Lie algebras

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What is this?

These are notes based on my reading of Humphreys's "Introduction to Lie Algebras and Representation Theory".

Contents

1 Definitions

Convention 1.0.1. All vector spaces are finite dimensional and no assumptions are made about the field they are over.

1.1 Lie algebras

Definition 1.1.1. A **Lie algebra** $\mathfrak{g}$ is a vector space equipped with a product

$$\begin{aligned} [\_,\_] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}, \\ (x, y) &\mapsto [xy], \end{aligned}$$

such that

(L1) $[\_,\_]$ is bilinear,

(L2) $[xx] = 0$ for all $x \in \mathfrak{g}$, and

(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0$.

We refer to $[xy]$ as the **bracket** or the **commutator** of x and y.

(L3) is referred to as the *Jacobi identity*.

As an exercise in using this definition, we show the following:

Proposition 1.1.2. Brackets are anticommutative, i.e

$$[xy] = -[yx]. \tag{L2'}$$

is a relation in any Lie algebra.

Proof. By (L2), we have that

$$[x+y, x+y] = 0,$$

and by (L1),

$$[xx] + [xy] + [yx] + [yy] = 0.$$

By (L2) again,

$$[xy] + [yx] = 0,$$

which completes the proof. □

We will look at our first example of a Lie algebra, which is closely related to the $\mathrm{GL}(V)$.

> **Definition 1.1.3** ($\mathfrak{gl}$, abstractly). Let V be a vector space. The **general linear algebra** $\mathfrak{gl}(V)$ is defined to be the Lie algebra with underlying vector space $\operatorname{End} V$ and bracket given by
> $$[xy] = xy - yx$$
> defined with $\operatorname{End} V$ natural ring structure.

Put in a more concrete sense, $\operatorname{End} V$'s aforementioned ring structure is exactly that of $n \times n$ matrices, where $n = \dim V$. Then, the following definition makes sense, and is in a sense "the only" finite dimensional general linear algebra.

> **Definition 1.1.4** ($\mathfrak{gl}$, concretely). Let $\mathbb{F}$ be some field and let n be a positive integer. The **general linear algebra** $\mathfrak{gl}_n(\mathbb{F})$ is defined to be the Lie algebra with underlying vector space the set of all $n \times n$ matrices over $\mathbb{F}$, with bracket given by
> $$[xy] = xy - yx.$$

We can ea

> **Proposition 1.1.5.** Let $\{e_{pq}\}_{p,q=0}^{n}$ be the standard basis of $\mathfrak{gl}_n(\mathbb{F})$. Then
> $$[e_{pq}e_{rs}] = \delta_{qr}e_{ps} - \delta_{sp}e_{rq},$$
> where δ is the Kronecker delta.

Proof. Using the Iverson bracket,

$$(e_{pq})_{ij} = [p = i \wedge q = j]^?$$

and so

$$\begin{aligned}
(e_{pq}e_{rs})_{ij} &= \sum_{k=1}^{n} (e_{pq})_{ik}(e_{rs})_{kj} \\
&= \sum_{k=1}^{n} [p = i \wedge q = k]^? \, [r = k \wedge s = j]^? \\
&= \left(\sum_{k=1}^{n} [q = r = k]^?\right) [p = i \wedge s = j]^?
\end{aligned}$$

$$= \delta_{qr}(e_{ps})_{ij}.$$

So $e_{pq}e_{rs} = \delta_{qr}e_{ps}$. Similarly, $e_{rs}e_{pq} = \delta_{sp}e_{rq}$. □

Importantly, many Lie algebras, and in fact all the Lie algebras we are concerned with, occur as subalgebras of the general linear algebra— a **subalgebra** of a Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ that is closed under the bracket.

Definition 1.1.6. A **linear Lie algebra** is a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$ for some n.

