DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TI2316 Lab Course
Solutions 8

deadline: June 24, 2019, 13:45

1. Use Rice's theorem to show that the following language is not decidable:

$$L = \{\langle M \rangle \mid \text{ if } M \text{ accepts } w \in \Sigma^*, \text{ then } M \text{ does not accept } w^2\}.$$

You may assume that all TMs have a common alphabet Σ.

Solution: L must satisfy the two conditions of Rice's theorem.

First, consider two machines M_1 and M_2 with equal languages. Assume $\langle M_1 \rangle \in L$. We have to show $\langle M_2 \rangle \in L$. Let w be an arbitrary word that M_2 accepts. Then M_1 also accepts it, because the languages are equal. Because $\langle M_1 \rangle \in L$, we know $w^2 \notin L(M_1)$. Since $L(M_1) = L(M_2)$, also $w^2 \notin L(M_2)$. Since w was arbitrary, $\langle M_2 \rangle \in L$.

Now assume $\langle M_1 \rangle \notin L$. We have to show $\langle M_2 \rangle \notin L$. Because $\langle M_1 \rangle \notin L$, there exists a word $w \in \Sigma^*$ such that M_1 accepts w as well as w^2. Since $L(M_1) = L(M_2)$, w is also a word such that M_2 accepts w as well as w^2. Hence $\langle M_2 \rangle \notin L$.

Second, L is not trivial: there is a machine that accepts the empty language, so that the machine is in L; and there is a machine with accepts the language $\{\varepsilon\}$, so that the machine is not in L. (Note that $\varepsilon^2 = \varepsilon$.)

The conditions of Rice's theorem are satisfied, so L is undecidable.

For the first part, we can alternatively write the property of L as

$$w \in L(M) \to w^2 \notin L(M).$$

Then it is immediate that the property only depends on $L(M)$, and not directly on M itself.

2. Suppose we have the following language:

$$L = \{\langle M_1, M_2 \rangle \mid \text{For all } w \in L(M_1), \text{ it holds } w^{|w|} \in L(M_2)\}.$$

You may assume that all TMs in this language have a common alphabet $\Sigma = \{a, b, c, d\}$.

Use mapping reduction to show that this language is undecidable. Give:

(a) A suitable reduction function $f : X \to L$ (also give a suitable problem X to reduce from).
Hint: don't be scared by the notation $w^{|w|}$. This does not make the reduction much more difficult than others: recall in what direction the reduction should go.

Solution: We reduce from $HALT_{\mathsf{TM}}$. The reduction f is computed by the following TM F:

F = "On input $\langle M, w \rangle$:

1. Construct the following TM M_1:

   M_1 = "On input x:

   1. If $x = abc$, accept.
   2. Run M on w.
   3. Accept."

2. Write $\langle M_1, M_1 \rangle$ to the output tape."

(b) A proof that f satisfies the requirements of a mapping reduction.

Solution:

Proof. We need to show that f is computable and reliable. It is clearly computable, because F computes it. To show reliability, we consider both directions:

- $\langle M, w\rangle \in HALT_{\mathsf{TM}} \Rightarrow f(\langle M, w\rangle) \in L$:

  Suppose M halts on w. Then $L(M_1) = \Sigma^*$. And indeed for each word $w \in \Sigma^*$, also $w^{|w|} \in \Sigma^*$. Hence $\langle M_1, M_1\rangle \in L$.

- $\langle M, w\rangle \notin HALT_{\mathsf{TM}} \Rightarrow f(\langle M, w\rangle) \notin L$:

  Suppose M does not halt on w. Then $L(M_1) = \{abc\}$. For the word $w = abc \in L(M_1)$, we have $w^{|w|} = abcabcabc$, and so $w^{|w|} \notin L(M_1)$. Hence $\langle M_1, M_1\rangle \notin L$.

□

Note that we use the same machine M_1 twice in the output. Also note that there are other possibilities to define the language of M_1 in the case $\langle M, w\rangle \notin HALT_{\mathsf{TM}}$ than $\{abc\}$.

3. One might claim that DFAs and LBAs are equivalent in terms of computation power because both can only use a bounded amount of memory.

(a) Explain why this reasoning fails to hold.

Solution: Although bounded, the memory of an LBA does depend on the input length: longer input implies more memory for an LBA. This is not true for a DFA.

(b) Give a language that can be decided by an LBA, but not by a DFA.

Solution: Consider the alphabet $\Sigma = \{a, b\}$ and language

$$L = \{ww^{\mathcal{R}} \mid w \in \Sigma^*\}.$$

This language is not regular (i.e., not decidable by a DFA), which you can prove using the pumping lemma.

To see that L is decidable by an LBA, consider a machine that zigzags over the input, marking symbols each time. More precisely, it marks the first symbol, then moves to the very right and checks if the symbol is equal. If not, it rejects. Else, it marks it, goes to the first unmarked symbol, and does the same thing (comparing the second symbol with the penultimate one), etc. When all symbols are marked, it accepts. The machine only uses the tape positions of the input, so it is indeed an LBA.