Rice's theorem

Let P be a set of Turing machines satisfying:

1. P is exclusively defined in terms of input/output behavior of TMs; that is to say, if $L(M_1) = L(M_2)$, then $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$;
2. P is not trivial, which is to say, $P \neq \emptyset$ and $\overline{P} \neq \emptyset$.

Then P is not decidable.

Proof (1)

Let $M_\emptyset$ be a TM that recognizes the empty language by rejecting every input. This means that $L(M_\emptyset) = \emptyset$.
If $\langle M_\emptyset \rangle \notin P$, then we perform the reduction $HALT_{\mathsf{TM}} \leq_m P$, otherwise $HALT_{\mathsf{TM}} \leq_m \overline{P}$. In both cases, the reduction is done the same way.
Suppose $\langle M_\emptyset \rangle \notin P$. Also let $\langle M_P \rangle \in P$ (we know that $P \neq \emptyset$).

Proof (2)

The reduction f is defined as follows:

$$f(\langle M, w\rangle) = \langle M_1\rangle,$$

with:

$$M_1 \quad = \quad \text{“On input } x\text{:}$$

1. Run M on input w.
2. Run M_P on input x.
3. Return the output of M_P on x.”

Proof (3)

Now we have:

- If $\langle M, w \rangle \in HALT_{\text{TM}}$, then $L(M_1) = L(M_P)$ (since M_1 always reaches steps 2 and 3). But then $\langle M_1 \rangle \in P$, since $\langle M_P \rangle \in P$.
- If $\langle M, w \rangle \notin HALT_{\text{TM}}$, then M_1 will not reach a halting state while executing step 1, so that $L(M_1) = \emptyset = L(M_\emptyset)$. Therefore, $\langle M_1 \rangle \notin P$, since $\langle M_\emptyset \rangle \notin P$.

Since f is a computable function, we have now presented a reduction. The other case is analogous.