Ordinary differential equations

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What is this?

These are notes I am taking for the class MATH-623, *Ordinary Differential Equations*, at Drexel University, taught by Yixin Guo.

Some notation is changed from her notes, and I try to add as many missing details from proofs as possible.

Contents

1 Basic theory **2**
1.1 Definitions 2
1.2 Existence and uniqueness 3

1 Basic theory

1.1 Definitions

Definition 1.1.1. Let $J \subseteq \mathbb{R}, U \subseteq \mathbb{R}, \Lambda \subseteq \mathbb{R}^k$ be open sets, and let $\mathbf{f} : J \times U \times \Lambda \to \mathbb{R}^n$ is a smooth function. An **ordinary differential equation** is an equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}) \tag{ODE}$$

where the dot denotes differentiation with respect to the independent variable t.

Morally, the individual parts of an ODE have the following meaning:

- t: an independent variable $t \in J$, typically time,
- $\mathbf{x}$: a dependent variable $\mathbf{x} \in U$,
- $\underline{\lambda}$: a vector of parameters $\underline{\lambda} \in \Lambda$,
- $\mathbf{f}$: a continuously differentiable function that encodes the (time) evolution of $\mathbf{x}$.

Definition 1.1.2. A **solution** of (ODE) is a function $\mathbf{F} : J_0 \to U$, where $J_0 \subseteq J \subseteq R$, such that

$$\frac{d}{dt}\mathbf{F}(t) = \mathbf{f}(t, \mathbf{F}(t), \underline{\lambda}), \qquad \forall t \in J_0$$

i.e, a function for which we can put $\mathbf{x} = \mathbf{F}(t)$ in (ODE) for all $t \in J_0$.

The **orbit** of the solution $\mathbf{F}$ is the set

$$\{\mathbf{F}(t) \in U : t \in J_0\} \subseteq \mathbb{R}^n.$$

This is also called the **trajectory**, **integral curve**, or **solution curve**.

Evidently, an ODE can have many solutions. For example, if $J = U = \Lambda = R$ and $f(t, x, \lambda) = \lambda x$, then it's well known that $F(t) = Ce^{\lambda t}$ is a solution to this ODE for all $C \in \mathbb{R}$.

Fortunately, both real-life and abstract experience tells us that in many cases, the following is a natural way to restrict solutions to an ODE.

Definition 1.1.3. An **initial-value problem** (IVP) is the system of equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}), \\ \mathbf{x}(t_0) = \mathbf{x}_0. \end{cases} \tag{IVP}$$

Namely, it is an (ODE) with additional data $(t_0, \mathbf{x}_0) \in J \times U$, encoding an **initial value** constraint.

A **solution** of (IVP) is, again a function $\mathbf{F} : J_0 \to U$ that solves the underlying ODE, but now subject to the condition that $\mathbf{F}(t_0) = \mathbf{x}_0$.

Example 1.1.4. The **forced Van der Pol** equation is defined to be

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = b(1 - x_1^2)x_2 - \omega^2 x_1 + a\cos\Omega t. \end{cases} \tag{ForcedVanDerPol}$$

1.2 Existence and uniqueness

Let $J, U \subseteq \mathbb{R}$, and consider the IVP

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 \tag{IVP-1D}$$

where $f : J \to U$ is continuously differentiable in t on some open interval containing t_0.

Supposing we can integrate (IVP-1D) from t_0 to a given point t,

$$\begin{aligned} \int_{t_0}^{t} \frac{dy}{d\tau} d\tau &= \int_{t_0}^{t} f(\tau, y) d\tau, \\ y(t) - y(t_0) &= \int_{t_0}^{t} f(\tau, y) d\tau, \\ y(t) &= y(t_0) + \int_{t_0}^{t} f(\tau, y) d\tau, \qquad (\textstyle\int \text{IVP-1D}) \\ y(t) &= y_0 + \int_{t_0}^{t} f(\tau, y) d\tau. \end{aligned}$$

If F satisfies ($\int$IVP-1D), it satisifes (IVP-1D) and vice versa, and this is easily seen using the fundamental theorem of calculus.

Definition 1.2.1. The **Picard iterates** y_i, given the data for (IVP-1D), are defined recursively as follows:

$$\begin{cases} y_0(t) := y_0, \\ y_{i+1}(t) := y_0 + \int_{t_0}^{t} f\big(\tau, y_i(\tau)\big) d\tau. \end{cases} \tag{PicardIter}$$

We have a weak but straightforward estimate on the y_i.

Lemma 1.2.2. For all $a > 0, b > 0$, define R to be the rectangle $[t_0, t_0 + a] \times [y_0 - b, y_0 + b]$. Then if we define

$$M := \max_{(t,y)\in R} |f(t, y)|, \qquad \alpha := \min\left\{a, \frac{b}{M}\right\}.$$

Then

$$|y_n(t) - y_0| \le M(t - t_0) \text{ for all } t_0 \le t \le t + \alpha \tag{1}$$

for all n.

Proof. We prove this by induction.

We note that if the statement holds for n, then for all $t_0 \le t \le \alpha$,

$$\begin{aligned} |y_n(t) - y_0| &\le M(t - t_0) \\ &\le M\alpha \\ &= M \cdot \min\left\{a, \frac{b}{M}\right\} \\ &= \min\left\{\frac{a}{M}, b\right\} \\ &\le b. \end{aligned}$$

Hence

$$|f(t, y_n(t))| \le M. \tag{2}$$

$n = 0$: We have that $|y_0(t) - y_0| = 0 \le M(t - t_0)$ trivially for *all* $t \ge t_0$.

$n > 0$: Suppose that $|y_n(t) - y_0| \le M(t - t_0)$ for all $t_0 \le t \le t_0 + \alpha$. Then

$$\begin{aligned} |y_{n+1}(t) - y_0| &= \left|\int_{t_0}^{t} f(\tau, y_n(\tau))d\tau\right| \\ &\le \int_{t_0}^{t} \underbrace{|f(\tau, y_n(\tau))|}_{\text{use } (2)} d\tau \\ &\le \int_{t_0}^{t} M d\tau \\ &= M(t - t_0) \end{aligned}$$

for all $t_0 \le t \le t_0 + \alpha$. This completes the proof.

□

Next we show that the the Picard iterates y_n converge to a function y on $[t_0, t_0+\alpha]$ which satisfies (IVP-1D).

Theorem 1.2.3. Let $f(t, y)$ be continuously differentiable in both t and y. Then

Proof. We have that

$$\begin{aligned} y_n(t) &= y_0(t) + [y_1(t) - y_0] + \cdots + [y_n(t) - y_{n-1}(t)] \\ &= y_0(t) + \sum_{i=1}^{n} y_i(t) - y_{i-1}(t). \end{aligned}$$

So, $y_n(t)$ converges if and only if $\sum_{i=1}^{n} y_i(t) - y_{i-1}(t)$ converges.

□