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(Note: no guarantee of correctness. Just an undergrad writing their answers down and archiving because notebooks have become unwieldy.)

Q1. Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.

A: Idea

x_0 may be covered with an interval of arbitrarily small width, and this segment will have an arbitrarily small weight in the sum due to α's continuity at x_0.

Proof that $f \in \mathscr{R}(\alpha)$:

Let $\varepsilon > 0$.

Since α is continuous at x_0, choose δ such that $|x - x_0| \le 2\delta$ implies $|\alpha(x) - \alpha(x_0)| \le \varepsilon$.

Then let $P = [a, x_0 - \delta, x_0 + \delta, b]$. Then

$$\begin{aligned} U(P,f,\alpha) &= \sum_{i=0}^{n} M_i \Delta\alpha_i \\ &= M_0\Delta\alpha_0 + M_1\Delta\alpha_1 + M_2\Delta\alpha_2 \end{aligned}$$

Since $f(x) = 0$ on $[a, x_0 - \delta]$ and $[x_0 + \delta, b]$, $M_0 = 0$ and $M_2 = 0$. Moreover, $M_1 = 1$

$$U(P,f,\alpha) = 0 + 1 \cdot \Delta\alpha_1 + 0\Delta = \alpha_1$$

As for L, the following holds for any partition at all, as any nonempty interval must contain a point that is not x_0

$$L(P,f,\alpha) = 0$$

So

$$U(P,f,\alpha) - L(P,f,\alpha) = \Delta\alpha_1 = \alpha(x+\delta) - \alpha(x-\delta) < \varepsilon$$

Then $f \in \mathscr{R}(\alpha)$.

Proof that $\int f d\alpha = 0$:

Since $L(P,f,\alpha) = 0$ for all P,

$$\underline{\int} f d\alpha = 0$$

Which lets us conclude

$$\int f d\alpha = 0$$

Q2. Suppose $f \ge 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)

A: Idea

We show a contradiction that a "bump" of nonzero area must exist if f is not identically zero, hence the integral must be nonzero.

Proof

Choose any arbitrary point q in $[a,b]$. Suppose $f(q) > 0$. Let m be a number such that $0 < m < f(q)$, and use the continuity of f to create an interval $[s,t]$ containing q such that $x \in [s,t]$ implies $f(x) \ge m$.

By additivity of the integral,

$$\int_a^b f dx = \int_a^s f dx + \int_s^t f dx + \int_t^b f dx$$

Hence

$$\int_a^b f dx \ge \int_s^t f dx$$

But $\int_s^t f dx \ge m(t-s) > 0$, so

$$\int_a^b f dx > 0$$

A contradiction to the assertion that $\int_a^b f dx = 0$

Q3. Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0^+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

(b) State and prove a similar result for β_2

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0

(d) If f is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

A: Idea

With β_i, the only intervals with non-zero weight in any partition will be those which contain 0. Moreover, these segments will have fixed weights due to our definitions of β_i. Then, bounds on the variation of the upper and lower Riemann sums automatically pass through as bounds on f in the segment itself.

Hence we can convert convergence of Riemann sums into convergence of f.

(a) Proof that $f \in \mathscr{R}(\beta_1)$ implies $f(0+) = f(0)$

Let $f \in \mathscr{R}(\beta_1)$. Let $\varepsilon > 0$. Choose a partition P such that $0 \in P$, and

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

The only nonzero term in either of the sums $U(P, f, \beta_1)$ or $L(P, f, \beta_1)$ comes from the segment containing $x = 0$ as a left endpoint. Let this segment be $[0, x_i]$. Then

$$U(P, f, \beta_1) = M_i \cdot \Delta x_i = M_i \cdot (\beta(x_i) - \beta(0)) = M_i \cdot 1 = M_i$$

$$L(P, f, \beta_1) = M_i \cdot \Delta x_i = m_i \cdot (\beta(x_i) - \beta(0)) = m_i \cdot 1 = m_i$$

So $U(P, f, \beta_1) - L(P, f, \beta_1) = M_i - m_i < \varepsilon$. This tells us that on $[0, x_i]$, f stays within ε of $f(0)$. Since ε was arbitrary, we conclude that $f(0+) = f(0)$

Proof that $f(0+) = f(0)$ implies $f \in \mathscr{R}(\beta_1)$

Let $f(0+) = f(0)$. Let $\varepsilon > 0$ Choose δ such that $x \in (0, \delta)$ implies $f(x) \in (f(0) - \varepsilon, f(0) + \varepsilon)$. Let P be the partition $\{a, 0, \delta, b\}$. Then for the same reasons as above,

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon$$

So $f \in \mathscr{R}(\beta_1)$.

