Trees and the Composition of Generating Functions

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What is this?

These are notes based on my study of Chapter 5 in Richard P. Stanley's "Enumerative Combinatorics".

Contents

1 The exponential formula

We recall the mechanics of *formal power series composition*.

Definition 1.0.1. Let $f, g \in \mathbb{K}[\![x]\!]$ such that $[x^0]g = 0$. The *composition* $f \circ g$ is defined to be

$$\begin{aligned}(f \circ g)(x) &:= \sum_{n \geq 0} f_n g^n \\ &= f_0 g^0 + f_1 g^1 + \cdots,\end{aligned}$$

where we define $f_n := [x^n]f$ for all $n \geq 0$.

The condition that $[x^0]g = 0$ ensures that $f \circ g$ is well-defined, as this means that, for any $n \geq 0$, $[x^n](f \circ g)$ is determined by only *finitely many* of the terms $f_k g^k$, namely those where $k \leq n$.

Now, let f and g be *exponential* generating functions, whose definition we recall.

Definition 1.0.2. Let $a = \{a_i\}_{i \geq 0}$ be a sequence of numbers (say, integers). We define its *exponential generating function* EGF_a to be the formal power series

$$\mathrm{EGF}_a(x) := \sum_{n \geq 0} a_n \frac{x^n}{n!}.$$

Is there any special structure, then, to a *composition of exponential generating functions* $\mathrm{EGF}_g \circ \mathrm{EGF}_f$? We first answer this "directly".

Theorem 1.0.3 (The compositional formula, take zero)**.** Let $\{f_i\}_{i \geq 0}$ be a sequence of numbers such that $f_0 = 0$, and let $\{g_i\}_{i \geq 0}$ be a sequence of numbers such that $g_0 = 1$. Then

$$(\mathrm{EGF}_g \circ \mathrm{EGF}_f)(x) = \sum_{m \geq 1} \left[\sum_{n \geq 0} \frac{g_n}{n!} \left[\sum_{\substack{(m_1, \dots, m_n) \in \mathbb{P}^n \\ m_1 + \cdots + m_n = m}} \frac{m!}{m_1! \cdots m_n!} f_{m_1} \cdots f_{m_n} \right] \right] \frac{x^m}{m!}.$$

Proof. This is just a long calculation.

$$\begin{aligned}
(\mathrm{EGF}_g \circ \mathrm{EGF}_f)(x) &= \sum_{n \geq 0} g_n \frac{\left[\mathrm{EGF}_f(x)\right]^n}{n!} \\
&= \sum_{n \geq 0} \frac{g_n}{n!} \underbrace{\left[\sum_{m \geq 1} f_m \frac{x^m}{m!} \right]^n}_{\text{expand product of sums}} \\
&= \sum_{n \geq 0} \frac{g_n}{n!} \underbrace{\left[\sum_{m_1, \dots, m_n \geq 1} f_{m_1} \frac{x^{m_1}}{m_1!} \cdots f_{m_n} \frac{x^{m_n}}{m_n!} \right]}_{\text{group terms by their index sum } m = \sum_i m_i}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 0} \frac{g_n}{n!} \left[\sum_{m\geq 1} \sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \underbrace{f_{m_1}\frac{x^{m_1}}{m_1!}\cdots f_{m_n}\frac{x^{m_n}}{m_n!}}_{\text{organize terms}} \right] \\
&= \sum_{n\geq 0} \frac{g_n}{n!} \left[\sum_{m\geq 1} \sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{x^{m_1}\cdots x^{m_n}}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n} \right] \\
&= \sum_{n\geq 0} \frac{g_n}{n!} \left[\sum_{m\geq 1} \sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{\overbrace{x^{m_1+\cdots+m_n}}^{\text{equals } x^m}}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n} \right] \\
&= \sum_{n\geq 0} \frac{g_n}{n!} \left[\sum_{m\geq 1} x^m \left[\sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{1}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n} \right]\right] \\
&= \sum_{n\geq 0} \frac{g_n}{n!} \left[\sum_{m\geq 1} \frac{x^m}{m!} \left[\sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{m!}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n} \right]\right] \\
&= \underbrace{\sum_{n\geq 0} \frac{g_n}{n!} \sum_{m\geq 1} \frac{x_m}{m!}}_{\text{interchange sums}} \sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{m!}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n} \\
&= \sum_{m\geq 1} \frac{x_m}{m!} \sum_{n\geq 0} \frac{g_n}{n!} \sum_{\substack{m_1,\ldots,m_n\geq 1\\ m_1+\cdots+m_n=m}} \frac{m!}{m_1!\cdots m_n!} f_{m_1}\cdots f_{m_n}.
\end{aligned}
$$

□

If, given $\{f_i\}_{i\geq 0}$ and $\{g_i\}_{i\geq 0}$, we defined the sequence $\{h_i\}_{i\geq 0}$ to be

$$
\begin{aligned}
h_n &:= \sum_{k\geq 0} \frac{g_k}{k!} \left[\sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{P}^k\\ n_1+\cdots+n_k=n}} \frac{n!}{n_1!\cdots n_k!} f_{n_1}\cdots f_{n_k} \right] \\
h_0 &:= 1,
\end{aligned}
$$

then the theorem can be restated as

$$\mathrm{EGF}_g \circ \mathrm{EGF}_f = \mathrm{EGF}_h .$$

But what *is h*? Can we *breathe some combinatorial life into it*? The answer is yes.

