Assignment 4

My score

93% (9.3/10)

Q1

9.3 / 10

Chapter 3: Problems 5, 14, 17, 23, 25, 27, 29, 35, 39, 60, 61

When submitting your solutions for assignment 4, please include the signed and dated Academic Integrity Checklist as the cover page, with your name clearly typed or printed.

5.

there are 3 w, 6 r, 5 b balls.

selec 6.

X = w selected

Y = b ball selec i white selected

$$E[X|Y=3] = \frac{C_3^i C_6^{3-i}}{C_9^3}$$

$$\cancel{E[X|Y=1] = \frac{C_3^i \cdot C_6^{3-i}}{C_9^5}} \quad i = 0,1,2,3.$$

$$\cancel{= \frac{C_6^3 + C_3^1 \cdot C_6^2 + C_3^2 \cdot C_6^1 + C_3^3 \cdot C_6^0 + 1}{C_9^5}}$$

$$\cancel{= \frac{6!}{3!\,3!} + \frac{3!}{2!} \cdot \frac{6!}{2!\,4!} + \frac{3!}{2!} \cdot \frac{6!}{5!} + 1} \Big/ C_9^3$$

$$\cancel{= \frac{4 \times 5 \times 8}{2 \times 3} + 3 \cdot \frac{5 \times 6}{2} + 3 \cdot 6 + 1} \Big/ C_9^3$$

$$= \cancel{\frac{20+45+18+1}{\frac{9!}{3!\,6!}}}$$

$$= \cancel{84}$$

Because ~~they~~ $Y=1$. means their probability is same

so

$$E(X \mid Y=1) = 5 \times 3/9 = 5/3$$

14. X is the unifor over (0.1).

$\therefore f(x) = \frac{1}{b-a}$

~~f(x)~~ $f(X \le \frac{1}{2}) = \frac{1}{\frac{1}{2}-0} = 2$

$E[X \mid X < \frac{1}{2}] = \int_0^{\frac{1}{2}} x f(x) dx$

$= \int_0^{\frac{1}{2}} x \cdot 2 \, dx$

$= 2 \frac{1}{2} x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{4}$

17.

$f_Y(y) = Ce^{-ay} y^{s-1}$

$p\{X=i \mid Y=y\} = e^{-y} y^i / i! \quad i \ge 0.$

So we want to calculate the $f_{X|Y}(X=i \mid Y=y)$ first.

~~$f_{X|Y}(X=i \mid Y=y) = dF(X,Y) / dy(X,Y)$~~

$f_{X|Y}(X=i \mid Y=y) = p\{X=i \mid Y=y\} \cdot f_Y(y) = e^{-y} y^i / i! \cdot Ce^{-ay} y^{s-1}$

$= e^{-y} y^i C e^{-ay} y^{s-1} / i!$

$$= e^{-(1+a)y} c y^{i+s-1} / i!$$

Then

$$f_X(X=i) = \int_0^\infty f_{XY}(X=i|Y=y)\, dy = \int_0^\infty e^{-(1+a)y} c \cdot y^{i+s-1} / i!\, dy$$

$$= c y \frac{1}{1+a} e^{-(1+a)y} \cdot \frac{1}{i+s} y^{i+s} \Big|_0^\infty / i!$$

$$= c y \frac{1}{1+a} e^{-(1+a)y} \cdot \frac{1}{i+s} y^{i+s} / i!$$

Because

$$f_Y(Y=y \mid X=i) = f_{XY}(i,y) \Big/ f_X(i)$$

$$= \frac{c\,y\,e^{-(1+a)y}\,y^{i+s-1} / i!}{c\,\frac{1}{1+a}\,e^{-(1+a)y}\cdot\frac{1}{i+s}\,y^{i+s} / i!}$$

$$\cancel{=(1+a)(i+s)\,c\,(y)}$$

$$= e^{-(a+1)y}\cdot y^{i+s-1} \Big/ \int_0^{\infty} \left(e^{-(a+1)y}\cdot y^{i+s-1}\right) dy$$

and the gamma distribution with parameter $(s+i, a+1)$

$\therefore$ we can find that

$$f_{Y|X}(Y=y \mid X=i) = e^{-(a+1)y}\cdot y^{i+s-1} \Big/ \int_0^{\infty} e^{-(a+1)y}\cdot y^{i+s-1}\,dy$$

is density function of $\Gamma(s+i, a+1)$.