Proof that $\int f d\beta_1 = 0$

By Theorem 6.7(b), letting $t_i = 0$, we have that

$$\left| f(0) - \int f d\beta_1 \right| < \varepsilon$$

Hence we conclude that

$$\int f d\beta_1 = f(0)$$

(b) Statement

$f \in \mathscr{R}(\beta_1)$ if and only if $f(0-) = f(0)$, and if it exists,

$$\int f d\beta_1 = f(0)$$

Proof

Extremely similar as before, but with $[x_{i-1}, 0]$ instead of $[0, x_i]$

(c) Proof that $f \in \mathscr{R}(\beta_3)$ implies f is continuous at 0

Let $f \in \mathscr{R}(\beta_3)$. Let $\varepsilon > 0$. Let P be a partition that such that $0 \in P$ and $U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$

Then, consider the intervals $[x_{i-1}, 0]$, $[0, x_i]$. We have that

$$\beta_3(0) - \beta_3(x_{i-1}) = \frac{1}{2}$$

$$\beta_3(x_i) - \beta_3(0) = \frac{1}{2}$$

Then

$$U(P, f, \beta_3) = \frac{M_i + M_{i-1}}{2}$$

$$L(P, f, \beta_3) = \frac{m_i + m_{i-1}}{2}$$

Let $M = \min\{M_i, M_{i-1}\}$ and $m = \max\{m_i, m_{i-1}\}$, then

$$U(P, f, \beta_3) \geq M$$

$$L(P, f, \beta_3) \leq m$$

Then

$$M - m \leq U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence $f(x)$ is within ε of $f(0)$ if $f(x)$ is in $[x_{i-1}, x_i]$. Hence f is continuous.

Proof that f continuous at 0 implies $f \in \mathscr{R}(\beta_3)$

Let f be continuous at 0. Let $\varepsilon > 0$. Choose δ such that $|x| \leq 2\delta$ implies $|f(x) - f(0)| \leq \varepsilon/2$ then take the partition $P = \{a, -\delta, \delta, b\}$. Then

$$U(P, f, \beta_3) - L(P, f, \beta_3) < \varepsilon$$

Hence $f \in \mathscr{R}(\beta_3)$. Also, with a similar argument as in (a),

$$\int f d\beta_3 = f(0)$$

(d) f being continuous at zero is a strong enough condition for all three previous proofs above to apply. Since all of them show equality of the integral with respect to β_i and $f(0)$, the result follows.

Q4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$

A: Idea

Any nonempty interval will contain both irrational and rational numbers, so the "gap" between the lower and upper sums never closes.

Proof

Let P be a partition. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for all x_i we can find two rational numbers p and q such that $x_{i-1} < p < q < x_i$. This tells us that $M_i = 1$ for all x_i. Furthermore, the number $s = p + (\sqrt{2}/2)(q - p)$ is irrational and between p and q. This tells us that $m_i = 1$ for all x_i. Hence it must be that

$$U(P, f) = 1$$

$$L(P, f) = 0$$

for any partition P. Then f is not Riemann-integrable.

Q5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

A: Idea

Because of squaring's non-injectivity, we can construct "terrible" discontinuities which can be undone via squaring. On the other hand, cubing is much nicer (it is a continuous bijection of the real line to itself).

PROOF

No. Consider the function

$$f(x) := \begin{cases} 1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases}$$

We have that $(f^2)(x) = 1$, so $\int_a^b f^2 dx = (b-a)$. But f itself is not Riemann-integrable, as the exercise above shows.

