Noncommutative Schur functions

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What is this?

This is (going to be) an "infinite napkin"-type set of notes I am taking about the Fomin-Greene theory of noncommutative Schur functions.

Note that this is distinct from the theory of the ring called NCSym, in which there exists structural analogues of monomial, elementary, homogeneous, power, and Schur functions. I am not currently aware of any connection between these two theories.

Contents

1 Some formalism

Convention 1.1. Rings are unital. Ideals are two-sided.

We will set up a minimal framework for stating theorems of the form

> Suppose the relations ______ hold among the variables ______, then ______.

where we will work with two obviously isomorphic objects:

- noncommutative monomials, and
- words,

which sacrifices only a little bit of generality for plenty of conceptual clarity.

1.1 Noncommutative monomials

We begin our construction from the "variables" side.

Definition 1.2. Let $\mathbf{u} = (u_1, \dots, u_N)$ be some list of elements, which we will call our **noncommuting variables**.

(a) The **monoid of noncommutative monomials** $\langle \mathbf{u} \rangle$ is the free monoid on the generating set $\mathbf{u}$.

(b) The **free associative ring** $\mathcal{U}$ is the monoid ring $\mathbb{Z}\langle \mathbf{u} \rangle$.

As one would expect, $\mathcal{U}$ is simply the associative unital $\mathbb{Z}$-algebra whose basis consists of elements

$$u_{i_1} \cdots u_{i_k} \in \langle \mathbf{u} \rangle$$

for some list of integers $(i_1, \dots, i_k)$, and whose multiplication on basis elements is given by

$$(u_{i_1} \cdots u_{i_k}) \cdot (u_{i_1} \cdots u_{j_\ell}) = u_{i_1} \cdots u_{i_k} u_{i_1} \cdots u_{j_\ell}.$$

1.2 Words

Next, continue from the "words" side.

Definition 1.3. Let $[N] = \{1, \dots, N\}$ be set of first N positive integers, which we will call our collection of **letters**.

(a) The **monoid of words** $\mathbb{W}$ is the free monoid on the generating set $[N]$.

(b) We define $\mathcal{U}^*$ to be the monoid ring $\mathbb{Z}\mathbb{W}$.

$\mathcal{U}^*$ is the associative unital $\mathbb{Z}$-algebra whose basis consists of words $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_k \in \mathbb{W}$ and whose multiplication is given by concatenation.

Of course, as a tuple of integers, they also index noncommutative monomials— as a shorthand, if w is a word, then we will denote by $\mathbf{u}_\mathsf{w}$ the monomial

$$\mathbf{u}_\mathsf{w} := u_{\mathsf{w}_1} \cdots u_{\mathsf{w}_k}.$$

This definition becomes an actual structure on $\mathcal{U}$ and $\mathcal{U}^*$ via pairing— we have simply set $\langle \mathbf{u} \rangle$ and $\mathbb{W}$ to be orthonormal bases.

Definition 1.4. We define a scalar product $\langle -, - \rangle : \mathcal{U} \times \mathcal{U}^* \to \mathbb{Z}$ by

$$\langle \mathbf{u}_\mathsf{v}, \mathsf{w} \rangle = \delta_{\mathsf{v}\mathsf{w}}.$$

for all $\mathbf{u}_\mathsf{v} \in \mathcal{U}$ and $\mathsf{w} \in \mathcal{U}^*$.

1.3 Ideals

Definition 1.5. For any commutative ring R, let $\mathcal{U}_R$ denote the extension of scalars $\mathcal{U} \otimes_\mathbb{Z} R$. We define $\mathcal{U}_R^*$ similarly.

Definition 1.6. Let I be an ideal of $\mathcal{U}$. The **orthogonal complement** $I^\perp$ of I is defined

$$I^\perp := \{\gamma \in \mathcal{U}^* \mid \big\langle I, \gamma \big\rangle = 0\}.$$

$I^\perp$ really provides a parameterization of coset representatives for $\mathcal{U}/I$— this is precise when we are working over, say, $\mathcal{U}_\mathbb{Q}$ or $\mathcal{U}_\mathbb{R}$.

1.4 Polynomial and formal power series rings over $\mathcal{U}$

Definition 1.7. If I is a two-sided ideal of $\mathcal{U}$, then we let $I[\![x]\!]$ be the ideal of $\mathcal{U}[\![x]\!]$ generated by I.

