Noncommutative Schur functions

Jasper Ty

What is this?

This is (going to be) an "infinite napkin" set of notes I am taking about noncommutative Schur functions.

Contents

1 Ideals and words

Let $\mathbf{u} = (u_1, \ldots, u_N)$ be a collection of variables. Let $\langle \mathbf{u} \rangle$ be the free semigroup on the generators $\mathbf{u}$. Then, let $\mathcal{U} = \mathbb{Z}\langle \mathbf{u} \rangle$ denote the corresponding semigroup ring— the free associative ring generated by $\mathbf{u}$.

We will denote by $\mathcal{U}^*$ the $\mathbb{Z}$-module spanned by words in the alphabet $\{1, \ldots, N\}$.

We will have a fundamental pairing $\langle -, - \rangle$ given by making noncommutative monomials dual to words.

Now, if I is an ideal of $\mathcal{U}$, we define $I^\perp$ by

$$I^\perp \coloneqq \{\gamma \in \mathcal{U}^* \mid \langle I, \gamma \rangle = 0\}.$$

2 Noncommutative e's and h's

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) \coloneqq \sum_{i_1 > i_2 > \cdots > i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

The **noncommutative complete homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) \coloneqq \sum_{i_1 \geq i_2 \geq \cdots \geq i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

2.1 The ideal I_C

Lemma 2.2. Let I be an ideal. If the e's commute modulo I, then the h's commute modulo I as well, and vice versa.

Definition 2.3. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \qquad (a < b), \qquad (1)$$

$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \qquad (a < b < c), \qquad (2)$$

$$u_c u_b u_c u_a + u_b u_c u_a u_c - u_c u_b u_a u_c - u_b u_c^2 u_a \qquad (a < b < c). \qquad (3)$$

Theorem 2.4. I_C is the smallest ideal in which the elementary symmetric functions $e_k(\mathbf{u}_S)$ and $e_\ell(\mathbf{u}_S)$ commute for any k, ℓ, S.

2.2 The homomorphism

Theorem 2.5 (Fundamental theorem of symmetric functions)**.** Let $\Lambda(\mathbf{x})$ denote the ring of symmetric polynomials in the commuting variables $\mathbf{x} = (x_1, \ldots, x_n)$. Then

$$\Lambda(\mathbf{x}) \simeq \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x})].$$

Proof. See Theorem 7.4.4 in [EC2]. One checks that products of the form. One can prove this via the *Gale-Ryser* theorem. □

Corollary 2.6. If I contains I_C, then the map

$$\begin{aligned} \Lambda_n(\mathbf{x}) &\to \mathcal{U}/I \\ e_k(\mathbf{x}) &\mapsto e_k(\mathbf{u}) \end{aligned}$$

extends to a ring homomorphism.

Proof. Combine Theorems 2.5 and 2.4. □

3 Noncommutative Schur functions

Definition 3.1. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u})$ is defined to be

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in S_t} \operatorname{sgn}(\pi) e_{\lambda_1^\top + \pi(1) - 1}(\mathbf{u}) e_{\lambda_2^\top + \pi(2) - 2}(\mathbf{u}) \cdots e_{\lambda_t^\top + \pi(t) - t}(\mathbf{u}),$$

where $t = \lambda_1$ is the number of parts of $\lambda^\top$.

Theorem 3.2 ([FG98], [BF16])**.** In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) := \sum_{T \in \mathrm{SSYT}(\lambda; N)} \mathbf{u}^{\mathsf{colword}\, T}.$$

3.1 Cauchy kernel

Definition 3.3. Let $\mathbf{x} = (x_1, x_2 \ldots)$ be a countable collection of commuting variables.

4 Stuff

Definition 4.1. A **combinatorial representation** of $\mathcal{U}/I$ is

Definition 4.2.

5 Appendix

5.1 Gessel's fundamental quasisymmetric function

5.2 The Edelman-Greene correspondence

References

[EC2] Sergey Fomin and Curtis Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), 179-200.

[FG98] Sergey Fomin and Curtis Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193** (1998), 179-200.

[BF16] Jonah Blasiak and Sergey Fomin, *Noncommutative Schur functions, switchboards, and Schur positivity*, Sel. Math. **23** (2017), 727-766.

Also available as arXiv:1510.00657.

[A15] Sami Assaf, *Dual equivalence graphs I: A new paradigm for Schur positivity*, Forum. Math. Sigma **3** (2015), e12.

Also available as arXiv:1506.03798.

[L04] Thomas Lam, *Ribbon Schur operators*, European J. Combin. **29** (2008), 343-359.

Also available as arXiv:math/0409463.