23.

This is a 0-1 distribution

$P(H) = p$
$p(T) = 1-p$

let X is the first H appear.

$E[N|X] = E[N|XHH] \cdot p^2 + E[N|XHT] \cdot p(1-p)$

$+ E[N|XTH] \cdot p(1-p) + E[N|XTT](1-p)^2$

$\because$ first two flips are head mean $N=2$

$\therefore E[N|XHH] = X+1$ $\quad E[N|XHT] = X+1$

$E[N|XTH] = X+2$ $\quad E[N|XTT] = X+2+E[N]$

$E[N|X] = (X+1)p^2 + (X+1)p(1-p) + (X+2)p(1-p) + (X+2+E[N])(1-p)^2$

$= (X+1)p^2 + p(1-p)(\ldots$

$+ \quad p(1-p)(2x+3) + (x+2+E[N])(1-p)^2$

$\cancel{E(E[N|X]) = E[N]} =$

$E[X] = \frac{1}{p}$

$E[N] = E[N|X] \cdot E[X]$??

$= (x+1)p + (x+1)(1-p) + (x+2)(1-p) + (x+2+E[N])(1-p)^2/p$

$=$

25

(a) let initial outcome is X

$E(N) = E(E(N|X))$

$$= \sum_{i=1}^{3} E[N|X=i]p_i$$

$$= E[N|X=1]p_1 + E[N|X=2]p_2 + E[N|X=3]p_3$$

$$= \sum_{i=1}^{3} (1+\frac{2}{p_i})p_i$$

(b) $X_{1,2}$ be the number of trials until both 1 and 2 occured.

$$E[X_{1,2}] = E[X_{1,2}|X=1]p_1 + E[X_{1,2}|X=2]p_2 + E[X_{1,2}|X=3]p_3$$

$$= (1+\frac{1}{p_2})p_1 + (1+\frac{1}{p_1})p_2 + (1+E[1,2])p_3$$

$$= 1 + p_1/p_2 + p_2/p_1 + p_3 E[X_{1,2}]$$

$$E[X_{1,2}] = \frac{1+\frac{p_1}{p_2}+\frac{p_2}{p_1}}{1-p_3}$$

$\therefore$

$\therefore$ $p_1 + p_2 + p_3 = 1$ $\qquad p_3 = 1-(p_1+p_2)$

$\because p_1+p_2+p_3=1 \qquad p_3 = 1-p_1-p_2$

$$\therefore E[X_{1,2}] = \frac{1+p_1/p_2+p_3/p_1}{p_1+p_2}.$$

2)

A = { first is H }
B = { second is TH }
C = { third is TT }

$E[X] = E[E[X \mid X_{TTH}]]$

$= \cancel{E[A] \cdot p + E}$

$= E[X \mid A] \cdot p(A) + E[X \mid B]\, p(B) + E[X \mid C] \cdot p(C)$

$p(A) = p$

$p(B) = p(1-p)$

$p(C) = (1-p)^2$

$E[X \mid A] = E[X] + 1$

$E[X \mid B] = E[X] + 2$

$E[X \mid C] = 2 + \frac{1}{p}$

$E[X] = (E[X]+1) \cdot p + (E[X]+2)\, p(1-p) + (2 + \frac{1}{p})(1-p)^2$

$E[X] = \frac{1}{p(1-2p+p^2)} = 1 / p(1-p)^2$

29.

(a) ~~$\mu_1 = E$~~

μ_i mean number of shot.

Shoot end if shot twice to target.

player 1 success is p_1 → $1-p_1$
player 2 success is p_2. → $1-p_2$.

~~the first success shot image number is X_1 player 1 → X_1~~
shot is X_1 if miss is X_2

~~$\mu_1 = E(E(\mu_1 | X_1))$~~

N is the total shooting number

~~$E[\mu_1 | X_1] = E[\mu_1] \mu_1 p_1 =$~~

$\mu_1 = E(N | X_1) p_1 + E(N | X_2)(1-p_1)$
$= E(N | X_1 X_1) p_2 p_1 + E(N | X_1, X_2) p_1 (1-p_2) + (1+\mu_2)(1-p_1)$
~~$= E(N|$~~ ~~p_1~~
$= 2 p_1 p_2 + (2+\mu_1) p_1 (1-p_2) + (1+\mu_2)(1-p_1)$

