# Ordinary differential equations

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## What is this?

These are notes I am taking for the class MATH-623, *Ordinary Differential Equations*, at Drexel University, taught by Yixin Guo.

Some notation is changed from her notes, and I try to add as many missing details from proofs as possible.

## Contents

# 1 Basic theory

## 1.1 Definitions

> **Definition 1.1.1.** Let $J \subseteq \mathbb{R}$, $U \subseteq \mathbb{R}^n$, $\Lambda \subseteq \mathbb{R}^k$ be open sets, and let $\mathbf{f} : J \times U \times \Lambda \to \mathbb{R}^n$ is a smooth function. An **ordinary differential equation** is an equation of the form
>
> $$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}) \tag{ODE}$$
>
> where the dot denotes differentiation with respect to the independent variable $t$.

Morally, the individual parts of an ODE have the following meaning:

$t \in J$: an independent variable, typically time,

$\mathbf{x} \in U$: a dependent variable,

$\underline{\lambda} \in \Lambda$: a vector of parameters,

$\mathbf{f}$: a continuously differentiable function that encodes the (time) evolution of $\mathbf{x}$.

> **Definition 1.1.2.** A **solution** of (ODE) is a function $\mathbf{F} : J_0 \to U$, where $J_0 \subseteq J \subseteq R$, such that
>
> $$\frac{d}{dt}\mathbf{F}(t) = \mathbf{f}(t, \mathbf{F}(t), \underline{\lambda}), \qquad \forall t \in J_0$$
>
> i.e, a function for which we can put $\mathbf{x} = \mathbf{F}(t)$ in (ODE) for all $t \in J_0$.
>     The **orbit** of the solution $\mathbf{F}$ is the set
>
> $$\{\mathbf{F}(t) \in U : t \in J_0\} \subseteq \mathbb{R}^n.$$
>
> This is also called the **trajectory**, **integral curve**, or **solution curve**.

Evidently, an ODE can have many solutions. For example, if $J = U = \Lambda = \mathbb{R}$ and $f(t, x, \lambda) = \lambda x$, then it's well known that $F(t) = Ce^{\lambda t}$ is a solution to this ODE for all $C \in \mathbb{R}$.

Fortunately, both real-life and abstract experience tells us that in many cases, the following is a natural way to restrict solutions to an ODE.

> **Definition 1.1.3.** An **initial-value problem** (IVP) is the system of equations
>
> $$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \underline{\lambda}), \\ \mathbf{x}(t_0) = \mathbf{x}_0. \end{cases} \tag{IVP}$$

Namely, it is an (ODE) with additional data $(t_0, \mathbf{x}_0) \in J \times U$, encoding an **initial value** constraint.

A **solution** of (IVP) is, again a function $\mathbf{F} : J_0 \to U$ that solves the underlying ODE, but now subject to the condition that $\mathbf{F}(t_0) = \mathbf{x}_0$.

**Example 1.1.4.** The **forced Van der Pol** equation is defined to be

$$
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = b(1 - x_1^2)x_2 - \omega^2 x_1 + a \cos \Omega t.
\end{cases}
\tag{fVdP}
$$

## 1.2 Existence and uniqueness

Let $J, U \subseteq \mathbb{R}$, and consider the IVP

$$
\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0
\tag{IVP-1D}
$$

where $f : J \to U$ is continuously differentiable in $t$ on some open interval containing $t_0$.

Supposing we can integrate (IVP-1D) from $t_0$ to a given point $t$,

$$
\int_{t_0}^{t} \frac{dy}{d\tau} d\tau = \int_{t_0}^{t} f(\tau, y) d\tau,
$$

$$
y(t) - y(t_0) = \int_{t_0}^{t} f(\tau, y) d\tau,
$$

$$
y(t) = y(t_0) + \int_{t_0}^{t} f(\tau, y) d\tau,
\tag{$\int$IVP-1D}
$$

$$
y(t) = y_0 + \int_{t_0}^{t} f(\tau, y) d\tau.
$$

If $F$ satisfies ($\int$IVP-1D), it satisifes (IVP-1D) and vice versa, and this is easily seen using the fundamental theorem of calculus.

**Definition 1.2.1.** The **Picard iterates** $y_i$, given the data for (IVP-1D), are defined recursively as follows:

$$
\begin{cases}
y_0(t) := y_0, \\
y_{i+1}(t) := y_0 + \int_{t_0}^{t} f\big(\tau, y_i(\tau)\big) d\tau.
\end{cases}
\tag{Picard}
$$

We have a weak but straightforward estimate on the $y_i$.

**Lemma 1.2.2.** For all $a > 0, b > 0$, define $R$ to be the rectangle $[t_0, t_0 + a] \times [y_0 - b, y_0 + b]$. Then if we define

$$M := \max_{(t, y) \in R} |f(t, y)|, \qquad \alpha := \min\left\{a, \frac{b}{M}\right\}.$$

Then

$$|y_n(t) - y_0| \leq M(t - t_0) \text{ for all } t_0 \leq t \leq t + \alpha \qquad (1)$$

for all $n$.