For f^3, use Theorem 6.11, which tells us that the composition of continuous functions with integrable functions is integrable. Let $\phi = \sqrt[3]{x}$. Then $\phi \circ (f^3) \in \mathscr{R}$ Since $\phi \circ f^3 = f$, $f \in \mathscr{R}$.

Q6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0,1]$. *Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

A: Each set of the sequence that constructs the Cantor set has 2^n intervals, each of width $1/3^n$. Let E_n be the nth set in the sequence. Cover each interval in E_n with an open interval whose endpoints have a distance of $1.1 \cdot (1/3^n)$. Then, the sum of all lengths of each cover will be $1.1 \cdot (2/3)^n$. This also covers the Cantor set itself, since the Cantor set E is a subset of E_n for all n. Hence, we have made a cover of the Cantor set with arbitrarily small area.

Take the endpoints of each interval and construct a partition, and proceed as in Theorem 6.10. This sequence of partitions eventually.

Q7. Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function F such that the above limit exists, although it fails to exist with $|f|$ in place of f

A:

(a) Lemma: If $f \in \mathscr{R}$ on $[a,b]$, then

$$\lim_{h\to a} \int_a^h f(x)dx = 0$$

Proof: Let $|f(x)| \leq M$. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then, letting $h \leq \delta$

$$\left| \int_a^h f(x)dx \right| \leq M((a+h)-a) = Mh \leq M\delta = \varepsilon$$

Since the ε was arbitrary, the lemma holds. Then, by Theorem 6.12, if $f \in \mathscr{R}$ on $[0,1]$,

$$\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx$$

Then, taking limits

$$\begin{aligned}
\lim_{c\to 0} \int_0^1 f(x)dx &= \lim_{c\to 0} \int_0^c f(x)dx + \lim_{c\to 0} \int_c^1 f(x)dx \\
\int_0^1 f(x)dx &= 0 + \lim_{c\to 0} \int_c^1 f(x)dx \\
\int_0^1 f(x)dx &= \lim_{c\to 0} \int_c^1 f(x)dx
\end{aligned}$$

Hence showing that the definition agrees.

(b) Let $n = 1, 2, \ldots$. Consider the function (with infinitely many cases)

$$f(x) = \Big\{ (-1)^{n+1} n + 1 \quad \text{if } \tfrac{1}{n+1} \leq x < \tfrac{1}{n}$$

Then the graph of f consists of boxes of width $(1/n) - (1/(n+1))$ and height $n+1$. The area of each box is then $1/n$. Then as the lower bound of the integral approaches 0, the integral approximates the alternating sum $S = 1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges to $\ln 2$. However, in the case of $|f|$, the integral apporximates the sum $S = 1 + 1/2 + 1/3 + 1/4 + \cdots$, which diverges.

Q8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*. Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges. (This is the so-called “integral test” for convergence of series.)

A: Suppose $\int_1^\infty f(x)dx$ converges. Then if we show that, for any positive integer b,

$$\sum_{n=1}^{b} f(n) \leq \int_1^b f(x)dx$$

we can prove convergence of the sum, as the partial sums form a monotonic sequence (f is non-negative). Let P be the partition consisting of the points $0, 1, 2, \ldots, b-1, b$. Then, since f is monotonically decreasing,

$$\inf_{x\in[n-1,n]} f(x) = f(n)$$

Hence

$$\sum_{n=1}^{b} f(n) = L(P, f) \leq \int_1^b f(x)dx$$

which was the inequality we wanted. Then the sum converges.

Suppose $\int_1^\infty f(x)dx$ converges. Then we make a similar argument, with $U(P, f)$ bounding $\int_1^b f(x)dx$ from above. Take the same partition of $[a, b]$, then

$$\sup_{x\in[n,n+1]} f(x) = f(n)$$

Hence

$$U(P, f) = \sum_{n=0}^{b-1} f(n)$$

Which proves that the integral converges.

Q9. Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}$$

Q10. Let p and q be positive real numbers such that

$$\frac{1}{q} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha$$

then

$$\int_a^b fg \, d\alpha \leq 1$$

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}}$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

A:

(a) Let, for $x \geq 0$ and $y \geq 0$,

$$f(x) := x^{p-1}$$
$$g(y) := y^{q-1}$$

Then f and g are both strictly increasing functions. Furthermore, since $(p-1)(q-1) = 1$, they are inverses to one another.

