DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

CSE2315 Lab Course
Solutions 7

deadline: March 31, 2020, 15:45

1. Prove that the following problem is in NP:

$$\{(V,E) \mid \text{The graph } G=(V,E) \text{ contains a simple path } \pi \text{ such that } \forall v \in V : v \in \pi\}$$

In a different notation we would write:

Name: Hamiltonian Path (HAMP)

Input: A graph $G = (V, E)$.

Question: Is there a simple path π in G such that all vertices $v \in V$ are in that path?

> **Solution:** Consider the following algorithm, given a graph $G = (V, E)$ and a path π.
>
> (a) Check if every vertex $v \in V$ occurs in π exactly once.
>
> (b) Check for every consecutive pair of vertices in $(v_i, v_{i+1}) \in \pi$ that $(v_i, v_{i+1}) \in E$.
>
> The first step takes $|V| \times |\pi|$ time, the second takes $|\pi| \times |E|$ time. Since the input $I = (V, E, \pi)$, this makes the time complexity $O(|I|^2)$ thus the problem can be verified in polynomial time, thus the problem is in NP.

2. Given the following information, what can we conclude about the problems A, B and C in their relation to the complexity classes P, NP, and NPC?

- $A \in$ NPC
- $B \leq A$
- $A \leq C$

> **Solution:** For A: since is it in NPC,it must also be in NP. For B: since it can be reduced to A, A must be "at least as hard as" B, so B is in NP. For C: since it is at least as hard as A, we do not know anything for certain in relation to P, NP, and NPC. It is however NP-hard. If we can show that C is also in NP, then C would be in NPC, but these statements do not give us this information.

3. (Recap) Use Rice's theorem to show that the following language is not decidable:

$$L = \{\langle M \rangle \mid \text{ if } M \text{ accepts } w \in \Sigma^*, \text{ then } M \text{ does not accept } w^2\}.$$

You may assume that all TMs have a common alphabet Σ.

Solution: L must satisfy the two conditions of Rice's theorem.

First, consider two machines M_1 and M_2 with equal languages. Assume $\langle M_1 \rangle \in L$. We have to show $\langle M_2 \rangle \in L$. Let w be an arbitrary word that M_2 accepts. Then M_1 also accepts it, because the languages are equal. Because $\langle M_1 \rangle \in L$, we know $w^2 \notin L(M_1)$. Since $L(M_1) = L(M_2)$, also $w^2 \notin L(M_2)$. Since w was arbitrary, $\langle M_2 \rangle \in L$.

Now assume $\langle M_1 \rangle \notin L$. We have to show $\langle M_2 \rangle \notin L$. Because $\langle M_1 \rangle \notin L$, there exists a word $w \in \Sigma^*$ such that M_1 accepts w as well as w^2. Since $L(M_1) = L(M_2)$, w is also a word such that M_2 accepts w as well as w^2. Hence $\langle M_2 \rangle \notin L$.

Second, L is not trivial: there is a machine that accepts the empty language, so that the machine is in L; and there is a machine with accepts the language $\{\varepsilon\}$, so that the machine is not in L. (Note that $\varepsilon^2 = \varepsilon$.)

The conditions of Rice's theorem are satisfied, so L is undecidable.

For the first part, we can alternatively write the property of L as

$$w \in L(M) \to w^2 \notin L(M).$$

Then it is immediate that the property only depends on $L(M)$, and not directly on M itself.