$\mu_2 = E(N | X_1) p_2 + E(N | X_2)(1-p_2)$
$= E(N | X_1 X_1) p_2 p_1 + E(N | X_1 X_2) p_2 (1-p_1) + (1+\mu_1)(1-p_2)$
$= 2 p_2 p_1 + (2+\mu_2) p_2 (1-p_1) + (1+\mu_1)(1-p_2)$

(b) h_i mean number of times. thargeet his

$$h_1 = E[M_1 | X_1] p_1 + E[M_1 | X_2](1-p_1)$$
$$= E[M_1 | X_1 X_1] p_1 p_2 + E[M_1 | X_1 X_2] p_1 (1-p_2) + E[M_1 | X_2](1-p_1)$$
$$= (2) p_1 p_2 + (M_1 + 2) p_1 (1-p_2) + (1+h_2)(1-p_1)$$
$$h_1 = 2p_1 p_2 + (M_1 + 2) p_1 (1-p_2) + (1+h_2)(1-p_1)$$

$h_2 = E[M_2|X_1]\,p_2 + E[M_2|X_2](1-p_2)$

$= E[M_2|X_1X_1]\,p_2 \cdot p_1 + E[M_2|X_1X_2]\,p_2(1-p_1) + (1+h_1)(1-p_2)$

$= 2p_2p_1 + (2+M_1)\,p_2(1-p_1) + (1+h_1)(1-p_2)$

35.

$E[X_1] = np_1$

$= E[X_1 | X_2=0](1-p_2)^n + E[X_1 | X_2>0]\,[1-(1-p_2)^n]^n$

$np_1 = np_1(1-p_2)^n + E[X_1|X_2>0]\,[1-(1-p_2)^n]^n$

$E[X_1|X_2>0] = np_1 - np_1(1-p_2)^n \Big/ [1-(1-p_2)^n]^n$

39.

(a) ~~N~~ ~~set~~ suppose N is the number of ~~that~~ cycle

X be the position of card 1

$M_k = \sum_{i=1}^{n} E[N|X_i]\cdot P(E[N|X_i]) = \sum_{i=1}^{n} E[N|X_i]\,\frac{1}{n}$

$$= \sum_{j=1} (m_{n-1} + 1) \frac{1}{n}$$

$$= 1 + \frac{1}{n} \sum_{j=1}^{n-1} m_j$$

(b)

$m_0 = 0.$

$m_1 = (m_0 + 1) 1 = 1$

$\cancel{m_2 = 1 + (m_1 + 1) \frac{1}{2} = 2}$ $\quad m_2 = 1 + \frac{1}{2} m_1 = \frac{3}{2}$

$\cancel{m_3 = 1 + 2 + (m_2 + 1) \frac{1}{3} = 3}$ $\quad m_3 = 1 + \frac{1}{3}(m_1 + m_2) = 1 + \frac{1}{3}(1 + \frac{3}{2}) = 11/6$

$m_4 = 1 + \frac{1}{4}(m_1 + m_2 + m_3) = 25/12$

(c) $m_0 = 0$

$m_1 = 1$

$m_2 = 1 + \frac{1}{2} = \frac{3}{2}$ $\Rightarrow 1 + \frac{1}{2}$

$m_3 = 1 + \frac{1}{3}(1 + \frac{3}{2}) = 11/6$ $\Rightarrow 1 + \frac{1}{2} + \frac{3}{2} \times \frac{1}{3} = 1 + \frac{1}{2} + \frac{1}{3}$

$m_4 = 1 + \frac{1}{4}(1 + \frac{3}{2} + 11/6) = 25/12$ $\Rightarrow 1 + \frac{1}{4} + \frac{3}{8} + \frac{11}{24}$

$= \frac{20}{24} = \frac{5}{6} \Rightarrow (\frac{1}{2} + \frac{1}{3})$

$\vdots$

$m_4 \Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$\Downarrow$

$\therefore m_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

(d)

$m_n = 1 + \frac{1}{n} \sum_{j=1}^{n-1} m_j$

$= 1 + \frac{1}{n} \sum_{j=1}^{n-1} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$

$= 1 +$

(e)

$N = \cancel{\sum N} \sum_{i=1}^{n} X_i$

(f)

$\cancel{m_n = \sum_{i=1}^{n} E[X_i] \cdot \frac{1}{n}}$ $m_n = \sum_{i=1}^{n} E[X_i]$

(g)

Yes

Because we donot know which one is the last one for each cycle.

