Noncommutative Schur functions

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What is this?

This is (going to be) an "infinite napkin"-type set of notes I am taking about the Fomin-Greene theory of noncommutative Schur functions.

Note that this is distinct from the theory of the ring called NCSym, in which there exists structural analogues of monomial, elementary, homogeneous, power, and Schur functions. I am not currently aware of any connection between these two theories.

Contents

1 Ideals and words

Let $\mathbf{u} = (u_1, \ldots, u_N)$ be a collection of noncommuting variables. Let $\langle \mathbf{u} \rangle$ be the free semigroup on the generators $\mathbf{u}$. Then, let $\mathcal{U} = \mathbb{Z}\langle \mathbf{u} \rangle$ denote the corresponding semigroup ring— the free associative ring generated by $\mathbf{u}$.

We will denote by $\mathcal{U}^*$ the $\mathbb{Z}$-module spanned by words in the alphabet $\{1, \ldots, N\}$.

We will have a fundamental pairing $\langle -, - \rangle$ given by making noncommutative monomials dual to words.

Now, if I is an ideal of $\mathcal{U}$, we define $I^\perp$ by

$$I^\perp := \{\gamma \in \mathcal{U}^* \mid \langle I, \gamma \rangle = 0\}.$$

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a collection of commuting variables. We will define $\mathcal{U}[\mathbf{x}] := \mathcal{U} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$, and for all ideals I of $\mathcal{U}$

2 Noncommutative elementary and homogeneous symmetric functions

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) := \underbrace{\sum_{i_1 > i_2 > \cdots > i_k}}_{\text{(decreasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{1}$$

The **noncommutative homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) := \underbrace{\sum_{i_1 \le i_2 \le \cdots \le i_k}}_{\text{(increasing!)}} u_{i_1} u_{i_2} \cdots u_{i_k}. \tag{2}$$

We will have the convention that $e_0(\mathbf{u}) = h_0(\mathbf{u}) = 1$, and that $e_k(\mathbf{u}) = h_k(\mathbf{u}) = 0$ if $k < 0$.

In symmetric function theory, the elementary and homogeneous symmetric functions can be seen as generating functions for semistandard Young tableaux whose shape is a single column and a single row respectively.

The same idea works here— except now one takes the *column word* of a tableau.

2.1 Newton's identities

We define noncommutative analogues of standard generating functions for the elementary and homogeneous symmetric functions.

Definition 2.2. Define the following functions:

$$E(x) := \sum_{k=0}^{N} x^k e_k(\mathbf{u}) = \prod_{i=N}^{1} (1 + xu_i) \tag{3}$$

and

$$H(x) := \sum_{k=0}^{\infty} x^k h_k(\mathbf{u}) = \prod_{i=1}^{N} (1 - xu_i)^{-1}. \tag{4}$$

in the ring $\mathcal{U}[x]$.

An immediate consequence of this definition is a noncommutative analogue of Newton's identities.

Proposition 2.3 (Noncommutative Newton-Girard formulas)**.** We have

$$E(x)H(-x) = H(x)E(-x) = 1. \tag{5}$$

In particular,

$$\sum_{k=0}^{n} (-1)^k e_k(\mathbf{u}) h_{n-k}(\mathbf{u}) = 0$$

and

$$\sum_{k=0}^{n} (-1)^k h_k(\mathbf{u}) e_{n-k}(\mathbf{u}) = 0$$

for all n.

Proof. Putting together (3) and (4) immediately gives us (5)

$$\begin{aligned} E(x)H(-x) &= \left[\prod_{i=N}^{1} (1 + xu_i)\right] \left[\prod_{i=1}^{N} (1 + xu_i)^{-1}\right] \\ &= \left[\prod_{i=N}^{1} (1 + xu_i)\right] \underbrace{\left[\prod_{i=N}^{1} (1 + xu_i)\right]^{-1}}_{\text{note reversal of product order!}} \end{aligned}$$

$$= 1.$$

And $H(x)E(-x) = 1$ is proved exactly the same way. □

Corollary 2.4. Let I be an ideal of $\mathcal{U}$. Then $E(x)E(y) \equiv_{I[x,y]} E(y)E(x)$ if and only if $H(x)H(y) \equiv_{I[x,y]} H(y)H(x)$.

