DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TI2316 Lab Course
Solutions 6

deadline: June 11, 2019, 13:45

1. Consider the alphabet $\Sigma = \{a, b\}$ and the language

$$L = \{ww^{\mathcal{R}} \mid w = (ab)^k, k \geq 1\}.$$

Give a high-level description for a three-tape enumerator that enumerates L. You may want to use the second and third tapes as counters. You do not have to give implementation details of the counters.

Solution: The following enumerator M enumerates L:
"On empty input:

1. Write 1 on the third tape.
2. Write 0 on the second tape.
3. Clear the first tape and move the head to the very left.
4. Write an a on the first tape and move the head to the right.
5. Write a b on the first tape and move the head to the right.
6. Increment the counter of the second tape.
7. If this counter is less than the counter on the third tape, go to step 4.
8. Write a b on the first tape and move the head to the right.
9. Write an a on the first tape and move the head to the right.
10. Decrement the counter of the second tape.
11. If this counter is more than zero, go to step 8.
12. At this point, a new word in L is on the first tape. Increment the counter on the third tape and go to step 2."

2. Suppose we have the following language:

$$L = \{\langle M \rangle \mid M \in \mathbb{N}^{k \times k} \text{ is an arrowhead matrix}\}.$$

An *arrowhead matrix* is a matrix for which the entries not in the first row, column or on the main diagonal are zero. See `https://en.wikipedia.org/wiki/Arrowhead_matrix`.

We are going to create a decider for L. A first step is designing an appropriate encoding.

(a) Consider the following way to encode an arbitrary matrix $M \in \mathbb{N}^{k \times k}$. We use the alphabet $\Sigma = \{(,), 0, 1, \ldots, 9\}$ and encode k as well as the entries $m_{i,j}$ in M (row i, column j) with the following format:

$$\langle M \rangle = (k)(m_{1,1} \ldots m_{1,k} \ldots m_{k,1} \ldots m_{k,k}),$$

where k and the $m_{i,j}$ are represented as decimal numbers.

Explain why this encoding is not appropriate.

Solution: We do not know where the entries start and end. For example, there are multiple matrices that encode to

$$(3)(1111110101).$$

Some of these matrices are arrowhead and some are not.

(b) Adapt the alphabet and encoding so that it becomes appropriate.

Solution: Separate the entries, e.g. with a comma (and add a comma to Σ):

$$\langle M \rangle = (k)(m_{1,1}, \ldots, m_{1,k}, \ldots, m_{k,1}, \ldots, m_{k,k}).$$

Note that separating the rows is not needed as we know k.

(c) The following is an incomplete high-level description of a two-tape Turing machine that decides L. Assume an appropriate encoding of the input.

"On input $\langle M \rangle$:

1. Read k out of the input.
2. Repeat the following for $i = \boxed{\text{A}}$.
   i. Repeat the following for $j = \boxed{\text{B}}$.
      a. If $\boxed{\text{C}}$, continue with the next j (if any). Else do the following steps.
      b. Move the head of the first tape to the $\boxed{\text{D}}$'s matrix element. Let x be this element.
      c. If $\boxed{\text{E}}$, reject.
3. Accept.

Please fill in each of the boxes. Note: $\boxed{\text{A}}$ and $\boxed{\text{B}}$ should represent intervals, $\boxed{\text{C}}$ should represent a condition on i and j, $\boxed{\text{D}}$ should represent a (single) one-based index, and $\boxed{\text{E}}$ should represent a condition.

Solution:

$\boxed{\text{A}}$: $2, \ldots, k$

$\boxed{\text{B}}$: $2, \ldots, k$

$\boxed{\text{C}}$: $i = j$

$\boxed{\text{D}}$: $(i-1)k + j$ or $(j-1)k + i$

$\boxed{\text{E}}$: $x \neq 0$

3. In the game of chess, consider an intermediate board configuration, and suppose it's white's move. We know that white is either guaranteed to win or not. Let

$$L = \{w \mid w \text{ represents a board configuration, and white is guaranteed to win if it is white's move and white plays optimally}\}.$$

Is L recognizable, decidable, or neither? Explain.

Solution: L is decidable because it is finite. We can create a TM with all words in L hardcoded in its state machine. Coming up with an explicit description for such a TM is practically difficult, but that's another story. We do know there exists some TM that decides L, so L is decidable.