1.2 Derivations, the adjoint representation

Definition 1.2.1. Let $\mathfrak{U}$ be a $\mathbb{F}$-algebra. A **derivation** of $\mathfrak{U}$ is a linear map $\mathrm{d} : \mathfrak{U} \to \mathfrak{U}$ which satisfies the *Leibniz rule*

$$\mathrm{d}(ab) = a(\mathrm{d}b) + (\mathrm{d}a)b.$$

The collection of all derivations of $\mathfrak{U}$ is denoted $\operatorname{Der}\mathfrak{U}$.

Definition 1.2.2. The **adjoint representation** of a Lie algebra $\mathfrak{g}$ is the mapping

$$x \mapsto \mathrm{ad}_x$$

where $\mathrm{ad}_x \in \operatorname{Der}\mathfrak{g}$ is defined to be

$$\begin{aligned} \mathrm{ad}_x : \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto [x, y]. \end{aligned}$$

Proposition 1.2.3. ad_x is a derivation.

Proof. We start with the Jacobi identity (L3)

$$[x\,[yz]] + [y\,[zx]] + [z\,[xy]] = 0,$$

which, using the anticommutation relations $[y\,[zx]] = -[y\,[xz]]$ and $[z\,[xy]] = -[[xy]\,z]$, is equivalent to

$$[x\,[yz]] = [y\,[xz]] + [[xy]\,z].$$

But this is saying that

$$\mathrm{ad}_x\,[yz] = [y, \mathrm{ad}_x\,z] + [\mathrm{ad}_x\,y, z]$$

which is exactly the defining identity for derivations. □

1.3 Examples

We have four distinguished families of Lie algebras:

$$\mathsf{A}_\ell, \qquad \mathsf{B}_\ell, \qquad \mathsf{C}_\ell, \qquad \mathsf{D}_\ell.$$

These classify all but five of the so-called **semisimple Lie algebras**.

1.3.1 Type A

Definition 1.3.1. Let V be a $\mathbb{F}$-vector space, and fix a basis $\{v_1, \ldots, v_n\}$ of V. The **trace** of an endomorphism $x \in \operatorname{End} V$ of V is defined to be the sum

$$\sum_{i=1}^{n} \langle v_i, x(v_i) \rangle$$

where $\langle -, - \rangle$ is the canonical pairing.

In other words, it is the sum of the diagonal entries of the matrix representation of x.

We say that x is **traceless** if $\operatorname{tr} x = 0$.

Definition 1.3.2. The **special linear algebra** $\mathfrak{sl}(V)$ is defined to be the set of all traceless endomorphisms of V.

Proposition 1.3.3. $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$.

Proof. The trace is a linear operator $\operatorname{tr} : \mathfrak{gl}(n, \mathbb{F}) \to \mathbb{F}$. Since the kernel of a linear operator is a vector subspace, we conclude that $\mathfrak{sl}(n, \mathbb{F})$ is a vector subspace of $\mathfrak{gl}$.

Finally, the fact that $\operatorname{tr}(xy - yx) = \operatorname{tr}(xy) - \operatorname{tr}(yx) = 0$ for *all* $x, y \in \mathfrak{gl}(n, \mathbb{F})$ means that $\mathfrak{gl}(n, \mathbb{F})$'s Lie bracket is closed in $\mathfrak{sl}(n, \mathbb{F})$. □

1.3.2 Type B

Definition 1.3.4. The **orthogonal algebra** $\mathfrak{o}(2n + 1, \mathbb{F})$ is defined to be

1.3.3 Type C

Definition 1.3.5. Let $\dim V = 2\ell$. We define a skew-symmetric bilinear form f on V via the matrix

$$s := \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

Namely,

$$f(u,v) := u^T s v.$$

The **symplectic algebra** $\mathfrak{sp}(V)$ is defined to be the set of all $x \in \operatorname{End} V$ such that

$$f\Big(x(u), v\Big) = -f\Big(u, x(v)\Big).$$

1.3.4 Type D

Definition 1.3.6. The **orthogonal algebra** $\mathfrak{o}(2n, \mathbb{F})$ is defined to be

1.4 Abstract Lie algebras

2 Ideals and homomorphisms

Definition 2.0.1. A subspace I of a Lie algebra L is called an **ideal** of L if $[xy] \in I$ for all $x \in L$ and $y \in I$.

