr{C}$ is **self-dual** if $\mathscr{C}^* = \mathscr{C}$.

Proposition 1.1.4. Let $\mathscr{C}$ be a cone.

(i) $\mathscr{C}^*$ is a cone.

(ii) $\mathscr{C}^{**} = \overline{\mathscr{C}}$.

Definition 1.1.5. A **extreme ray** of a cone $\mathscr{C}$ is

1.2 The cone PSD_n

Definition 1.2.1. Let Mat_n be the Hilbert space over $\mathbb{C}$ consisting of $n \times n$ complex matrices, with inner product given by

$$\langle A, B \rangle_{\mathrm{Mat}_n} := \mathrm{tr}(AB^*)$$

Let Herm_n be the *real* Hilbert space consisting of Hermitian matrices.

Namely, $\mathrm{Herm}_n := \mathbb{R}.\, \mathrm{Mat}_n$.

It's a neat fact that $\mathrm{Herm}_n \oplus i\mathrm{Herm}_n = \mathrm{Mat}_n$.

Definition 1.2.2. The **cone of positive semidefinite matrices** PSD_n is the subset of Herm_n consisting of positive semidefinite matrices.

1.3 Trigonometric polynomials

Definition 1.3.1. Let $\mathbb{F}$ be a field and let X be any set. We denote the $\mathbb{F}$-vector space with basis X by $\mathbb{F}.X$.

Moreover, if $M \in \mathrm{PSD}_{|X|}$

Definition 1.3.2. Let $\mathbf{x}_d$ denote a column vector of variables

$$\mathbf{x}_d = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}.$$