LEMMA

Let f be a strictly increasing, invertible function on $[a, b]$ that is also $\mathscr{R}(\alpha)$ on $[a, b]$. Then

$$(b-a)(f(b) - f(a)) = \int_a^b f dx + \int_{f(a)}^{f(b)} f^{-1} dx - (b-a)(f(a)) - (f(b) - f(a))(a)$$

Proof

Let P be any partition at all of $[a, b]$. Construct a corresponding partition P' of $[f(a), f(b)]$ made of points $y_i := f(x_i)$. This is a valid partition as f is increasing, and the endpoints map correctly. Now consider the facts

$$b - a = \sum_{i=1}^{n} \Delta x_i$$

$$f(b) - f(a) = \sum_{i=1}^{n} \Delta y_i$$

Hence

$$(b-a)(f(b) - f(a)) = \sum_{1 \leq i,j \leq n} \Delta x_i \Delta y_j$$

Let m_i denote the inf of f in $[x_{i-1}, x_i]$, and let m_i' denote the inf of f^{-1} on $[y_{i-1}, y_i]$. Define M_i and M_i' similarly. By the monotonicity of f, we know that

$$m_i = y_{i-1} = f(x_{i-1})$$
$$m_i' = x_{i-1} = f^{-1}(y_{i-1})$$
$$M_i = y_i = f(x_i)$$
$$M_i' = x_i = f^{-1}(y_i)$$

We also know that

$$x_i - a = \sum_{j=1}^{i} \Delta x_j$$

$$y_i - f(a) = \sum_{j=1}^{i} \Delta y_j$$

Hence $U(P, f)$ may be written as

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} y_i \Delta x_i = \sum_{i=1}^{n} \left[\sum_{j=1}^{i} \Delta y_j + f(a) \right] \Delta x_i$$

Simplifying the sums

$$U(P, f) = \left[\sum_{1 \le j \le i \le n} \Delta x_i \Delta y_j \right] + (b - a)(f(a))$$

Similarly,

$$U(P', f^{-1}) = \left[\sum_{1 \le i \le j \le n} \Delta x_i \Delta y_j \right] + (f(b) - f(a))(a)$$

Consider the term $\Delta x_i \Delta y_k$. Either $j < i$, in which case it is part of the sum of $U(P, f)$ but not of $U(P', f^{-1})$, or $i > j$ in which case it is part of the sum of $U(P', f^{-1})$ but not of $U(P, f)$. Or $i = j$, in which case it is part of both. Hence, by inclusion-exclusion,

$$U(P, f) + U(P', f^{-1}) = \left[\sum_{1 \le i, j \le n} \Delta x_i \Delta y_j - \sum_{k=1}^{n} \Delta x_k \Delta y_k \right] + (b - a) \cdot f(a) + (f(b) - f(a)) \cdot a$$

But since $\Delta y_k = y_k - y_{k-1} = M_k - m_k$,

$$\sum_{k=1}^{n} \Delta x_k \Delta y_k = U(P, f) - L(P, f)$$

And in the same way, with $\Delta x_k = x_k - x_{k-1} = M_i' - m'i$

$$\sum_{k=1}^{n} \Delta x_k \Delta y_k = U(P', f^{-1}) - L(P', f^{-1})$$

Hence choose a partition P such that

$$U(P, f) - L(P, f) \le \varepsilon$$

We automatically get a partition such that

$$U(P', f) - L(P', f) \le \varepsilon$$

and we show that

$$U(P, f) + U(P', f^{-1}) = [(b - a)(f(b) - f(a)) - \varepsilon] + (b - a) \cdot f(a) + (f(b) - f(a)) \cdot a$$

Similarly,

$$L(P, f) + L(P', f^{-1}) = [(b - a)(f(b) - f(a)) + \varepsilon] + (b - a) \cdot f(a) + (f(b) - f(a)) \cdot a$$

Then we have the inequalities

$$U(P,f)+U(P',f^{-1}) \leq (b-a)(f(b)-f(a))+(b-a)\cdot f(a)+(f(b)-f(a))\cdot a$$

$$L(P,f)+L(P',f^{-1}) \geq (b-a)(f(b)-f(a))+(b-a)\cdot f(a)+(f(b)-f(a))\cdot a$$

Hence

$$\int_a^b f dx + \int_{f(a)}^{f(b)} f^{-1} dx = (b-a)(f(b)-f(a))+(b-a)\cdot f(a)+(f(b)-f(a))\cdot a$$

Which is the result we wanted.