Definition 2.0.2. The **quotient of a Lie algebra** L by an ideal I, denoted L/I, is defined to be the quotient of L as a vector space by I as a subspace, equipped with the product

$$[x + I, y + I] := [xy] + I.$$

Proposition 2.0.3. L/I is a Lie algebra.

Proof. These are all easy to check.

$$\begin{aligned}
&[ax + by + I, z + I] = \\
&\Big([ax + by, z]\Big) + I = \Big(a[x,z] + b[y,z]\Big) + I \\
&\qquad = \Big(a[x,z] + I\Big) + \Big(b[y,z] + I\Big) \\
&\qquad = a[x + I, z + I] + b[y + I, z + I].
\end{aligned}$$

$$[x + I, x + I] = [xx] + I = 0 + I$$

□

2.1 Homomorphisms

There is a natural definition of a Lie algebra homomorphism— it's a map that respects brackets.

Definition 2.1.1. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. We say that a map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a **Lie algebra homomorphism** if it is a linear map for which

$$\phi\Big([xy] \Big) = [\phi(x)\phi(y)]$$

for all $x, y \in \mathfrak{g}$. A **Lie algebra isomorphism** is a Lie algebra morphism that is also an isomorphism of vector spaces.

Definition 2.1.2. A **representation** of a Lie algebra $\mathfrak{g}$ is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$.

Theorem 2.1.3 (Lie algebra isomorphism theorems)**.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algerbas.

(a) If $\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism, then $\mathfrak{g}/\ker\phi \simeq \operatorname{im}\phi$. If $\mathfrak{i} \subseteq \ker\phi$ is an ideal of $\mathfrak{g}$, there exists a unique homomorphism $\overline{\phi} : \mathfrak{g}/\mathfrak{i} \to \mathfrak{h}$ that makes the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathfrak{h} \\ \mathfrak{g}/\mathfrak{i} \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

(b) if $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of $\mathfrak{g}$ such that $\mathfrak{i} \subseteq \mathfrak{j}$, then $\mathfrak{j}/\mathfrak{i}$ is an ideal of $\mathfrak{g}/\mathfrak{i}$ and there is a natural isomorphism

$$(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \simeq \mathfrak{g}/\mathfrak{j}.$$

(c) if $\mathfrak{i}, \mathfrak{j}$ are ideals of $\mathfrak{g}$, there is a natural isomorphism

$$(\mathfrak{i} + \mathfrak{j})(\mathfrak{j}) \simeq \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j}).$$

Proof. (a) The map

$$\begin{aligned} \overline{\phi} : \mathfrak{g}/\ker\phi &\to \operatorname{im}\phi \\ x + \ker\phi &\mapsto \phi(x) \end{aligned}$$

is the desired isomorphism $\mathfrak{g}/\ker\phi \simeq \operatorname{im}\phi$. We verify that it is well defined: let $x+\ker\phi = x'+\ker\phi$. Then there exists $k, k' \in \ker\phi$ such that $x+k = x'+k'$, and we have that

$$\phi(x) = \phi(x+k) = \phi(x+k') = \phi(x'),$$

so $\overline{\phi}$ is a well-defined function on the cosets in $\mathfrak{g}/\ker\phi$.

