Noncommutative Schur functions

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What is this?

This is (going to be) an "infinite napkin"-type set of notes I am taking about the Fomin-Greene theory of noncommutative Schur functions.

Note that this is distinct from the theory of the ring called NCSym, in which there exists structural analogues of monomial, elementary, homogeneous, power, and Schur functions. I am not currently aware of any connection between these two theories.

Contents

1 Some formalism

Convention 1.1. Rings are unital. Ideals are two-sided.

We will set up a minimal framework for stating theorems of the form

Suppose the relations ______ hold among the variables ______, then ______.

where, paradoxically, we will *distinguish* two obviously isomorphic objects:

- noncommutative monomials, and
- words,

which sacrifices only a little bit of generality for plenty of conceptual clarity— the particular isomorphism here is very important!

1.1 Noncommutative monomials

We begin our construction from the "variables" side.

Definition 1.2. Let $\mathbf{u} = (u_1, \ldots, u_N)$ be some list of elements, which we will call our **noncommuting variables**.

(a) The **monoid of noncommutative monomials** $\langle \mathbf{u} \rangle$ is the free monoid on the generating set $\mathbf{u}$.

(b) The **free associative ring** $\mathcal{U}$ is the monoid ring $\mathbb{Z}\langle \mathbf{u} \rangle$.

As one would expect, $\mathcal{U}$ is simply the unital $\mathbb{Z}$-algebra whose basis consists of elements

$$u_{i_1} \cdots u_{i_k} \in \langle \mathbf{u} \rangle$$

for some list of integers $(i_1, \ldots, i_k)$, and whose multiplication on basis elements is given by

$$(u_{i_1} \cdots u_{i_k}) \cdot (u_{i_1} \cdots u_{j_\ell}) = u_{i_1} \cdots u_{i_k} u_{i_1} \cdots u_{j_\ell}.$$

1.2 Words

Next, continue from the "words" side.

Definition 1.3. Let $[N] = \{1, \dots, N\}$ be set of first N positive integers, which we will call our collection of **letters**.

(a) The **monoid of words** $\mathbb{W}$ is the free monoid on the generating set $[N]$.

(b) The **free associative ring** $\mathcal{U}^*$ is the monoid ring $\mathbb{Z}\mathbb{W}$.

As before, $\mathcal{U}^*$ is simply the unital $\mathbb{Z}$-algebra whose basis consists of elements

$$u_{i_1} \cdots u_{i_k} \in \langle \mathbf{u} \rangle$$

for some list of integers $(i_1, \dots, i_k)$, and whose multiplication on basis elements is given by

$$(u_{i_1} \cdots u_{i_k}) \cdot (u_{i_1} \cdots u_{j_\ell}) = u_{i_1} \cdots u_{i_k} u_{i_1} \cdots u_{j_\ell}.$$

1.3 Ideals

We will define an inner product $\langle -, - \rangle$ given by making noncommutative monomials dual to words, that is, $\langle \mathbf{u} \rangle$ and $\mathbb{W}$ are orthonormal bases under $\langle -, - \rangle$.

Definition 1.4. The inner product $\langle -, - \rangle : \mathcal{U} \times \mathcal{U}^* \to \mathbb{Z}$ is defined by

$$\langle \mathbf{u}_\mathsf{v}, \mathsf{w} \rangle = \delta_{\mathsf{vw}}.$$

for all $\mathbf{u}_\mathsf{v} \in \mathcal{U}$ and $\mathsf{w} \in \mathcal{U}^*$.

Now, if I is a ideal of $\mathcal{U}$, we define $I^\perp$ by

$$I^\perp := \{\gamma \in \mathcal{U}^* \mid \langle I, \gamma \rangle = 0\}.$$

2 Noncommutative elementary and homogeneous symmetric functions

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) := \underbrace{\sum_{i_1 > i_2 > \cdots > i_k}}_{\text{(decreasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{1}$$

The **noncommutative homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) := \underbrace{\sum_{i_1 \le i_2 \le \cdots \le i_k}}_{\text{(weakly increasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{2}$$

We will have the convention that $e_0(\mathbf{u}) = h_0(\mathbf{u}) = 1$, and that $e_k(\mathbf{u}) = h_k(\mathbf{u}) = 0$ if $k < 0$.

In symmetric function theory, the elementary and homogeneous symmetric functions can be interpreted as generating functions for semistandard Young tableaux whose shape is a single column and a single row respectively.

The same idea works here— except now one takes the *column word* of a tableau, hence the content of a single row is read left to right— weakly increasing, whereas the content of a single column is read bottom up— decreasing.

2.1 Newton's identities

We define noncommutative analogues of standard generating functions for the elementary and homogeneous symmetric functions.

Definition 2.2. We define the generating functions for e_k's and h_k's

$$E(x) := \sum_{k=0}^{N} x^k e_k(\mathbf{u}) = \prod_{i=N}^{1} (1 + x u_i), \tag{3}$$

$$H(x) := \sum_{k=0}^{\infty} x^k h_k(\mathbf{u}) = \prod_{i=1}^{N} (1 - x u_i)^{-1}. \tag{4}$$

in the rings $\mathcal{U}[x]$ and $\mathcal{U}[\![x]\!]$ respectively.