2 Noncommutative elementary and homogeneous symmetric functions

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) := \underbrace{\sum_{i_1 > i_2 > \cdots > i_k}}_{\text{(decreasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{1}$$

The **noncommutative homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) := \underbrace{\sum_{i_1 \le i_2 \le \cdots \le i_k}}_{\text{(weakly increasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{2}$$

We will have the convention that $e_0(\mathbf{u}) = h_0(\mathbf{u}) = 1$, and that $e_k(\mathbf{u}) = h_k(\mathbf{u}) = 0$ if $k < 0$.

In other words, the elementary symmetric functions are generating functions for columns, and the homogeneous symmetric functions are generating functions for rows.

2.1 Newton's identities

We define noncommutative analogues of standard generating functions for the elementary and homogeneous symmetric functions.

Definition 2.2. We define the generating functions for $e_k(\mathbf{u})$'s and $h_k(\mathbf{u})$'s

$$E(x) := \sum_{k=0}^{N} x^k e_k(\mathbf{u}) = \prod_{i=N}^{1} (1 + x u_i), \tag{3}$$

$$H(x) := \sum_{k=0}^{\infty} x^k h_k(\mathbf{u}) = \prod_{i=1}^{N} (1 - x u_i)^{-1}. \tag{4}$$

in $\mathcal{U}[\![x]\!]$.

An immediate consequence of this definition is a noncommutative analogue of Newton's identities.

Proposition 2.3 (Noncommutative Newton-Girard formulas)**.** We have

$$E(x)H(-x) = H(x)E(-x) = 1. \tag{5}$$

In particular,

$$\sum_{k=0}^{n}(-1)^k e_k(\mathbf{u})h_{n-k}(\mathbf{u}) = 0, \tag{6}$$

$$\sum_{k=0}^{n}(-1)^k h_k(\mathbf{u})e_{n-k}(\mathbf{u}) = 0 \tag{7}$$

for all $n > 1$.

Proof. Putting together (3) and (4) immediately gives us (5). One obtains (6) and (7) by comparing coefficients in (5). □

Corollary 2.4. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $E(x)E(y) \overset{I[x,y]}{\equiv} E(y)E(x)$.

(b) $H(x)H(y) \overset{I[x,y]}{\equiv} H(y)H(x)$.

Proof. Suppose $E(x)E(y) = E(y)E(x)$ for all commuting x, y. Then $H(x)H(y) = E(-x)^{-1}E(-y)^{-1} = E(-y)^{-1}E(-x)^{-1} = H(y)H(x)$. The reverse implication is proved identically. □

2.2 The ideal I_C

Evidently, in the quotient

$$\mathcal{U}/[\mathcal{U}\mathcal{U}],$$

the elementaries commute, since all monomials commute:

$$e_k(\mathbf{u})e_j(\mathbf{u}) \overset{[\mathcal{U}\mathcal{U}]}{\equiv} e_j(\mathbf{u})e_k(\mathbf{u}).$$

This ideal is pretty big— it turns the entirety of $\mathcal{U}$ into a commutative algebra. It turns out that, not only can we find smaller ideals, but we can find the *smallest ideal* in which the elementaries commute.

Definition 2.5. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \qquad (a < b), \qquad (8)$$
$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \qquad (a < b < c), \qquad (9)$$
$$u_c u_b u_c u_a + u_b u_c u_a u_c - u_c u_b u_a u_c - u_b u_c^2 u_a \qquad (a < b < c). \qquad (10)$$

Equivalently, I_C is the ideal generated by the relations

$$[u_a u_b] u_a \equiv u_b [u_a u_b] \qquad (a < b), \qquad (11)$$
$$[u_a u_c] u_b \equiv u_b [u_a u_c] \qquad (a < b < c), \qquad (12)$$
$$[u_b u_c][u_a u_c] \equiv 0 \qquad (a < b < c). \qquad (13)$$

We now have the following deep, difficult calculation— the proof which is deferred to the Appendix, §9.1.

Theorem 2.6. If $I \supseteq I_C$, then $e_k(\mathbf{u}) e_j(\mathbf{u}) \stackrel{I}{\equiv} e_j(\mathbf{u}) e_k(\mathbf{u})$ for all j, k.

With this, we now precisely know in which quotients of $\mathcal{U}$ the $e_k(\mathbf{u})'s$'s generate a commutative subalgebra.

Moreover, these are the exact same quotients in which the $h_k(\mathbf{u})$'s generate a commutative subalgebra.

Lemma 2.7. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $e_k(\mathbf{u}) e_j(\mathbf{u}) \stackrel{I}{\equiv} e_j(\mathbf{u}) e_k(\mathbf{u})$ for all j, k.