Next, we check that it respects brackets:

$$\begin{aligned}
\overline{\phi}\Big(\big[x+\ker\phi, y+\ker\phi\big] \Big) &= \overline{\phi}\Big(\big[xy\big] + \ker\phi\Big) \\
&= \phi\Big(\big[xy\big] \Big) \\
&= \big[\phi(x)\phi(y)\big] \\
&= \Big[\overline{\phi}\Big(x+\ker\phi\Big), \overline{\phi}\Big(y+\ker\phi\Big)\Big].
\end{aligned}$$

Then, it is a homomorphism. To show that it is an isomorphism, we note that it has a trival kernel, trivially:

$$\ker\overline{\phi} = \{x+\ker\phi : x+\ker\phi = \ker\phi\} = \{0+\ker\phi\}.$$

□

Theorem 2.1.4. The adjoint representation $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of $\mathfrak{g}$.

Proof. ad is evidently linear. Next, we just check that it is a homomorphism:

$$\begin{aligned}
\big[\operatorname{ad}_x \operatorname{ad}_y\big](z) &= \Big(\operatorname{ad}_x \operatorname{ad}_y - \operatorname{ad}_y \operatorname{ad}_x\Big)(z) \\
&= \Big(\operatorname{ad}_x \operatorname{ad}_y\Big)(z) - \Big(\operatorname{ad}_y \operatorname{ad}_x\Big)(z) \\
&= \operatorname{ad}_x \big[yz\big] - \operatorname{ad}_y [xz] \\
&= \big[x\big[yz\big]\big] - \big[y[xz]\big] \\
&= \big[x\big[yz\big]\big] + \big[y[zx]\big] \\
&= \big[\big[xy\big]z\big] \\
&= \operatorname{ad}_{[xy]}(z).
\end{aligned}$$

□

Corollary 2.1.5. Any simple Lie algebra is isomorphic to a linear Lie algebra.

Proof. Let $\mathfrak{g}$ be a Lie algebra. We have that

$$\ker \operatorname{ad} = \left\{x \in \mathfrak{g} : \operatorname{ad}_x = 0\right\} = \left\{x \in \mathfrak{g} : [xy] = 0 \text{ for all } y \in \mathfrak{g}\right\} = Z(\mathfrak{g}).$$

Hence, if $\mathfrak{g}$ is simple, i.e if $Z(\mathfrak{g}) = 0$, then ad has a trivial kernel, so it is an isomorphism. □

3 Automorphisms

Definition 3.0.1. A **automorphism** of a Lie algebra $\mathfrak{g}$ is an isomorphism $\mathfrak{g} \to \mathfrak{g}$.

Proposition 3.0.2. Let V be a vector space and let $g \in \mathrm{GL}(V)$ be an invertible element of End V. Then the map

$$x \mapsto gxg^{-1}$$

is an automorphism of $\mathfrak{gl}(V)$.

Proof. The aforementioned map is a vector space isomorphism, with explicit inverse

$$x \mapsto g^{-1}xg$$

and it is a homomorphism, as

$$\begin{aligned} g\,[xy]\,g^{-1} &= g\left(xy - yx\right)g^{-1} \\ &= \left(gxyg^{-1}\right) - \left(gyxg^{-1}\right) \\ &= \left(gxg^{-1}gyg^{-1}\right) - \left(gyg^{-1}gxg^{-1}\right) \\ &= \left[gxg^{-1}, gyg^{-1}\right]. \end{aligned}$$

□

4 Solvable and nilpotent Lie algebras

4.1 The derived series, solvability

Definition 4.1.1. The **derived series** of a Lie algebra $\mathfrak{g}$ is a sequence of ideals $\mathfrak{g}^{(0)}, \mathfrak{g}^{(1)}, \ldots$ defined

$$\begin{cases} \mathfrak{g}^{(0)} := \mathfrak{g} \\ \mathfrak{g}^{(i)} := \left[\mathfrak{g}^{(i-1)}\mathfrak{g}^{(i-1)}\right] \end{cases}.$$

In other words, $\mathfrak{g}^{(i)}$ is all those elements of $\mathfrak{g}$ which can be written as linear combinations of i "binary trees" of brackets in $\mathfrak{g}$.