An immediate consequence of this definition is a noncommutative analogue of Newton's identities.

Proposition 2.3 (Noncommutative Newton-Girard formulas)**.** We have

$$E(x)H(-x) = H(x)E(-x) = 1. \tag{5}$$

In particular,

$$\sum_{k=0}^{n} (-1)^k e_k(\mathbf{u}) h_{n-k}(\mathbf{u}) = 0, \tag{6}$$

$$\sum_{k=0}^{n}(-1)^k h_k(\mathbf{u})e_{n-k}(\mathbf{u}) = 0 \tag{7}$$

for all $n > 1$.

Proof. Putting together (3) and (4) immediately gives us (5)

$$\begin{aligned} E(x)H(-x) &= \left[\prod_{i=N}^{1}(1+xu_i)\right]\left[\prod_{i=1}^{N}(1+xu_i)^{-1}\right] \\ &= \left[\prod_{i=N}^{1}(1+xu_i)\right]\underbrace{\left[\prod_{i=N}^{1}(1+xu_i)\right]^{-1}}_{\text{note reversal of product order}} \\ &= 1. \end{aligned}$$

And $H(x)E(-x) = 1$ is proved exactly the same way. One obtains (6) and (7) by comparing coefficients in (5). □

Corollary 2.4. Let I be an ideal of $\mathcal{U}$. Then $E(x)E(y) \equiv_{I[x,y]} E(y)E(x)$ if and only if $H(x)H(y) \equiv_{I[x,y]} H(y)H(x)$.

Proof. Suppose $E(x)E(y) = E(y)E(x)$ for all commuting x, y. Then $H(x)H(y) = E(-x)^{-1}E(-y)^{-1} = E(-y)^{-1}E(-x)^{-1} = H(y)H(x)$. The reverse implication is proved identically. □

2.2 When do the elementaries commute?

Lemma 2.5. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $E(x)E(y) \equiv_{I[x,y]} E(y)E(x)$.

(b) $e_k(\mathbf{u})e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

Proof. Expand and compare coefficients. □

Lemma 2.6. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $e_k(\mathbf{u})e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

(b) $h_k(\mathbf{u})h_j(\mathbf{u}) \equiv_I h_j(\mathbf{u})h_k(\mathbf{u})$ for all j, k.

Proof. Combine Corollary 2.4 and Lemma 2.5. □

Definition 2.7. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \qquad (a < b), \tag{8}$$
$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \qquad (a < b < c), \tag{9}$$
$$u_c u_b u_c u_a + u_b u_c u_a u_c - u_c u_b u_a u_c - u_b u_c^2 u_a \qquad (a < b < c). \tag{10}$$

Compactly, these are the relations

$$[u_a u_b] u_a \equiv u_b [u_a u_b], \quad [u_a u_c] u_b \equiv u_b [u_a u_c], \quad [u_b u_c][u_a u_c] \equiv 0.$$

for all $a < b < c$.

We now come to the key theorem about I_C— namely that it is the smallest ideal which allows the noncommutative elementaries to commute.

We will follow A.N Kirillov's proof [K16, Theorem 2.26]. Blasiak and Fomin also have a proof carried out in much higher generality in [BF18].

Theorem 2.8. If $I \supseteq I_C$, then $e_k(\mathbf{u}) e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u}) e_k(\mathbf{u})$ for all j, k.

Proof. First we show that, in I_C, the elementaries commute. Define $E_n(x)$ by

$$E_n(x) = \sum_{k=1}^{n} x^k e_k(\mathbf{u}) = \prod_{i=n}^{1} (1 + x u_i).$$

Then $E_N(x) = E(x)$, and $E_{n+1}(x) = (1 + x u_n) E(x)$. We will prove the statement inductively.

Suppose □

2.3 The map Ψ

Theorem 2.9 (Fundamental theorem of symmetric functions). Let $\Lambda(\mathbf{x})$ denote the ring of symmetric polynomials in the commuting variables $\mathbf{x} = (x_1, \ldots, x_N)$. Then the elementary symmetric functions in $\mathbf{x}$ are algebraically independent, and moreover

$$\Lambda(\mathbf{x}) \simeq \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})].$$

Proof. One can prove this via the *Gale-Ryser* theorem, which is a certain combinatorial result about zero-one matrices which implies that the transition matrix (an integer matrix) from the elementary to monomial basis is upper unitriangular— it is upper triangular with diagonals consisting of only 1's, hence it is invertible, with inverse also an integer matrix. This proof is carried out in Theorem 7.4.4 in [EC2]. □

Corollary 2.10. If I contains I_C, then the map

$$\begin{aligned}\Psi : \Lambda(\mathbf{x}) &\to \mathcal{U} \\ e_k(\mathbf{x}) &\mapsto e_k(\mathbf{u})\end{aligned}$$

induces a ring homomorphism $\Lambda(\mathbf{x}) \to \mathcal{U}/I$.

Proof. Combine Theorems 2.9 and 2.8. □

3 Noncommutative Schur functions

Definition 3.1. Let $I \supseteq I_C$. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u}) \in \mathcal{U}/I$ is defined to be

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{\pi \in S_t} \mathrm{sgn}(\pi) e_{\lambda_1^\top + \pi(1) - 1}(\mathbf{u}) e_{\lambda_2^\top + \pi(2) - 2}(\mathbf{u}) \cdots e_{\lambda_t^\top + \pi(t) - t}(\mathbf{u}),$$

where $t = \lambda_1$ is the number of parts of $\lambda^\top$. Alternatively, since the h's commute whenever the e's do,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{\pi \in S_t} \mathrm{sgn}(\pi) h_{\lambda_1 + \pi(1) - 1}(\mathbf{u}) h_{\lambda_2 + \pi(2) - 2}(\mathbf{u}) \cdots h_{\lambda_t + \pi(t) - t}(\mathbf{u}).$$

The first definition is based on the **Kostka-Naegelsbach identity**

$$s_\lambda(\mathbf{x}) = \det\left(e_{\lambda_i^\top + j - i}(\mathbf{x})\right)_{i,j=1}^n,$$

and the second is based on the **Jacobi-Trudi identity**

$$s_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i + j - i}(\mathbf{x})\right)_{i,j=1}^n.$$

Since these are purely polynomials of elementary symmetric and complete homogeneous polynomials, one sees the following

Definition 3.2. If $I \supseteq I_C$, then

$$\Psi\big(s_\lambda(\mathbf{x})\big) \overset{I}{\equiv} \mathfrak{J}_\lambda(\mathbf{u}).$$

Proof.

$$\begin{aligned}
\Psi\big(s_\lambda(\mathbf{x})\big) &= \Psi\left(\det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n\right) \\
&= \Psi\left(\sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{x})\cdots h_{\pi_n+\pi(n)-n}(\mathbf{x})\right) \\
&\overset{I}{\equiv} \sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{u})\cdots h_{\pi_n+\pi(n)-n}(\mathbf{u}) \\
&= \mathfrak{J}_\lambda(\mathbf{u}).
\end{aligned}$$

□

Theorem 3.3 ([FG98], [BF16])**.** In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{T\in \mathrm{SSYT}(\lambda;N)} \mathbf{u}^{\mathsf{colword}\,T}.$$

3.1 Cauchy kernel

Theorem 3.4. If I contains I_C, then for all $\gamma \in I_C^\perp$,

$$\left\langle \prod_{i=1}^{N}\prod_{j=1}^{N}(1-x_iu_j)^{-1}, \gamma\right\rangle = \sum_\lambda s_\lambda(\mathbf{x})\left\langle \mathfrak{J}_\lambda(\mathbf{u}),\gamma\right\rangle, \tag{11}$$

$$\left\langle \prod_{i=1}^{N}\prod_{j=N}^{1}(1+x_iu_j), \gamma\right\rangle = \sum_\lambda s_\lambda(\mathbf{x})\left\langle \mathfrak{J}_{\lambda^\top}(\mathbf{u}),\gamma\right\rangle. \tag{12}$$

4 The symmetric function F_γ

We will now give the definition of the symmetric function associated to a vector in $I^\perp$, first defined in [FG98].

Definition 4.1. Fix an ideal I containing I_C, and let $\gamma \in \mathcal{U}^*$. We define the **symmetric function γ** to be

$$F_\gamma(\mathbf{x}) := \langle \Omega(\mathbf{x}, \mathbf{u}), \gamma \rangle .$$

Definition 4.2.

5 Applications

5.1 Recovering known results in the plactic algebra

Theorem 5.1 (Littlewood-Richardson rule)**.**

5.2 Stanley symmetric functions via the nilCoxeter algebra

The connection between Schubert polynomials and the nilCoxeter ideal was first explored by Richard Stanley and

Definition 5.2. The **nilCoxeter** ideal

5.3 LLT polynomials via the algebra of Ribbon Schur operators

6 Linear programming

Consider the positive cones $\mathcal{U}_{\geq 0}$ and $\mathcal{U}^*_{\geq 0}$.

7 Algebras of operators

Definition 7.1. A **combinatorial representation** of $\mathcal{U}/I$ is

8 Switchboards

Definition 8.1. Let $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_n \in \mathcal{U}^*$. We define the **fundamental quasisymmetric function** $Q_{\mathrm{Des}(\mathsf{w})}(\mathbf{x})$ by

$$Q_{\mathrm{Des}(\mathsf{w})}(\mathbf{x}) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ j \in \mathrm{Des}(\mathsf{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

In general
$$Q_{\mathrm{Des}(\mathrm{w})}(\mathbf{x}) := \sum_{\substack{i_1 \le \cdots \le i_n \\ j \in \mathrm{Des}(\mathrm{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

9 Appendix

9.1 Formal power series over noncommutative rings

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