Ordinary differential equations

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What is this?

These are notes I am taking for the class MATH-623, *Ordinary Differential Equations*, at Drexel University, taught by Yixin Guo.

Some notation is changed from her notes, and I try to add as many missing details from proofs as possible.

Contents

1 Basic theory

1.1 Definitions

Definition 1.1.1. Let $J \subseteq \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $\Lambda \subseteq \mathbb{R}^k$ be open sets, and let $\mathbf{f} : J \times U \times \Lambda \to \mathbb{R}^n$ is a smooth function. An **ordinary differential equation** is an equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}) \tag{ODE}$$

where the dot denotes differentiation with respect to the independent variable t.

Morally, the individual parts of an ODE have the following meaning:

$t \in J$: an independent variable, typically time,

$\mathbf{x} \in U$: a dependent variable,

$\underline{\lambda} \in \Lambda$: a vector of parameters,

$\mathbf{f}$: a continuously differentiable function that encodes the (time) evolution of $\mathbf{x}$.

Definition 1.1.2. A **solution** of (ODE) is a function $\mathbf{F} : J_0 \to U$, where $J_0 \subseteq J \subseteq R$, such that

$$\frac{d}{dt}\mathbf{F}(t) = \mathbf{f}(t, \mathbf{F}(t), \underline{\lambda}), \qquad \forall t \in J_0$$

i.e, a function for which we can put $\mathbf{x} = \mathbf{F}(t)$ in (ODE) for all $t \in J_0$.

The **orbit** of the solution $\mathbf{F}$ is the set

$$\{\mathbf{F}(t) \in U : t \in J_0\} \subseteq \mathbb{R}^n.$$

This is also called the **trajectory**, **integral curve**, or **solution curve**.

Evidently, an ODE can have many solutions. For example, if $J = U = \Lambda = \mathbb{R}$ and $f(t, x, \lambda) = \lambda x$, then it's well known that $F(t) = Ce^{\lambda t}$ is a solution to this ODE for all $C \in \mathbb{R}$.

Fortunately, both real-life and abstract experience tells us that in many cases, the following is a natural way to restrict solutions to an ODE.

Definition 1.1.3. An **initial-value problem** (IVP) is the system of equations

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}), \\ \mathbf{x}(t_0) = \mathbf{x}_0. \end{cases} \tag{IVP}$$

Namely, it is an (ODE) with additional data $(t_0, \mathbf{x}_0) \in J \times U$, encoding an **initial value** constraint.

A **solution** of (IVP) is, again a function $\mathbf{F} : J_0 \to U$ that solves the underlying ODE, but now subject to the condition that $\mathbf{F}(t_0) = \mathbf{x}_0$.

Example 1.1.4. The **forced Van der Pol** equation is defined to be

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = b(1 - x_1^2)x_2 - \omega^2 x_1 + a \cos \Omega t. \end{cases} \tag{fVdP}$$

1.2 Local existence and uniqueness

TODO: I don't like the way these theorems are laid out in the notes— I think I reorganize these much better

Let $J, U \subseteq \mathbb{R}$, and consider the IVP

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 \tag{IVP-1D}$$

where $f : J \to \mathbb{R}$ is continuously differentiable in t on some open interval containing t_0.

Supposing we can integrate (IVP-1D) from t_0 to a given point t,

$$\begin{aligned} \int_{t_0}^{t} \frac{dy}{d\tau} d\tau &= \int_{t_0}^{t} f(\tau, y) d\tau, \\ y(t) - y(t_0) &= \int_{t_0}^{t} f(\tau, y) d\tau, \\ y(t) &= y(t_0) + \int_{t_0}^{t} f(\tau, y) d\tau, \qquad (\textstyle\int \text{IVP-1D}) \\ y(t) &= y_0 + \int_{t_0}^{t} f(\tau, y) d\tau. \end{aligned}$$

If F satisfies ($\int$IVP-1D), it satisifes (IVP-1D) and vice versa, and this is easily seen using the fundamental theorem of calculus.

Definition 1.2.1. The **Picard iterates** y_i, given the data for (IVP-1D), are defined recursively as follows:

$$\begin{cases} y_0(t) := y_0, \\ y_{i+1}(t) := y_0 + \int_{t_0}^{t} f(\tau, y_i(\tau)) d\tau. \end{cases} \tag{Picard}$$

We have a weak but straightforward estimate on the y_i.

