DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

CSE2315 Lab Course
Solutions 6

deadline: March 24, 2020, 15:45

1. Suppose we have the following language:

$$E_{\textsf{REX}} = \{\langle R\rangle \mid R \text{ is a regular expression such that } L(R) = \varnothing\}.$$

Show that $E_{\textsf{REX}}$ is decidable. You are allowed to make use of the deciders implied by the text in the book (e.g., Section 4.1).

Solution: The following TM M decides $E_{\textsf{REX}}$:

$M =$ "On input $\langle R\rangle$:

1. Convert R into a DFA D.
2. Run the decider for $E_{\textsf{DFA}}$ on $\langle D\rangle$.
3. If the decider accepts, accept. Otherwise, reject."

2. Consider the language

$$L = \{\langle M, w\rangle \mid M \text{ halts on } w \text{ and visits only the first } |w|^2 \text{ tape positions}\}.$$

Is this language decidable? Explain.

Solution: L is decidable. We can show this much like the proof of Theorem 5.9 in Sipser.

Namely, given M and w, there are only a finite number of configurations for M that use the first $|w|^2$ tape positions, say k of them, and we can compute k based on M and w. We can let M run on w for k steps. If M has halted without using tape after position $|w|^2$, we accept. Else M either has looped, has used tape after position $|w|^2$, or will do so in the next step. In all these cases, we can reject with certainty.

3. We define $EQC_{\textsf{TM}}$ as:

$$EQC_{\textsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are Turing machines with } L(M_1) = \overline{L(M_2)}\}.$$

Use direct reduction to show that $A_{\textsf{TM}} \leq EQC_{\textsf{TM}}$.

Solution: Suppose we have a decider R for $EQC_{\textsf{TM}}$. We construct a TM S that decides $A_{\textsf{TM}}$ as follows:

$S =$ "On input $\langle M, w\rangle$:

1. Construct the TM M_1:

   $M_1 =$ "On input x :

   1. Run M on w.

2. If M accepts w, accept. Else reject."

2. Construct the TM M_2:

   $M_2 =$ "On input x :

   1. Reject."

3. Run R on $\langle M_1, M_2 \rangle$.

4. If R accepts, accept. Otherwise, reject."

We claim S is indeed a decider for A_{TM}. That is, we claim S accepts $\langle M, w \rangle$ if $\langle M, w \rangle \in A_{\mathsf{TM}}$ and rejects it otherwise.

First note that $L(M_2) = \varnothing$.

If $\langle M, w \rangle \in A_{\mathsf{TM}}$, then M accepts w. This implies $L(M_1) = \Sigma^*$. Hence, $L(M_1) = \overline{L(M_2)}$ and R accepts. Consequently, S accepts also, as required.

If $\langle M, w \rangle \notin A_{\mathsf{TM}}$, then M does not accept w: it either loops or rejects w. In both cases, this implies $L(M_1) = \varnothing$. Hence, $L(M_1) \neq \overline{L(M_2)}$, and R rejects. Consequently, S rejects also, as required.

Therefore, $A_{\mathsf{TM}} \leq EQC_{\mathsf{TM}}$, as desired.

(Neither M_1 nor M_2 use the input they get!)