(b) $h_k(\mathbf{u}) h_j(\mathbf{u}) \stackrel{I}{\equiv} h_j(\mathbf{u}) h_k(\mathbf{u})$ for all j, k.

2.3 The map Ψ

Theorem 2.8 (Fundamental theorem of symmetric functions)**.** Let $\Lambda(\mathbf{x})$ denote the ring of symmetric polynomials in the commuting variables $\mathbf{x} = (x_1, \ldots, x_N)$. Then the elementary symmetric functions in $\mathbf{x}$ are algebraically independent, and moreover generate $\Lambda(\mathbf{x})$ as an algebra:

$$\Lambda(\mathbf{x}) = \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})].$$

Proof. One can prove this via the *Gale-Ryser* theorem, to show that the transition matrix from the elementary symmetric function basis to the monomial symmetric function basis is invertible. This proof is carried out in [EC2, Theorem 7.4.4]. □

Corollary 2.9. If I contains I_C, then the map

$$\begin{aligned}\Psi : \Lambda(\mathbf{x}) &\to \mathcal{U}\\ e_k(\mathbf{x}) &\mapsto e_k(\mathbf{u})\end{aligned}$$

induces a ring homomorphism $\Lambda(\mathbf{x}) \to \mathcal{U}/I$.

Proof. Combine Theorems 2.8 and 2.6. □

Proposition 2.10. If I contains I_C, any identity in $\Lambda(\mathbf{x})$ that can be purely expressed in terms of polynomials in $h_k(\mathbf{x})$'s and $e_k(\mathbf{x})$'s holds in $\Lambda(\mathbf{u})/I$.

Proof. □

3 Noncommutative Schur functions

Definition 3.1. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u})$ is defined

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{\pi \in S_t} \operatorname{sgn}(\pi) e_{\lambda_1^\top + \pi(1) - 1}(\mathbf{u}) e_{\lambda_2^\top + \pi(2) - 2}(\mathbf{u}) \cdots e_{\lambda_t^\top + \pi(t) - t}(\mathbf{u}),$$

where $t = \lambda_1$ is the number of parts of $\lambda^\top$. Alternatively, we also define

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{\pi \in S_t} \operatorname{sgn}(\pi) h_{\lambda_1 + \pi(1) - 1}(\mathbf{u}) h_{\lambda_2 + \pi(2) - 2}(\mathbf{u}) \cdots h_{\lambda_t + \pi(t) - t}(\mathbf{u}),$$

and it's easy to see that they are congruent modulo any ideal containing I_C.

The first definition is based on the **Kostka-Naegelsbach identity**

$$s_\lambda(\mathbf{x}) = \det\left(e_{\lambda_i^\top + j - i}(\mathbf{x})\right)_{i,j=1}^n,$$

and the second is based on the **Jacobi-Trudi identity**

$$s_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i + j - i}(\mathbf{x})\right)_{i,j=1}^n.$$

Since these are purely polynomials of elementary symmetric and complete homogeneous polynomials, one sees the following

Definition 3.2. If $I \supseteq I_C$, then

$$\Psi(s_\lambda(\mathbf{x})) \stackrel{I}{\equiv} \mathfrak{J}_\lambda(\mathbf{u}).$$

Proof.

$$\begin{aligned}
\Psi(s_\lambda(\mathbf{x})) &= \Psi\left(\det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n\right) \\
&= \Psi\left(\sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{x}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{x})\right) \\
&\stackrel{I}{\equiv} \sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{u}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{u}) \\
&= \mathfrak{J}_\lambda(\mathbf{u}).
\end{aligned}$$

□

Theorem 3.3 ([FG98], [BF16])**.** In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{T\in \mathrm{SSYT}(\lambda;N)} \mathbf{u}^{\text{colword } T}.$$

3.1 Cauchy kernel

Theorem 3.4. If I contains I_C, then for all $\gamma \in I_C^\perp$,

$$\left\langle \prod_{i=1}^N \prod_{j=1}^N (1 - x_i u_j)^{-1}, \gamma \right\rangle = \sum_\lambda s_\lambda(\mathbf{x}) \left\langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \right\rangle, \tag{14}$$

$$\left\langle \prod_{i=1}^N \prod_{j=N}^1 (1 + x_i u_j), \gamma \right\rangle = \sum_\lambda s_\lambda(\mathbf{x}) \left\langle \mathfrak{J}_{\lambda^\top}(\mathbf{u}), \gamma \right\rangle. \tag{15}$$

