Mechanics

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These are notes written as I self-studied Mechanics by Landau and Lifshitz.
I'm a math student, but I didn't bother to "upgrade" anything to be more rigorous.

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1 Equations of Motion

1.1 Generalized coordinates

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Definition 1.2.1. The path taken in going from $\mathbf{q}(t_1)$ to $\mathbf{q}(t_2)$ must satisfy

$$\delta S = \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) \mathrm{d}x = 0.$$

This is the *principle of least action*, or *Hamilton's principle*.

Theorem 1.2.2. Let $\mathbf{q}(t) = \big(q_1(t), \ldots, q_s(t)\big)$. Then equations $\mathbf{q}$ must satisfy so that $\delta S = 0$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad 1 \leq i \leq s.$$

This is called the *Euler-Lagrange equation*.

Proof. We note that

$$L(q+\delta q, \dot{q}+\delta\dot{q}, t) \approx L(q,\dot{q},t) + \underbrace{\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}}_{\text{First order terms}}.$$

Then

$$\begin{aligned}
\delta S &= \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t)\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\delta\dot{\mathbf{q}} \right)\mathrm{d}t \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q}\mathrm{d}t + \underbrace{\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{\mathbf{q}}}\delta\dot{\mathbf{q}}\mathrm{d}t}_{\text{integrate by parts}} \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q}\mathrm{d}t + \left(\underbrace{\left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\delta\mathbf{q}\right]_{t_1}^{t_2}}_{=\delta\mathbf{q}(t_2)-\delta\mathbf{q}(t_1)=0} - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\delta\mathbf{q}\mathrm{d}t \right) \\
&= \int_{t_1}^{t_2} \frac{\partial L}{\partial \mathbf{q}}\delta\mathbf{q}\mathrm{d}t - \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\delta\mathbf{q}\mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\delta\mathbf{q}\mathrm{d}t,
\end{aligned}$$

and this integral must be zero for *any* variation δq effected. So it must be that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0.$$

□

1.3 Galilean transformations

Definition 1.3.1. The *Galilean group* is the group of transformations of $\mathbb{R}^3 \times \mathbb{R}$ generated by the following families of transformations:

(a) Space translations:

$$(\mathbf{x}, t) \mapsto (\mathbf{x} + \mathbf{a}, t), \quad \mathbf{a} \in \mathbb{R}^3$$

(b) Time translations:

$$(\mathbf{x}, t) \mapsto (\mathbf{x}, t + s), \quad s \in \mathbb{R}$$

(c) Uniform motion:

$$(\mathbf{x}, t) \mapsto (\mathbf{x} + t\mathbf{v}, t), \quad \mathbf{v} \in \mathbb{R}^3$$

(d) Rotations:

$$(\mathbf{x}, t) \mapsto (R\mathbf{x}, t), \quad R \in \mathrm{SO}(3)$$