Consider the quantities u^{p-1} and v^{q-1}. If they are not equal, then there are two possibilities:

$$u^{p-1} > v \text{ and } u > v^{q-1}$$

$$u^{p-1} < v \text{ and } u < v^{q-1}$$

Suppose, without loss of generality, that $u^{p-1} \geq v$

Then

$$uv = (u - v^{q-1} + v^{q-1})v = v^{q-1}\cdot v + (u-v^{q-1})v$$

By the lemma,

$$v^{q-1}\cdot v = \int_0^{v^{q-1}} x^{p-1}dx + \int_0^v y^{q-1}dy$$

so

$$uv = \int_0^{v^{q-1}} x^{p-1}dx + \int_0^v y^{q-1}dy + (u-v^{q-1})v$$

By the monotonicity of x^{p-1}, the inf of x^{p-1} on $[v^{q-1}, u]$ is $v^{(q-1)(p-1)} = v$, so

$$(u-v^{q-1})v \leq \int_{v^{q-1}}^u x^{p-1}dx$$

Then

$$uv \leq \int_0^{v^{q-1}} x^{p-1}dx + \int_0^v y^{q-1}dy + \int_{v^{q-1}}^u x^{p-1}dx$$

Combining the integrals with the common integrand x^{p-1},

$$uv \leq \int_0^u x^{p-1} + \int_0^v y^{q-1}dy$$

Finally, evaluating the integrals,

$$uv \leq \frac{u^p}{p} + \frac{y^q}{q}$$

(b) By the previous part, we have that

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

Then

$$\int_a^b fg\, d\alpha \leq \int_a^b \left[\frac{f^p}{p} + \frac{g^q}{q}\right] dx$$

Using properties of the integral,

$$\int_a^b fg\, d\alpha \leq \frac{1}{p}\int_a^b f^p\, dx + \frac{1}{q}\int_a^b g^q\, dx$$

Since the integrals on the right hand side equal 1

$$\int_a^b fg\, d\alpha \leq \frac{1}{p} + \frac{1}{q} = 1$$

(c) Let f and g be any $\mathscr{R}(\alpha)$ functions at all. Then $|f|$ and $|g|$ are also $\mathscr{R}(\alpha)$. Then the integrals.

$$\int_a^b |f|^p \, d\alpha$$

$$\int_a^b |g|^q \, d\alpha$$

exist and are finite. Suppose they are both positive. Define

$$P := \int_a^b |f|^p \, d\alpha$$

$$Q := \int_a^b |g|^q \, d\alpha$$

Then define the functions

$$h := \frac{|f|}{P^{\frac{1}{p}}}$$

$$k := \frac{|g|}{Q^{\frac{1}{q}}}$$

We also have that $\overline{f}$ and $\overline{g}$ are nonnegative, hence the inequality in (b) applies, and

$$\int_a^b hk \, d\alpha \leq 1$$

Then

$$\int_a^b \frac{|f||g|}{P^{\frac{1}{p}} Q^{\frac{1}{q}}} \, d\alpha \leq 1$$

$$\frac{1}{P^{\frac{1}{p}} Q^{\frac{1}{q}}} \int_a^b |f||g| d\alpha \leq 1$$

$$\int_a^b |f||g| d\alpha \leq P^{\frac{1}{p}} Q^{\frac{1}{q}}$$

Since we know that

$$\left| \int_a^b fg \, d\alpha \right| \leq \int_a^b |fg| \, d\alpha \leq \int_a^b |f||g| \, d\alpha$$

and that

$$P^{\frac{1}{p}} Q^{\frac{1}{q}} = \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}}$$

we conclude that

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}}$$

(d)