A neat infinite sum identity

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1 Introduction

This identity I'm writing about is entirely routine. But, I do think it's a cute starting point for the combinatorial topics it touches on.

We wish to compute, say, the infinite series

$$\sum_{n=0}^{\infty} \frac{4n^3 + 2n^2 - 8n - 23}{2^n}.$$

Alright, whatever, seems a bit tricky.

Consider the numerator, a polynomial in n, and write down its values for $n = 1, 2, 3, \ldots$ in a row.

$$-23 \quad -25 \quad 1 \quad 79 \quad 233 \quad 487 \quad 865 \quad \ldots$$

Still doesn't look so nice. But now, write a row below it, whose numbers are the difference of the number on its top right and its top left.

$$\begin{array}{ccccccccccc}
-23 & & -25 & & 1 & & 79 & & 233 & & \cdots \\
& -2 & & 26 & & 78 & & 154 & & \cdots & \\
& & 28 & & 52 & & 76 & & \cdots & & \\
& & & 24 & & 24 & & \cdots & & & \\
& & & & 0 & & \cdots & & & & \\
& & & & & \ddots & & & & &
\end{array}$$

If you take the first column of numbers, add them up, and multiply them by two, this turns out to be the answer: it happens to be that

$$\sum_{n=0}^{\infty} \frac{4n^3 + 2n^2 - 8n - 23}{2^n} = 2(-23 - 2 + 28 + 24) = 54.$$

Moreover, this "always happens" to be, which we will now show.

2 Finite differences and computing polynomials

But first, we define a discrete analogue of the derivative, the *difference operator*, or the *first difference operator*.

Definition 2.1. The *difference operator* Δ is defined by the equation

$$(\Delta f)(x) := f(x+1) - f(x).$$

We define a related operator which is interesting in its own right, but whose purpose right now is to one specific calculation easier.

Definition 2.2. The *shift operator* T is defined by

$$(Tf)(x) := f(x+1).$$

Then, we can write $T = \Delta + \mathbf{1}$, where $\mathbf{1}$ is the identity operator. This *immediately* gives us the following result.

Theorem 2.3. Let f be some function. We have that

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0).$$

Proof.

$$\begin{aligned}
f(n) &= f(0+n) \\
&= T^n f(0) \\
&= (\Delta + \mathbf{1})^n f(0) \\
&= \left[\sum_{k=0}^{n} \binom{n}{k} \Delta^k \mathbf{1}^{n-k} \right] f(0) \\
&= \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0).
\end{aligned}$$

□

One remark is in order: just as differentiating enough times kills off polynomials, taking enough finite differences does the same thing as well.

Remark 2.4. If $p(x)$ is a polynomial, then for all $m > \deg p$,

$$(\Delta^m p) \equiv 0.$$

Proof. Left to the reader. □

This gives us an easy corollary to Theorem 2.3.

Corollary 2.5. If $p(x)$ is a polynomial in x, we have that

$$p(n) = \sum_{k=0}^{\deg p} \binom{n}{k} (\Delta^k p)(0).$$

Proof. Left to the reader. □

This result is one of the easier pieces of proving the identity, which we can now finally precisely state using the difference operator.

3 The identity

Constructing the triangular array we constructed earlier.

Now instead of numbers, we can write it with notation.

$$\begin{array}{ccccccccc}
f(0) & & f(1) & & f(2) & & f(3) & & \cdots \\
& \Delta f(0) & & \Delta f(1) & & \Delta f(2) & & \cdots & \\
& & \Delta^2 f(0) & & \Delta^2 f(1) & & \cdots & & \\
& & & \Delta^3 f(0) & & \cdots & & & \\
& & & & \ddots & & & &
\end{array}$$

Now, we can see that "taking the first column" accounts to looking at $(\Delta^k f)(0)$ for all k.