Proof. Suppose $E(x)E(y) = E(y)E(x)$ for all commuting x, y. Then $H(x)H(y) = E(-x)^{-1}E(-y)^{-1} = E(-y)^{-1}E(-x)^{-1} = H(y)H(x)$. The reverse implication is proved identically. □

2.2 When do the elementaries commute?

Lemma 2.5. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $E(x)E(y) \equiv_{I[x,y]} E(y)E(x)$.

(b) $e_k(\mathbf{u})e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

Proof. Expand and compare coefficients. □

Lemma 2.6. Let I be an ideal of $\mathcal{U}$. The following are equivalent:

(a) $e_k(\mathbf{u})e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

(b) $h_k(\mathbf{u})h_j(\mathbf{u}) \equiv_I h_j(\mathbf{u})h_k(\mathbf{u})$ for all j, k.

Proof. Combine Corollary 2.4 and Lemma 2.5. □

Definition 2.7. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2u_a + u_au_bu_a - u_bu_au_b - u_bu_a^2 \qquad (a < b), \tag{6}$$

$$u_bu_cu_a + u_au_cu_b - u_bu_au_c - u_cu_au_b \qquad (a < b < c), \tag{7}$$

$$u_cu_bu_cu_a + u_bu_cu_au_c - u_cu_bu_au_c - u_bu_c^2u_a \qquad (a < b < c). \tag{8}$$

Compactly, these are the relations

$$[u_au_b]u_a \equiv u_b[u_au_b], \quad [u_au_c]u_b \equiv u_b[u_au_c], \quad [u_bu_c][u_au_c] \equiv 0.$$

for all $a < b < c$.

We now come to the key theorem about I_C— namely that it is the smallest ideal which allows the noncommutative elementaries to commute.

We will follow A.N Kirillov's proof [K16, Theorem 2.26]. Blasiak and Fomin also have a proof carried out in much higher generality [BF18].

Theorem 2.8. If $I \supseteq I_C$, then $e_k(\mathbf{u})e_j(\mathbf{u}) \equiv_I e_j(\mathbf{u})e_k(\mathbf{u})$ for all j, k.

Proof. First we show that, in I_C, the elementaries commute. Define $E_n(x) =$

□

2.3 The map Ψ_I

Theorem 2.9 (Fundamental theorem of symmetric functions)**.** Let $\Lambda(\mathbf{x})$ denote the ring of symmetric polynomials in the commuting variables $\mathbf{x} = (x_1, \ldots, x_n)$. Then the elementary symmetric functions in $\mathbf{x}$ are algebraically independent, and moreover

$$\Lambda(\mathbf{x}) \simeq \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})].$$

Proof. See Theorem 7.4.4 in [EC2]. One can prove this via the *Gale-Ryser* theorem. □

Corollary 2.10. If I contains I_C, then the map

$$\begin{aligned}\Psi_I : \Lambda_n(\mathbf{x}) &\to \mathcal{U}/I \\ e_k(\mathbf{x}) &\mapsto e_k(\mathbf{u})\end{aligned}$$

extends to a ring homomorphism.

