Mathematical logic notes

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What is this?

These are notes based on my reading of Mileti's "Modern Mathematical Logic", accompanied by sitting in Henry Towsner's MATH 5700 class at Penn.

I mix in some other notation along with those in the book. For example, I take $\mathbb{P}$ to be the set of positive integers, and I prefer to notate sequences as symbols with subscripts rather than as functions.

The biggest differences are that I use _ as a wildcard variable, and I like to use the $\mapsto$ notation to specify functions.

I have an intense dislike for `mathcal`, and actively try to replace uses of it whenver I can.

Contents

2 Induction and recursion

We put the notions of induction and recursion in a more general framework.

2.1 Induction and recursion on $\mathbb{N}$

First, we recall well-known avatars of induction and recursion.

Definition 2.1.1. We define the *successor function* S to be

$$\begin{aligned} S : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto n + 1. \end{aligned}$$

Theorem 2.1.2 (Induction on $\mathbb{N}$ – steps)**.** Let $X \subseteq N$ such that $0 \in X$ and $S(n) \in X$ whenever $n \in X$. Then it must be that $X = \mathbb{N}$.

Theorem 2.1.3 (Recursion on $\mathbb{N}$ – steps)**.** Let X be a set. If $y \in X$, and $g : \mathbb{N} \times X \to X$, there exists a unique function $f : \mathbb{N} \to X$ such that

(i) $f(0) = y$, and

(ii) $f(S(n)) = g(n, f(n))$ for all $n \in \mathbb{N}$

Theorem 2.1.4 (Induction theorem – order)**.** Let $X \subseteq N$ such that $0 \in X$ and for all $n \in X$, $m \in X$ whenever $m < n$. Then $X = \mathbb{N}$.

The utility in the previous formalization of the recursion theorem is apparent in its order form.

Theorem 2.1.5 (Recursion theorem – order)**.** Let X be a set. If $g : X^* \to X$, then there is a unique function f such that

$$f(n) = g(f \upharpoonright [n]).$$

Example 2.1.6 (Fibonacci numbers)**.** Let $X = \mathbb{N}$, and define

$$\begin{aligned} g : \mathbb{N}^* &\to \mathbb{N} \\ \{a_i\}_{1 \le i \le n} &\mapsto \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ a_{n-2} + a_{n-1} & n \ge 2 \end{cases} \end{aligned}$$

Then by Theorem 2.1.5, there is a unique function f such that $f(n) = g(f \upharpoonright [n])$. We call this function the *Fibonacci sequence*.

2.2 Generation

Definition 2.2.1. Let A be a set, and fix an *arity* $k \in \mathbb{P}$. We define *k-ary functions on* A to be those functions of the form

$$f : A^k \to A.$$

Common shorthands are *unary*, *binary*, and *ternary* functions for 1-ary, 2-ary, and 3-ary functions respectively.

Definition 2.2.2. Let A be a set, $B \subseteq A$, and let $\mathcal{H}$ be a collection such that each $h \in \mathcal{H}$ is a _-ary function on A.

We call $(A, B, \mathcal{H})$ a *simple generating system*.

To be able to pick out all the functions $h \in \mathcal{H}$ of a *specific arity* $k \in \mathbb{P}$, we denote the set of all such functions $\mathcal{H}_k$.

The interpretation is that A is our background set, and B is the set which we wish to generate some larger set using all the operations of $\mathcal{H}$.

Example 2.2.3 (Subgroups generated by a subset)**.** Let A be a group, and let $B \subset A$ be a subset that contain's the identity of A.

We are interested in the *subgroup generated by A generated by B*. In this case, $\mathcal{H} = \{h_1, h_2\}$, where

$$\begin{aligned} h_1 : A^2 &\to A \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

and

$$\begin{aligned} h_2 : A &\to A \\ x &\mapsto x^{-1}. \end{aligned}$$

Example 2.2.4 (Vector subspaces generated by a subset)**.** Now if V is a vector space over an infinite field $\mathbb{F}$, and $B \subseteq V$ is a subset that contains the zero vector, the correct $\mathcal{H}$ that identifies the *subspace generated by B* is now an *infinite family*.

It contains vector addition, namely the map

$$\begin{aligned} h_+ : V^2 &\to V \\ (u, v) &\mapsto u + v \end{aligned}$$

and the *scaling maps*

$$\begin{aligned} h_\alpha : V &\to V \\ v &\mapsto \alpha v \end{aligned}$$

for all $\alpha \in \mathbb{F}$.

Definition 2.2.5. Fix $k \in \mathbb{P}$. A *set-valued k-ary function* is a k-ary function of the form

$$h : A^k \to \mathscr{P}(\_) .$$

Definition 2.2.6. Let A be a set, $B \subseteq A$, and now let $\mathcal{H}$ be a collection of *set-valued* _-ary functions on A.

We call $(A, B, \mathcal{H})$ a *generating system*.