Lemma 1.2.2. For all $a > 0, b > 0$, define R to be the rectangle $[t_0, t_0+a] \times [y_0 - b, y_0 + b]$. Then if we define

$$M := \max_{(t,y)\in R} |f(t,y)|, \qquad \alpha := \min\left\{a, \frac{b}{M}\right\}.$$

Then

$$|y_n(t) - y_0| \leq M(t - t_0) \text{ for all } t_0 \leq t \leq t + \alpha \tag{1}$$

for all n.

Proof. We prove this by induction.

We note that if (1) holds for n, then for all $t_0 \leq t \leq \alpha$,

$$\begin{aligned} |y_n(t) - y_0| &\leq M(t - t_0) \\ &\leq M\alpha \\ &= M \cdot \min\left\{a, \frac{b}{M}\right\} \\ &= \min\left\{\frac{a}{M}, b\right\} \\ &\leq b. \end{aligned}$$

Hence

$$|f(t, y_n(t))| \leq M. \tag{2}$$

$n = 0$: We have that $|y_0(t) - y_0| = 0 \leq M(t - t_0)$ trivially for *all* $t \geq t_0$.

$n > 0$: Suppose that $|y_n(t) - y_0| \leq M(t - t_0)$ for all $t_0 \leq t \leq t_0 + \alpha$. Then

$$\begin{aligned} |y_{n+1}(t) - y_0| &= \left|\int_{t_0}^t f(\tau, y_n(\tau))\, d\tau\right| \\ &\leq \int_{t_0}^t \underbrace{\left|f(\tau, y_n(\tau))\right|}_{\text{use (2)}} d\tau \\ &\leq \int_{t_0}^t M\, d\tau \end{aligned}$$

$$= M(t - t_0)$$

for all $t_0 \leq t \leq t_0 + \alpha$. This completes the proof.

□

Next we show that the the Picard iterates y_n converge.

Theorem 1.2.3. Let $f(t, y)$ be continuously differentiable in both t and y. Then, on $[t_0, t_0+\alpha]$, the Picard iterates y_n converge pointwise to a function y that satisifes (IVP-1D).

Proof. We have that

$$\begin{aligned} y_n(t) &= y_0(t) + [y_1(t) - y_0] + \cdots + [y_n(t) - y_{n-1}(t)] \\ &= y_0(t) + \sum_{i=1}^{n} y_i(t) - y_{i-1}(t). \end{aligned}$$

So, $y_n(t)$ converges if and only if $\sum_{i=1}^{n} y_i(t) - y_{i-1}(t)$ converges.
It suffices to prove that $\sum_{i=1}^{n} |y_i(t) - y_{i-1}(t)|$ converges.
We compute that

$$\begin{aligned} &|y_i(t) - y_{i-1}(t)| \\ &= \left| \left[y_0 + \int_{t_0}^{t} f(\tau, y_{i-1}(\tau)) d\tau \right] - \left[y_0 + \int_{t_0}^{t} f(\tau, y_{i-2}(\tau)) d\tau \right] \right| \\ &= \left| \int_{t_0}^{t} f(\tau, y_{i-1}(\tau)) - f(\tau, y_{i-2}(\tau)) d\tau \right| \\ &\leq \int_{t_0}^{t} \underbrace{\left| f(\tau, y_{i-1}(\tau)) - f(\tau, y_{i-2}(\tau)) \right|}_{\text{apply mean value theorem}} d\tau \\ &= \int_{t_0}^{t} \left| \frac{\partial f(\tau, c(\tau))}{\partial y} \right| \left| y_{i-1}(\tau) - y_{i-2}(\tau) \right| d\tau \\ &\leq P \int_{t_0}^{t} \left| y_{i-1}(\tau) - y_{i-2}(\tau) \right| d\tau. \end{aligned}$$

Now, we can show inductively that

$$|y_i(t) - y_{i-1}(t)| \leq M P^{i-1} \frac{(t - t_0)^i}{i!} \tag{3}$$

for all $i \geq 1$.

The base case $i = 1$ ends up being (1)

$$|y_1(t) - y_0(t)| = |y_1(t) - y_0| \leq M(t - t_0).$$

For the induction step, supppose it were true for i, then

$$\begin{aligned}
|y_{i+1}(t) - y_i(t)| &\leq P \int_{t_0}^{t} \underbrace{\left|y_{i-1}(\tau) - y_i(\tau)\right|}_{\text{apply induction hypothesis}} \, d\tau \\
&\leq P \int_{t_0}^{t} \left[M P^{i-1} \frac{(\tau - t_0)^i}{i!} \right] d\tau \\
&= M P^i \int_{t_0}^{t} \frac{(t - t_0)^i}{i!} d\tau \\
&= M P^i \frac{(t - t_0)^{i+1}}{(i+1)!}.
\end{aligned}$$

Hence, (3) holds for all i.