Definition 4.1.2. A Lie algebra $\mathfrak{g}$ is said to be **solvable** if $\mathfrak{g}^{(n)} = 0$ for some n.

For example, abelian Lie algebras are solvable, whereas simple Lie algebras are never solvable.

Proposition 4.1.3. The Lie algebra of upper triangular matrices $\mathfrak{t}_n(\mathbb{F})$ is solvable.

Proof. □

4.2 The descending central series, nilpotency

Definition 4.2.1. The **descending central series** of a Lie algebra $\mathfrak{g}$ is a sequence of ideals $\mathfrak{g}^0, \mathfrak{g}^1, \ldots$ defined

$$\begin{cases} \mathfrak{g}^0 := \mathfrak{g} \\ \mathfrak{g}^i := \left[\mathfrak{g}\mathfrak{g}^{i-1}\right] \end{cases}.$$

Definition 4.2.2. A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $\mathfrak{g}^n = 0$ for some n.

Proposition 4.2.3. All nilpotent Lie algebras are solvable.

Definition 4.2.4. Let $\mathfrak{g}$ be a Lie algebra. We say that $x \in \mathfrak{g}$ is **ad-nilpotent** if $(\mathrm{ad}_x)^n = 0$ for some n.

4.3 Engel's theorem

We will prove **Engel's theorem**

Theorem 4.3.1 (Engel)**.** Let $\mathfrak{g}$ be a Lie algebra. Then the following are equivalent:

(i) $\mathfrak{g}$ is nilpotent,

(ii) All the elments of $\mathfrak{g}$ are ad-nilpotent.

Proof of Engel's theorem. □

5 Solutions to exercises

Exercise 5.1 (Humphreys 1.1)**.** Verify that $\mathbb{R}^3$ with the bracket given by the *cross product*

$$[xy] := x \times y$$

is a Lie algebra, and write down its structure constants relative to the usual basis of $\mathbb{R}^3$.

Let

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

The cross product is defined

$$x \times y = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Then we directly verify the Lie algebra axioms.

For (L1),

$$\begin{aligned} \left(ax + by\right) \times z &= \begin{pmatrix} (ax_2 + by_2)z_3 - (ax_3 + by_3)z_2 \\ (ax_3 + by_3)z_1 - (ax_1 + by_1)z_3 \\ (ax_1 + by_1)z_2 - (ax_2 + by_2)z_1 \end{pmatrix} \\ &= \begin{pmatrix} \left(ax_2z_3 + by_2z_3\right) - \left(ax_3z_2 + by_3z_2\right) \\ \left(ax_3z_1 + by_3z_1\right) - \left(ax_1z_3 + by_1z_3\right) \\ \left(ax_1z_2 + by_1z_2\right) - \left(ax_2z_1 + by_2z_1\right) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} a\left(x_2z_3 - x_3z_2\right) + b\left(y_2z_3 + y_3z_2\right) \\ a\left(x_3z_1 - x_1z_3\right) + b\left(y_3z_1 + y_1z_3\right) \\ a\left(x_1z_2 - x_2z_1\right) + b\left(y_1z_2 + y_2z_1\right) \end{pmatrix} \\
&= a(x \times z) + b(y \times z).
\end{aligned}
$$

And, via an almost identical calculation,

$$
x \times \left(a y \times bz\right) = a(x \times y) + b(x \times z).
$$

Next, we verify (L2)

$$
x \times x = \begin{pmatrix} x_2x_3 - x_3x_2 \\ x_3x_1 - x_1x_3 \\ x_1x_2 - x_2x_1 \end{pmatrix} = 0.
$$

And finally, we verify the Jacobi identity (L3)

$$
x \times x = \begin{pmatrix} x_2x_3 - x_3x_2 \\ x_3x_1 - x_1x_3 \\ x_1x_2 - x_2x_1 \end{pmatrix} = 0.
$$