Proof. Combine Theorems 2.9 and 2.8. □

3 Noncommutative Schur functions

Definition 3.1. Let $I \supseteq I_C$. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u}) \in \mathcal{U}/I$ is defined to be

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{\pi \in S_t} \operatorname{sgn}(\pi) e_{\lambda_1^\top + \pi(1) - 1}(\mathbf{u}) e_{\lambda_2^\top + \pi(2) - 2}(\mathbf{u}) \cdots e_{\lambda_t^\top + \pi(t) - t}(\mathbf{u}),$$

where $t = \lambda_1$ is the number of parts of $\lambda^\top$. Alternatively, since the h's commute whenever the e's do,

$$\mathfrak{J}_\lambda(\mathbf{u}) \coloneqq \sum_{\pi \in S_t} \operatorname{sgn}(\pi) h_{\lambda_1+\pi(1)-1}(\mathbf{u}) h_{\lambda_2+\pi(2)-2}(\mathbf{u}) \cdots h_{\lambda_t+\pi(t)-t}(\mathbf{u}).$$

The first definition is based on the **Kostka-Naegelsbach identity**

$$s_\lambda(\mathbf{x}) = \det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n,$$

and the second is based on the **Jacobi-Trudi identity**

$$s_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i+j-i}(\mathbf{x})\right)_{i,j=1}^n.$$

Since these are purely polynomials of elementary symmetric and complete homogeneous polynomials, one sees the following

Definition 3.2. If $I \supseteq I_C$, then

$$\Psi_I(s_\lambda(\mathbf{x})) \equiv \mathfrak{J}_\lambda(\mathbf{u}) \mod I.$$

Proof.

$$\begin{aligned}
\Psi_I(s_\lambda(\mathbf{x})) &= \Psi_I\left(\det\left(e_{\lambda_i^\top+j-i}(\mathbf{x})\right)_{i,j=1}^n\right) \\
&= \Psi_I\left(\sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{x}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{x})\right) \\
&\equiv \sum_{\pi\in S_n} \operatorname{sgn}(\pi) h_{\pi_1+\pi(1)-1}(\mathbf{u}) \cdots h_{\pi_n+\pi(n)-n}(\mathbf{u}) \mod I \\
&\equiv \mathfrak{J}_\lambda(\mathbf{u}) \mod I.
\end{aligned}$$

□

Theorem 3.3. If I contains I_C, then for all $\gamma \in I_C^\perp$,

$$\left\langle \prod_{i=1}^m \prod_{j=n}^1 (1 + x_i u_j), \gamma \right\rangle = \sum_\lambda s_\lambda(\mathbf{x}) \left\langle \mathfrak{J}_{\lambda^\top}(\mathbf{u}), \gamma \right\rangle.$$

Proof. □

Theorem 3.4 ([FG98], [BF16]). In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{T \in \mathrm{SSYT}(\lambda;N)} \mathbf{u}^{\mathsf{colword}\,T}.$$

3.1 Cauchy kernel

Definition 3.5. Let $\mathbf{x} = (x_1, x_2 \ldots)$ be a countable collection of commuting variables.

4 Applications

4.1 Recovering known results in the plactic algebra

Theorem 4.1 (Littlewood-Richardson rule).

4.2 Stanley symmetric functions via the nilCoxeter algebra

The connection between Schubert polynomials and the nilCoxeter ideal was first explored by Richard Stanley and

Definition 4.2. The **nilCoxeter** ideal

4.3 LLT polynomials via the algebra of Ribbon Schur operators

5 Linear programming

Consider the positive cones $\mathcal{U}_{\geq 0}$ and $\mathcal{U}^*_{\geq 0}$.

6 Algebras of operators

Definition 6.1. A **combinatorial representation** of $\mathcal{U}/I$ is

7 Switchboards

8 Appendix

8.1 Formal power series

Corollary 8.1.

Proposition 8.2.

8.2 Gessel's fundamental quasisymmetric function

Definition 8.3. Let $\mathsf{w} \in \mathcal{U}^*$ be a word. We define the **fundamental quasisymmetric function** $Q_{\mathrm{Des}(\mathsf{w})}$ by

$$Q_{\mathrm{Des}(\mathsf{w})} := \sum_{\substack{1 \le i_1 \le \cdots \le i_n \\ j \in \mathrm{Des}(\mathsf{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

8.3 The Edelman-Greene correspondence

References

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