So

$$\begin{aligned}
\sum_{i=1}^{n} |y_i(t) - y_{i-1}(t)| &\leq \sum_{i=1}^{n} M P^{i-1} \frac{(t - t_0)^i}{i!} \\
&= M P^{-1} \left(\sum_{i=1}^{n} P^i \frac{(t - t_0)^i}{i!} \right) \\
&= M P^{-1} \left(-1 + \sum_{i=0}^{n} P^i \frac{(t - t_0)^i}{i!} \right) \\
&= M \frac{e^{P(t - t_0)} - 1}{P},
\end{aligned}$$

which is a finite number.

So, we have shown that $y_n(t) \longrightarrow y(t)$ for some function y.

Next, we show that y satisifes (IVP-1D), by showing that it satisfies ($\int$IVP-1D).

Since f is continuous, $f\big(\tau, y_n(\tau)\big)$ converges pointwise to $f\big(\tau, y(\tau)\big)$ as $n \to \infty$. Moreover, f is bounded on R, so by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_{t_0}^{t} f\big(\tau, y_n(\tau)\big) d\tau = \int_{t_0}^{t} f\big(\tau, y(\tau)\big) d\tau.$$

So finally, we conclude that

$$\begin{aligned} y(t) &= \lim_{n\to\infty} y_{n+1}(t) \\ &= \lim_{n\to\infty}\left[y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))d\tau\right] \\ &= y_0 + \lim_{n\to\infty}\left[\int_{t_0}^{t} f(\tau, y_n(\tau))d\tau\right]. \end{aligned}$$

Hence $y(t)$ satisfies ($\int$ IVP-1D), so it satisfies (IVP-1D).

□

Theorem 1.2.4. Let R be defined as before, given an IVP. Then the solution produced by Theorem (1.2.3) is unique.

Proof. Let $y(t)$ and $z(t)$ be two solutions to the IVP on R. The function $|y(t)-z(t)|$ is nonnegative. Using the same P in the proof of Theorem (1.2.3), we have

$$|y(t) - z(t)| \leq \int_{t_0}^{t} |y(\tau) - z(\tau)| s\tau,$$

so we use Lemma (1.2.5) with $w(t) = |y(t) - z(t)|$, to show that $|y(t) - z(t)| = 0$ for all $t_0 \leq t \leq t_0 + \alpha$, hence $y(t) = z(t)$ on $t_0 \leq t \leq t_0 + \alpha$. □

Lemma 1.2.5. Let $P \in \mathbb{R}$ and let $w : [t_0, t_1] \to \mathbb{R}$ such that $w(t) \geq 0$ and suppose

$$w(t) \leq P \int_{t_0}^{t} w(\tau)d\tau$$

for all $t_0 \leq t \leq t_1$. Then $w(t) = 0$ for all $t_0 \leq t \leq t_1$.

Proof. Let $U(t) := \int_{t_0}^{t} w(\tau)d\tau$. Then

$$\frac{dU(t)}{dt} = w(t) \leq P \int_{t_0}^{t} w(\tau)d\tau = PU(t),$$

so

$$\begin{aligned} e^{-P(t-t_0)}\frac{dU(t)}{dt} &\leq e^{-P(t-t_0)}PU(t), \\ e^{-P(t-t_0)}\frac{dU(t)}{dt} - e^{-P(t-t_0)}PU(t) &\leq 0, \end{aligned}$$

$$\begin{aligned}
\int_{t_0}^{t} e^{-P(s-t_0)} \frac{dU(s)}{ds} - e^{-P(s-t_0)} PU(s) ds &\leq 0, \\
\int_{t_0}^{t} \frac{d}{ds} \left[e^{-P(s-t_0)} U(s) \right] ds &\leq 0, \\
e^{-P(t-t_0)} U(t) - e^{-P(t_0-t_0)} U(t_0) &\leq 0, \\
e^{-P(t-t_0)} U(t) &\leq 0.
\end{aligned}$$

So,

$$0 \leq e^{-P(t-t_0)} U(s) \leq 0,$$

and we conclude that $e^{-P(t-t_0)} U(t) = 0$, since $e^{-P(t-t_0)} \neq 0$, it must be that $U(t) = 0$, so

$$0 \leq w(t) \leq PU(t) = 0$$

for all $t_0 \leq t \leq t_1$. □

1.3 Extending solutions; global uniqueness