Proof. We will prove (14) directly:

$$\left\langle \prod_{i=1}^N \prod_{j=1}^N (1 - x_i u_j)^{-1}, \gamma \right\rangle = \left\langle \prod_{i=1}^N \sum_{k=0}^\infty x^k h_k(\mathbf{u}), \gamma \right\rangle$$

$$= \left\langle \Psi\left(\prod_{i=1}^{N} \sum_{k=0}^{\infty} x^k h_k(\mathbf{y})\right), \gamma \right\rangle$$
$$= \left\langle \Psi\left(\sum_{\lambda} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})\right), \gamma \right\rangle$$
$$= \left\langle \sum_{\lambda} s_\lambda(\mathbf{x}) \mathfrak{J}_\lambda(\mathbf{u}), \gamma \right\rangle$$
$$= \sum_{\lambda} s_\lambda(\mathbf{x}) \left\langle \mathfrak{J}_\lambda(\mathbf{u}), \gamma \right\rangle.$$

(15) is proven the exact same way. □

4 The symmetric function F_γ

We will now give the definition of the symmetric function associated to a vector in $I^\perp$, first defined in [FG98].

Definition 4.1. Fix an ideal I containing I_C, and let $\gamma \in \mathcal{U}^*$. We define F_γ to be

$$F_\gamma(\mathbf{x}) := \left\langle \Omega(\mathbf{x}, \mathbf{u}), \gamma \right\rangle.$$

for all $\gamma \in I^\perp$.

Definition 4.2.

Theorem 4.3. If $\mathfrak{J}_\lambda(\mathbf{u})$ has a monomial-positive representative in $\mathcal{U}$ for all λ, then $F_\gamma(\mathbf{x})$ is Schur positive.

If, moreover, one can has a procedure for computing this representative, one automatically has a procedure for computing the coefficients of $F_\gamma(\mathbf{x})$'s Schur expansion.

5 Applications

5.1 Recovering known results in the plactic algebra

5.2 Stanley symmetric functions via the nilCoxeter algebra

The connection between Schubert polynomials and the nilCoxeter ideal was first explored by Richard Stanley and

Definition 5.1. The **nilCoxeter** ideal

5.3 LLT polynomials

6 Linear programming

Consider the positive cones $\mathcal{U}_{\geq 0}$ and $\mathcal{U}^*_{\geq 0}$.

7 Algebras of operators

Definition 7.1. A **combinatorial representation** of $\mathcal{U}/I$ is

8 Switchboards

Definition 8.1. The **switchboard ideal** I_S is

Definition 8.2. Let $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_n \in \mathcal{U}^*$. We define the **fundamental quasisymmetric function** $Q_{\mathrm{Des}(\mathsf{w})}(\mathbf{x})$ by

$$Q_{\mathrm{Des}(\mathsf{w})}(\mathbf{x}) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \mathrm{Des}(\mathsf{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

In general

$$Q_{\mathrm{Des}(\mathsf{w})}(\mathbf{x}) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \mathrm{Des}(\mathsf{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

9 Appendix

9.1 Proof of Theorem 2.6

We will follow A.N Kirillov's proof [K16, Theorem 2.26]. It has a few errors, but it's recoverable. Blasiak and Fomin prove the assertion in the context of a much more general framework in [BF18].

First, we recast the problem in terms of generating functions. Recall that $E(x) := (1 + xu_N) \cdots (1 + xu_1)$ is the generating function for the $e_k(\mathbf{u})'s$, that is,

$$E(x) = \sum_{k=1}^{N} x^k e_k(\mathbf{u}).$$

Then, we can recast our problem by working with the E's.

Lemma 9.1. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $E(x)E(y) \overset{I[x,y]}{\equiv} E(y)E(x)$.

(b) $e_k(\mathbf{u})e_j(\mathbf{u}) \overset{I}{\equiv} e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

Proof. Expand and compare coefficients. □

To do this inductively, we put $E_{ji}(x) = (1 + xu_j) \cdots (1 + xu_i)$ for all $i < j$.