*Proof.* We prove this by induction.

We note that if (1) holds for $n$, then for all $t_0 \leq t \leq \alpha$,

$$\begin{aligned}
|y_n(t) - y_0| &\leq M(t - t_0) \\
&\leq M\alpha \\
&= M \cdot \min\left\{a, \frac{b}{M}\right\} \\
&= \min\left\{\frac{a}{M}, b\right\} \\
&\leq b.
\end{aligned}$$

Hence

$$|f(t, y_n(t))| \leq M. \qquad (2)$$

$n = 0$: We have that $|y_0(t) - y_0| = 0 \leq M(t - t_0)$ trivially for *all* $t \geq t_0$.

$n > 0$: Suppose that $|y_n(t) - y_0| \leq M(t - t_0)$ for all $t_0 \leq t \leq t_0 + \alpha$. Then

$$\begin{aligned}
|y_{n+1}(t) - y_0| &= \left|\int_{t_0}^{t} f(\tau, y_n(\tau)) d\tau\right| \\
&\leq \int_{t_0}^{t} \underbrace{|f(\tau, y_n(\tau))|}_{\text{use (2)}} d\tau \\
&\leq \int_{t_0}^{t} M d\tau \\
&= M(t - t_0)
\end{aligned}$$

for all $t_0 \leq t \leq t_0 + \alpha$. This completes the proof.

□

Next we show that the the Picard iterates $y_n$ converge.

**Theorem 1.2.3.** Let $f(t, y)$ be continuously differentiable in both $t$ and $y$. Then, on $[t_0, t_0 + \alpha]$, the Picard iterates $y_n$ converge pointwise to a function $y$ that satisifes (IVP-1D).

*Proof.* We have that

$$y_n(t) = y_0(t) + [y_1(t) - y_0] + \cdots + [y_n(t) - y_{n-1}(t)]$$

$$= y_0(t) + \sum_{i=1}^{n} y_i(t) - y_{i-1}(t).$$

So, $y_n(t)$ converges if and only if $\sum_{i=1}^{n} y_i(t) - y_{i-1}(t)$ converges.
It suffices to prove that $\sum_{i=1}^{n} |y_i(t) - y_{i-1}(t)|$ converges.
We compute that

$$|y_i(t) - y_{i-1}(t)| = \left| \left[ y_0 + \int_{t_0}^{t} f(\tau, y_{i-1}(\tau))d\tau \right] - \left[ y_0 + \int_{t_0}^{t} f(\tau, y_{i-2}(\tau))d\tau \right] \right|$$

$$= \left| \int_{t_0}^{t} f(\tau, y_{i-1}(\tau)) - f(\tau, y_{i-2}(\tau))d\tau \right|$$

$$\leq \int_{t_0}^{t} \underbrace{\left| f(\tau, y_{i-1}(\tau)) - f(\tau, y_{i-2}(\tau)) \right|}_{\text{apply mean value theorem}} d\tau$$

$$= \int_{t_0}^{t} \left| \frac{\partial f(\tau, c(\tau))}{\partial y} \right| \left| y_{i-1}(\tau) - y_{i-2}(\tau) \right| d\tau$$

$$\leq P \int_{t_0}^{t} \left| y_{i-1}(\tau) - y_{i-2}(\tau) \right| d\tau.$$

Now, we can show inductively that

$$|y_i(t) - y_{i-1}(t)| \leq M P^{i-1} \frac{(t - t_0)^i}{i!} \tag{3}$$

for all $i \geq 1$.
The base case $i = 1$ ends up being (1)

$$|y_1(t) - y_0(t)| = |y_1(t) - y_0| \leq M(t - t_0).$$

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For the induction step, supppose it were true for $i$, then

$$|y_{i+1}(t) - y_i(t)| \leq P \int_{t_0}^{t} \underbrace{|y_{i-1}(\tau) - y_i(\tau)|}_{\text{apply induction hypothesis}} d\tau$$

$$\leq P \int_{t_0}^{t} \left[ M P^{i-1} \frac{(\tau - t_0)^i}{i!} \right] d\tau$$

$$= M P^i \int_{t_0}^{t} \frac{(t - t_0)^i}{i!} d\tau$$

$$= M P^i \frac{(t - t_0)^{i+1}}{(i+1)!}.$$

Hence, (3) holds for all $i$.

So

$$\sum_{i=1}^{n} |y_i(t) - y_{i-1}(t)| \leq \sum_{i=1}^{n} M P^{i-1} \frac{(t - t_0)^i}{i!}$$

$$= M P^{-1} \left( \sum_{i=1}^{n} P^i \frac{(t - t_0)^i}{i!} \right)$$

$$= M P^{-1} \left( -1 + \sum_{i=0}^{n} P^i \frac{(t - t_0)^i}{i!} \right)$$

$$= M \frac{e^{P(t - t_0)} - 1}{P},$$

which is a finite number.

So, we have shown that $y_n(t) \rightarrow y(t)$ for some function $y$.

Next, we show that $y$ satisifes (IVP-1D).

$\square$