We've looked at $(\Delta^k f)(0)$ for a bit now— in Theorem 2.3, in Corollary 2.5, and now in the identity we want to prove. This is another "finite calculus" analogy— Theorem 2.3 is an analogue of *Taylor series expansion*. Instead of expanding f as a sum of *derivatives* $f^{(k)}(0)$, we expand f as a sum of *differences* $(\Delta^k f)(0)$.

In its more general form, the identity looks very similar to Corollary 2.5. Without further ado, here it is.

Theorem 3.1 (The theorem, more generally)**.** Let p be some polynomial. Fix a number $a > 1$. Then,

$$\sum_{n=0}^{\infty} \frac{p(n)}{a^n} = \sum_{k=0}^{\deg p} \frac{a}{(a-1)^{k+1}} (\Delta^k p)(0).$$

We can't quite prove this yet, though.

For the meantime, we note that something *awesome* happens in the $a = 2$ case, which was demonstrated in the introduction.

Corollary 3.2 (The theorem, less generally)**.** Let p be some polynomial. Then,

$$\sum_{n=0}^{\infty} \frac{p(n)}{2^n} = 2 \sum_{k=0}^{\deg p} (\Delta^k p)(0).$$

Proof. Left to the reader. ◻

Here's a quick example:

Example 3.3. Consider the sum

$$\sum_{n=0}^{\infty} \frac{n}{2^n},$$

so now $p(n) = n$, and $a = 2$.

The finite differences are $(\Delta^0 p)(0) = 0$, $(\Delta^1 p)(0) = 1$, so

$$\sum_{n=0}^{\infty} \frac{n}{2^n} = 2(0+1)$$

4 Falling factorials and Newton's binomial formula

Next, we define an operation that is *like* taking powers, just like how differences are *like* taking derivatives.

Definition 4.1. Let $m \geq 0$ be a number. The *falling factorial*, $x^{\underline{m}}$, is defined by

$$x^{\underline{m}} := \underbrace{x \cdot (x-1) \cdots (x-m+1)}_{m \text{ factors}}.$$

Note that the ordinary factorial $n!$ is $n^{\underline{n}}$. On the flip side, if $m < n$ are two integers, then $n^{\underline{m}} = n!/(n-m)!$. With that said, we can see then that $\binom{n}{k} = n^{\underline{k}}/k!$, when n and k are positive integers.

Even better, falling factorials allow us to give a *more general definition* of the binomial coefficient, in which the upper index is no longer required to be a nonnegative integer.

Definition 4.2. Let $k \in \mathbb{N}$, and let n be *any* number. The *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n^{\underline{k}}}{k!}.$$

Now we have identities which we couldn't have *dreamed of* without a more general binomial coefficient, for example:

Lemma 4.3 (Upper negation)**.** Let $k \in \mathbb{N}$ and let n be any number again. Then

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Proof. We start by expanding the left hand side, which is

$$\binom{-n}{k} = \frac{(-n)^{\underline{k}}}{k!}.$$

Then, we manipulate the product $(-n)^{\underline{k}}$ with our bare hands:

$$\begin{aligned}
(-n)^{\underline{k}} &= \Big((-n)\Big)\Big((-n)-1\Big)\Big((-n)-2\Big)\cdots\Big((-n)-k+1\Big) \\
&= \Big((-1)(n)\Big)\Big((-1)(n+1)\Big)\Big((-1)(n+2)\Big)\cdots\Big((-1)(n+k-1)\Big) \\
&= (-1)^k\Big(n\Big)\Big(n+1\Big)\Big(n+2\Big)\cdots\Big(n+k-1\Big) \\
&= (-1)^k (n+k-1)^{\underline{k}}.
\end{aligned}$$

Then,

$$\binom{-n}{k} = \frac{(-n)^{\underline{k}}}{k!} = \frac{(-1)^k (n+k-1)^{\underline{k}}}{k!} = (-1)^k \binom{n+k-1}{k}.$$

□

What this tells us is that negating the upper index can be "straightened out" this way into an expression *without* a negative upper index.

