Noncommutative Schur functions

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What is this?

This is (going to be) an "infinite napkin" set of notes I am taking about noncommutative Schur functions.

Contents

1 Ideals and words

Let $\mathbf{u} = (u_1, \ldots, u_N)$ be a collection of variables. Let $\langle \mathbf{u} \rangle$ be the free semigroup on the generators $\mathbf{u}$. Then, let $\mathcal{U} = \mathbb{Z}\langle \mathbf{u} \rangle$ denote the corresponding semigroup ring— the free associative ring generated by $\mathbf{u}$.

We will denote by $\mathcal{U}^*$ the $\mathbb{Z}$-module spanned by words in the alphabet $\{1, \ldots, N\}$.

We will have a fundamental pairing $\langle -, - \rangle$ given by making noncommutative monomials dual to words.

Now, if I is an ideal of $\mathcal{U}$, we define $I^\perp$ by

$$I^\perp := \{\gamma \in \mathcal{U}^* \mid \langle I, \gamma \rangle = 0\}.$$

So that the elements of $\mathcal{U}/I$ are in correspondence with words in $I^\perp$.

2 Noncommutative e's and h's

Definition 2.1. The **noncommutative elementary symmetric function** $e_k(\mathbf{u})$ is defined to be

$$e_k(\mathbf{u}) := \sum_{i_1 > i_2 > \cdots > i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

The **noncommutative complete homogeneous symmetric function** $h_k(\mathbf{u})$ is defined to be

$$h_k(\mathbf{u}) := \sum_{i_1 \geq i_2 \geq \cdots \geq i_k} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

2.1 Commutation relations

Lemma 2.2. Let I be an ideal. If the e's commute modulo I, then the h's commute modulo I as well, and vice versa.

Definition 2.3. We define the ideal I_C to be the ideal consisting of exactly the elements

$$u_b^2 u_a + u_a u_b u_a - u_b u_a u_b - u_b u_a^2 \qquad (a < b), \qquad (1)$$

$$u_b u_c u_a + u_a u_c u_b - u_b u_a u_c - u_c u_a u_b \qquad (a < b < c), \qquad (2)$$

$$u_c u_b u_c u_a + u_b u_c u_a u_c - u_c u_b u_a u_c - u_b u_c^2 u_a \qquad (a < b < c). \qquad (3)$$

Theorem 2.4. I_C is the smallest ideal in which the elementary symmetric functions $e_k(\mathbf{u}_S)$ and $e_\ell(\mathbf{u}_S)$ commute for any k, ℓ, S.

3 Noncommutative Schur functions

Definition 3.1. The **noncommutative Schur function** $\mathfrak{J}(\mathbf{u})$ is defined to be

$$\mathfrak{J}_\lambda(\mathbf{u}) \coloneqq \sum_{T \in \mathrm{SSYT}(\lambda;N)} \mathbf{u}^{\mathsf{colword}\,T}.$$

Theorem 3.2 ([FG98], [BF16]). In the ideal $I_\varnothing$,

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{\pi \in S_t} \mathrm{sgn}(\pi) e_{\lambda'_1+\pi(1)-1}(\mathbf{u}) e_{\lambda'_2+\pi(2)-2}(\mathbf{u}) \cdots e_{\lambda'_t+\pi(t)-t}(\mathbf{u}),$$

where $t = \lambda_1$.

Corollary 3.3 ([FG98]). If I contains the ideal $I_\varnothing$, the map

$$\begin{aligned} \Lambda_n &\to \mathcal{U}/I \\ s_\lambda &\mapsto \mathfrak{J}_\lambda \end{aligned}$$

defines an algebra homomorphism.

3.1 Cauchy kernel

Definition 3.4. Let $\mathbf{x} = (x_1, x_2 \ldots)$ be a countable collection of commuting variables.

4 Stuff

Definition 4.1. A **combinatorial representation** of $\mathcal{U}/I$ is

Definition 4.2.

5 Appendix

5.1 Gessel's fundamental quasisymmetric function

5.2 The Edelman-Greene correspondence

References

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