(h)

~~$Var(X) = \sigma_x^2 = E[X^2] - \mu_x^2$~~

$Var(X) = \sum_{i=1}^{n} Var(X_i)$

6.

(a) Yes and increasing.

Because of 2 player A and B, If A first. this win probability is bigger than B. A must flip more than B.

So this is an increaseing.

(b)

if $p \to 1$. $f(p)$ is 1

(c) if

$p \to 0$ $f(p)$ is $\frac{1}{2}$ Because the first one have $\frac{1}{2}$ probability

(d) $f(p) = P\{A \text{ win} \mid H\} \times p + P\{A \text{ win} \mid T\} \times (1-p)$

$= P\{A \text{ win} \mid H\} \times p + P\{\underline{A w} \mid TH\} \times (1-p)$ ~~$\times p \{ A w \mid TT$~~

$= p + (1 - f(p))(1-p)$

$f(p) = 1/2-p$

61.

Shooting end of target hit twice.

$m_i \to$ shot need to play i first.

$p_i \nearrow$ player i probability.

N total shot number. X_1 success shot. X_2 fail shot.

(a) $m_1 = E[N|X_1]p_1 + E[N|X_2](1-p_1)$

$~~= E[N|X_1 X_1]p_1 + E[N|X_1X_2](1-p_1) + E[N|X_2 X_1](1-p_1)~~$

$= p_1 + (1+m_2)(1-p_1)$

$m_2 = E[N|X_1]p_2 + E[N|X_2](1-p_2)$

$= p_2 + (1+m_1)(1-p_2)$

(b) $p_1 + 1 - p_1 + m_2 - m_2 p_1 = m_1$

$p_1 = (1+m_2-m_1)/m_2$

$$m_2 = p_2 + 1 - p_2 + m_1 - m_1 p_2$$

$$p_2 = (1 + m_1 - m_2)/m_1$$

(C)
~~(d)~~ Y_i means ~~player i~~ probabity final hit by 1 when i shoot first.

$$Y_1 = P_1 P_2 + (1-P_1) Y_2$$

p or P?

$$Y_2 = P_2 P_1 + (1-P_2) Y_1$$

(d) let Z_i is the probabity both hit by i ~~when~~

$$\cancel{Z_1} = \cancel{P_1 P_2 P_1 + P_1 P_2 P_1 +} \; P_1$$

$$Z_1 = P_1 (1-P_2) P_1 + (1-P_1)(1-P_2) Z_1$$

$$Z_1 = P_1^2 (1-P_2) + (1-P_2)(1-P_1) Z_1$$

p or P?

$$Z_1 = P_1^2 (1-P_2) / [1-(1-P_2)(1-P_1)]$$

(e) $z_2 = (1-p_1)\, p_2\, (1-p_1) + (1-p_1)(1-p_2)(1-p_1)\, z_2$

$z_2 = (1-p_1)^2 p_2 + (1-p_1)^2 (1-p_2)\, z_2$

$z_2 = (1-p_1)^2 p_2 \,/\, [1-(1-p_1)^2(1-p_2)]$

(f)

$$E[N] = E[N|X_1X_1]\,p_1p_2 + E[N|X_1X_2]\,p_1(1-p_2) + E[N|X_2X_2](1-p_1)(1-p_2) + E[N|X_2X_1](1-p_1)p_2$$

$$E[N] = 2p_1p_2 + (2+m_1)p_1(1-p_2) + (2+E[N])(1-p_1)(1-p_2) + (1+m_2)p_2(1-p_1)$$

$$E[N] = 2p_1p_2 + (2+m_1)p_1(1-p_2) + 2(1-p_1)(1-p_2) + E[N](1-p_1)(1-p_2) + (1+m_2)p_2(1-p_1)$$

$$\cancel{E[N] = \frac{2p_1p_2 + (2+m_1)p_1(1-p_2) + (1+m_2)p_2(1-p_1)}{1 - 2(1-p_1)(1-p_2)}}$$

$$E[N] = \frac{2p_1p_2 + (2+m_1)p_1(1-p_2) + 2(1-p_1)(1-p_2) + (1+m_2)p_2(1-p_1)}{1-(1-p_1)(1-p_2)}$$