As another application of our new binomial coefficient, we have a powerful and important generalization of the *binomial formula*, which involves a *series* rather than a sum.

Theorem 4.4 (Newton's binomial formula)**.** For any $a \in \mathbb{R}$,

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

Proof. Doing this rigorously takes *forever*, so I refer to [GrinbergAC], Theorem 3.8.3.

□

However, in proving our identity, we'll want the above sum to run over the *upper index* of the binomial coefficient. Luckily, we *do* have a version that runs over the upper index.

Theorem 4.5. Fix $k \in \mathbb{N}$. Then

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$

Proof. We begin with Newton's binomial formula

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k,$$

And we, superficially for now, replace k with n, so we have

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n.$$

Now k is back in our pool of free variables, so put $a = -k-1$. Then,

$$\frac{1}{(1+x)^{k+1}} = \sum_{n=0}^{\infty} \binom{-(k+1)}{n} x^n.$$

Now we hit it with upper negation,

$$\binom{-(k+1)}{n} = (-1)^n \binom{(k+1)+n-1}{n} = (-1)^n \binom{n+k}{n},$$

so now we finally have a n in the top index,

$$\frac{1}{(1+x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n} (-1)^n x^n.$$

To get rid of the n in the bottom index, we use binomial coefficient symmetry,

$$\binom{n+k}{n} = \binom{n+k}{(n+k)-n} = \binom{n+k}{k},$$

and now we're almost done, since we have

$$\frac{1}{(1+x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^n x^n.$$

To cancel out the $(-1)^n$, we introduce another $(-1)^n$ by substituting $-x$ for x, so

$$\begin{aligned}\frac{1}{(1-x)^{k+1}} &= \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^n (-x)^n \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^n (-1)^n x^n \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^{2n} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} x^n.\end{aligned}$$

And finally, we shift by a x^k term, so that we can do a re-indexing of the sum,

$$\begin{aligned}\frac{x^k}{(1+x)^{k+1}} &= x^k \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} x^{n+k} \\ &= \sum_{n=k}^{\infty} \binom{n}{k} x^n.\end{aligned}$$

$\square$

Now we basically have the *other* piece we need to prove our identity.

Corollary 4.6. Fix k. We have that

$$\sum_{n=0}^{\infty} \binom{n}{k} \frac{1}{a^n} = \frac{a}{(a-1)^{k+1}}.$$

Proof. Left to the reader. $\square$

5 Proof of the identity

Now we have all the tools to prove this!

Proof of Theorem 3.1. By Theorem 2.3, we have that

$$p(n) = \sum_{k=0}^{\deg p} \binom{n}{k} (\Delta^k p)(0)$$

Next, we grind it out a little.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{p(n)}{a^n} &= \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{\deg p} \binom{n}{k} (\Delta^k p)(0)}{a^n} && \text{By Corollary 2.5} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\deg p} \frac{\binom{n}{k} (\Delta^k p)(0)}{a^n} && \text{Push the } a^{-n} \text{ inside the sum} \\
&= \sum_{k=0}^{\deg p} \sum_{n=0}^{\infty} \frac{\binom{n}{k} (\Delta^k p)(0)}{a^n} && \text{Interchange sums} \\
&= \sum_{k=0}^{\deg p} (\Delta^k p)(0) \sum_{n=0}^{\infty} \frac{\binom{n}{k}}{a^n} && \text{Pull out } (\Delta^k) p(0) \text{ from sum running over } n \\
&= \sum_{k=0}^{\deg p} (\Delta^k p)(0) \frac{a}{(a-1)^{k+1}} && \text{By Corollary 4.6} \\
&= \sum_{k=0}^{\deg p} \frac{a}{(a-1)^{k+1}} (\Delta^k p)(0).
\end{aligned}$$

□

References

[GrinbergAC] Darij Grinberg, *An Introduction to Algebraic Combinatorics*, http://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf