# The Vandermonde Determinant

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**Theorem 0.0.1** (Vandermonde determinant)**.** Let $n \in \mathbb{N}$. Then,

$$\det \begin{bmatrix} a_1^0 & a_1^1 & \cdots & a_1^{n-1} & a_1^n \\ a_2^0 & a_2^1 & \cdots & a_2^{n-1} & a_2^n \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1}^0 & a_{n-1}^{n-1} & \cdots & a_{n-1}^{n-1} & a_{n-1}^n \\ a_n^0 & a_n^1 & \cdots & a_n^{n-1} & a_n^n \end{bmatrix} = \prod_{1 \le i < j \le n} (a_i - a_j).$$

where $a_1, \ldots, a_n$ are elements of some commutative ring.

**Definition 0.0.2.** A *tournament* $D$ is a subset of $[n] \times [n]$ such that for all $i < j$, exactly one of $(i, j)$ or $(j, i)$ is in $D$.

We define the *scoreboard* $\operatorname{scb} D$ of a tournament $D$ to be the $n$-tuple $(s_1, \ldots, s_n)$ defined by

$$s_j := (\#i \text{ such that } (i, j) \in D).$$

We say that a tournament is *injective* if all of the entries of its scoreboard are distinct.

We denote the set of all tournaments $T$.

**Definition 0.0.3.** Let $\sigma \in S_n$. Define $P_\sigma \in T$ to be

$$P_\sigma := \{(\sigma(i), \sigma(j)) : 1 \le i < j \le n\}.$$

**Lemma 0.0.4.** A tournament $D \in T$ is injective if and only if it is equal to $P_\sigma$ for some $\sigma \in S_n$.

*Proof.* Let $\sigma \in S_n$. Then consider scb $P_\sigma$. We have that

$$\begin{aligned} s_{\sigma(1)} &= 0 \\ s_{\sigma(2)} &= 1 \\ &\vdots \\ s_{\sigma(n)} &= n-1. \end{aligned}$$

Then $P_\sigma$ is injective.

Suppose $D$ is injective. Then

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_2 \\ &\vdots \\ s_n &= a_n, \end{aligned}$$

where $a_1, \ldots, a_n$ is some labeling of $\{0, \ldots, n-1\}$ □

*Proof of Theorem 0.0.1.* □