First, you will have noticed that our manipulations have introduced a *multinomial coefficient*, so we can rewrite

$$\begin{aligned}
h_n &= \sum_{k \geq 0} \frac{g_k}{k!} \sum_{\substack{(n_1,\dots,n_k) \in \mathbb{P}^k \\ n_1+\dots+n_k=n}} \overbrace{\frac{n!}{n_1! \cdots n_k!}}^{=\binom{i}{i_1,\dots,i_n}} f_{n_1} \cdots f_{n_k} \\
&= \sum_{k \geq 0} \frac{g_k}{k!} \sum_{\substack{(n_1,\dots,n_k) \in \mathbb{P}^k \\ n_1+\dots+n_k=n}} \binom{n}{n_1,\dots,n_k} f_{n_1} \cdots f_{n_k} \\
&= \sum_{k \geq 0} g_k \sum_{\substack{(n_1,\dots,n_k) \in \mathbb{P}^k \\ n_1+\dots+n_k=n}} \left[\frac{1}{k!}\binom{n}{n_1,\dots,n_k}\right] f_{n_1} \cdots f_{n_k}.
\end{aligned}$$

We recall $\binom{n}{n_1,\dots,n_k}$'s most direct combinatorial interpretation: ways of putting n balls into k *labeled* boxes, such that the first box has n_1 balls, the second has n_2 balls, and so on. Equivalently, we are considering *ordered partitions of a set of size n into k parts*.

Dividing this quantity by $k!$ *forgets* the labeling on the boxes. In terms of partitions, this means we are considering *unordered partitions of a set of size n* now.

> **Definition 1.0.4.** Let X be a finite set. Denote by $\mathbf{Par}(X)$ the set of all *ordered partitions of X into k parts*. Denote by $\mathbf{Par}^{\mathrm{Sym}}(X)$ the set of all *unordered partitions of X into k parts*.

Then

$$\begin{aligned}
h_n &= \sum_{k \geq 0} g_k \sum_{\substack{(n_1,\dots,n_k) \in \mathbb{P}^k \\ n_1+\dots+n_k=n}} \left[\frac{1}{k!}\binom{n}{n_1,\dots,n_k}\right] f_{n_1} \cdots f_{n_k} \\
&= \sum_{k \geq 0} \frac{g_k}{k!} \sum_{\substack{(n_1,\dots,n_k) \in \mathbb{P}^k \\ n_1+\dots+n_k=n}} \binom{\text{\# of ordered partitions of } [n]}{\text{with part sizes } \{n_1,\dots,n_k\}} f_{n_1} \cdots f_{n_k}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k \geq 0} \frac{g_k}{k!} \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{P}^k \\ n_1+\cdots+n_k=n}} \left(\sum_{\substack{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{P}(n) \\ |\pi_i|=n_i}} 1 \right) f_{n_1}\cdots f_{n_k} \\
&= \sum_{k \geq 0} \frac{g_k}{k!} \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{P}^k \\ n_1+\cdots+n_k=n}} \left(\sum_{\substack{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{P}(n) \\ \#\pi_i=n_i}} f_{n_1}\cdots f_{n_k} \right) \\
&= \sum_{k \geq 0} g_k \frac{1}{k!} \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{P}^k \\ n_1+\cdots+n_k=n}} \left(\sum_{\substack{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{P}(n) \\ \#\pi_i=n_i}} f_{\#\pi_1}\cdots f_{\#\pi_k} \right) \\
&= \sum_{k \geq 0} g_k \frac{1}{k!} \sum_{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{P}(n)} f_{\#\pi_1}\cdots f_{\#\pi_k} \\
&= \sum_{k \geq 0} g_k \sum_{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{S}(n)} f_{\#\pi_1}\cdots f_{\#\pi_k} \\
&= \sum_{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{S}(n)} f_{\#\pi_1}\cdots f_{\#\pi_k} g_k
\end{aligned}
$$

Now, we make a psychological shift— we consider our sequences not as sequences, but as *weight functions*, which gives us a rule.

Namely, we will consider counting the number of *structures*

Theorem 1.0.5 (The compositional formula)**.** Let $f(n)$ and $g(n)$ be two weight functions. Define

$$h(n) := \sum_{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{S}(n)} f(\#\pi_1)\cdots f(\#\pi_k)g(k)$$

Then

$$\mathrm{EGF}_g \circ \mathrm{EGF}_f = \mathrm{EGF}_h .$$

The compositional formula reflects putting a f-structure on a partition of X, and a g-structure on the partitions.

Theorem 1.0.6 (The exponential formula)**.** Let $f : \mathbb{P} \to K$.

Then if we define

$$h(n) := \sum_{\pi=(\pi_1,\ldots,\pi_k)\in\mathbf{S}(n)} f(\#\pi_1)\cdots f(\#\pi_k)g(k)$$

We have that

$$\mathrm{EGF}_h = \exp \mathrm{EGF}_f$$