Again, $\mathcal{H}_k$ denotes all $h \in \mathcal{H}$ of arity $k \in \mathbb{P}$.

Example 2.2.7 (Subfields generated by a set)**.** Let $\mathbb{F}$ be a field, and let $B \subseteq \mathbb{F}$ such that $0, 1 \in B$.

Let $\mathcal{H}$ be the collection of functions

$$\begin{aligned} h_1 : \mathbb{F}^2 &\to \mathscr{P}(\mathbb{F}) \\ (a, b) &\mapsto \{a + b\} \end{aligned}$$

$$\begin{aligned} h_2 : \mathbb{F}^2 &\to \mathscr{P}(\mathbb{F}) \\ (a, b) &\mapsto \{a \cdot b\} \end{aligned}$$

$$\begin{aligned} h_3 : \mathbb{F} &\to \mathscr{P}(\mathbb{F}) \\ a &\mapsto \{-a\} \end{aligned}$$

$$\begin{aligned} h_4 : \mathbb{F} &\to \mathscr{P}(\mathbb{F}) \\ a &\mapsto \begin{cases} \{a^{-1}\} & a \neq 0 \\ \varnothing & a = 0 \end{cases} . \end{aligned}$$

Then $(\mathbb{F}, B, \mathcal{H})$ identifies the *subfield of* $\mathbb{F}$ *generated by* B.

Example 2.2.8 (Subgraphs generated by reachability)**.** If $G = (V, E)$ is a directed graph, reachability can be characterized by $\mathcal{H} = \{h\}$, where

$$\begin{aligned} h : V &\to \mathscr{P}(V) \\ v &\mapsto \{w \in V : (v, w) \in E\}. \end{aligned}$$

We remark that every simple generating system can be represented as a generating system, by converting each k-ary function h on A to a set-valued k-ary function h' on A by putting

$$\begin{aligned} h' : A^k &\to \mathscr{P}(A) \\ (a_1, \ldots, a_k) &\mapsto \{h(a_1, \ldots, a_k)\}. \end{aligned}$$

Now we explicitly define what it is exactly that a generating system generates.

From above

Definition 2.2.9. Let $(A, B, \mathcal{H})$ be a generating system.

We call a subset J of A *inductive* if

(i) $B \subseteq J$.

(ii) If $k \in \mathbb{P}$, $h \in \mathcal{H}_k$, and $a_1, \ldots, a_k \in J$, then $h(a_1, \ldots, a_k) \subseteq J$.

We do a common set-theoretic trick here— we take intersections to get the smallest set in a family.

Proposition 2.2.10. Let $(A, B, \mathcal{H})$ be a generating system. Then there exists a unique inductive set I such that $I \subseteq J$ for all inductive sets J.

Proof. We claim that

$$I = \bigcap_{J \text{ is an inductive set}} J.$$

Clearly, this satisfies $I \subseteq J$ for all inductive sets J. Next, we prove that it is inductive.

Since $B \subseteq J$ for all inductive sets J, we have that $B \subseteq I$.

Fix $k \in \mathbb{P}$, and take some $h \in \mathcal{H}_k$. If $a_1, \ldots, a_k \in I$, then $a_1, \ldots, a_k \in J$ for all inductive sets J. Hence $h(a_1, \ldots, a_k) \subseteq J$ for all inductive sets J. By the same principle from which we concluded $B \subseteq I$, we must conclude that $h(a_1, \ldots, a_k) \subseteq I$.

Finally, uniqueness follows from the fact that if I_1 and I_2 are inductive sets such that $I_1 \subseteq J$ and $I_2 \subseteq J$ for all inductive sets J, it must be that $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$, and therefore $I_1 = I_2$. $\square$

From below: levels

This approach is more in the spirit of induction.

Definition 2.2.11. Let $(A, B, \mathcal{H})$ be a generating system. We define a sequence $\{V_n\}_{n=0}^{\infty}$ of subsets of A recursively as follows

$$\begin{aligned} V_0 &= B \\ V_{n+1} &= V_n \cup \{c \in A; c = h(a_1, \ldots, a_k) \text{ for some } a_1, \ldots, a_k \in V_n, h \in \mathcal{H}_k\}. \end{aligned}$$

Namely, the n-th subset in the sequence is all the elements that can be obtained by applying functions in $\mathcal{H}$ *at most n times*.

From below: witnessing sequences

2.3 Step induction

Definition 2.3.1 (Step induction)**.** Let $(A, B, \mathcal{H})$ be a generating system. If $X \subseteq A$ satisfies

(i) $B \subseteq X$

(ii) $h(a_1, \ldots, a_k) \in X$ whenever $k \in \mathbb{P}$, $h \in \mathcal{H}_k$, and $a_1, \ldots, a_k \in X$,

then $G \subseteq X$.

This implies that if $X \subseteq G$ additionally, it must be that $X = G$.

2.4 Freeness and step recursion

The problem with attempting "definition by recursion" with something like a generating system is that *there might be conflicting witnessing sequences*.