Proof of Theorem 2.6. We will first show this inductively— we claim that

$$[E_{n,1}(x)E_{n,1}(y)] \overset{I_C[x,y]}{\equiv} 0,$$

for all $n \geq 0$. The case $n = 1$ is easily verified. Now, suppose $[E_n(x)E_n(y)] \equiv 0$. Then

$$
\begin{aligned}
&[E_{n+1,1}(x), E_{n+1,1}(y)] \\
&= \big[(1 + xu_{n+1})E_{n,1}(x), (1 + yu_{n+1})E_{n,1}(y)\big] \\
&= (1 + xu_{n+1})E_{n,1}(x)(1 + yu_{n+1})E_{n,1}(y) - (1 + yu_{n+1})E_{n,1}(y)(1 + xu_{n+1})E_{n,1}(x) \\
&= (1 + xu_{n+1})\Big(\big[E_{n,1}(x), (1 + yu_{n+1})\big] + (1 + yu_{n+1})E_{n,1}(x)\Big)E_{n,1}(y) \\
&\qquad - (1 + yu_{n+1})\Big(\big[E_{n,1}(y), (1 + xu_{n+1})\big] + (1 + xu_{n+1})E_{n,1}(y)\Big)E_{n,1}(x) \\
&= (1 + xu_{n+1})\big[E_{n,1}(x), (1 + yu_{n+1})\big]E_{n,1}(y) + (1 + xu_{n+1})(1 + yu_{n+1})E_{n,1}(x)E_{n,1}(y) \\
&\qquad - (1 + yu_{n+1})\big[E_{n,1}(y), (1 + xu_{n+1})\big]E_{n,1}(x) - (1 + yu_{n+1})(1 + xu_{n+1})E_{n,1}(y)E_{n,1}(x) \\
&\equiv (1 + xu_{n+1})\big[E_{n,1}(x), (1 + yu_{n+1})\big]E_{n,1}(y) - (1 + yu_{n+1})\big[E_{n,1}(y), (1 + xu_{n+1})\big]E_{n,1}(x) \\
&= (1 + xu_{n+1})\ \big[(1 + xu_n)\cdots(1 + xu_1), (1 + yu_{n+1})\big]\ E_{n,1}(y) \\
&\qquad - (1 + yu_{n+1})\ \big[(1 + yu_n)\cdots(1 + yu_1), (1 + xu_{n+1})\big]\ E_{n,1}(x) \\
&= (1 + xu_{n+1})\left(\sum_{i=1}^{n} E_{n,i+1}(x)\big[(1 + xu_i), (1 + yu_{n+1})\big]E_{i-1,1}(x)\right)E_n(y) \\
&\qquad - (1 + yu_{n+1})\left(\sum_{i=1}^{n} E_{n,i+1}(y)\big[(1 + yu_i), (1 + xu_{n+1})\big]E_{i-1,1}(y)\right)E_n(x) \\
&= (1 + xu_{n+1})\left(xy\sum_{i=1}^{n} E_{n,i+1}(x)\big[u_i, u_{n+1}\big]E_{i-1,1}(x)\right)E_n(y) \\
&\qquad - (1 + yu_{n+1})\left(xy\sum_{i=1}^{n} E_{n,i+1}(y)\big[u_i, u_{n+1}\big]E_{i-1,1}(y)\right)E_n(x)
\end{aligned}
$$

$$
\begin{aligned}
&= xy\sum_{i=1}^{n}\Big((1+xu_{n+1})E_{n,i+1}(x)\left[u_i,u_{n+1}\right]E_{i-1,1}(x)E_n(y)\\
&\qquad -(1+yu_{n+1})E_{n,i+1}(y)\left[u_i,u_{n+1}\right]E_{i-1,1}(y)E_n(x)\Big)\\
&\equiv xy\sum_{i=1}^{n}\Big((1+xu_{n+1})[u_i,u_{n+1}]E_{n,i+1}(x)E_{i-1,1}(x)E_n(y)\\
&\qquad -(1+yu_{n+1})[u_i,u_{n+1}]E_{n,i+1}(y)E_{i-1,1}(y)E_n(x)\Big)\\
&\equiv xy\sum_{i=1}^{n}\Big([u_i,u_{n+1}](1+xu_i)E_{n,i+1}(x)E_{i-1,1}(x)E_n(y)\\
&\qquad -[u_i,u_{n+1}](1+yu_i)E_{n,i+1}(y)E_{i-1,1}(y)E_n(x)\Big)\\
&= xy\sum_{i=1}^{n}[u_i,u_{n+1}]\Big((1+xu_i)E_{n,i+1}(x)E_{i-1,1}(x)E_n(y)-(1+yu_i)E_{n,i+1}(y)E_{i-1,1}(y)E_n(x)\Big)\\
&\equiv xy\sum_{i=1}^{n}[u_i,u_{n+1}]\Big(E_{n,1}(x)E_{n,1}(y)-E_{n,1}(y)E_{n,1}(x)\Big)\\
&= xy\sum_{i=1}^{n}[u_i,u_{n+1}][E_{n,1}(x),E_{n,1}(y)]\\
&= 0.
\end{aligned}
$$

□

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