SC42145: Preliminaries

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SC42145: Robust Control

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Purpose of the course

- Formulate control objectives in a mixed-sensitivity design
- Define stability and performance for MIMO LTI systems
- Construct a generalized plant for complex system interconnections
- Design MIMO controllers on the basis of the mixed-sensitivity
- Describe parametric and dynamic uncertainties
- Translate concrete controller synthesis problem into the abstract framework of robust control
- Reproduce definition, properties and computation of the structured singular value
- Master application of structured singular value for robust stability and performance analysis
- Design robust controllers on the basis of the $\mathcal{H}_\infty$ control algorithm (D-K iterations)

Organization

- 7 Lectures (L), see BS
- Communication via BrightSpace (BS)
- We are available during the scheduled assisted lecture hours for questions
- Design exercise, see BS

Design exercise: Control of floating wind turbine

- Part 1: Design a manually tuned SISO controller
- Part 2: Design nominal $\mathcal{H}_\infty$ controller (mixed sensitivity)
- Part 3a: Add uncertainties to your model
- Part 3b: Test robustness
- Part 3c: Design robust $\mathcal{H}_\infty$ controller

(Use: robust control toolbox)

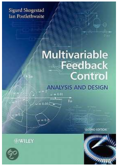
WindFloat (Principle Power USA)

Deliverables & Grade

1. Report deadlines:
   Part 1: 8th of Dec.,
   Part 2: 14th of Jan.,
2. 1 Report per team (team size max. 2)
3. Submit on BS
4. Grade: 40% Part 1 + 60% Part 2

Study material

- Chapter 1-9 and 12 of Skogestad & Postlethwaite (see BS for more details)
- Overhead slides of classes
- Assignments



Sigurd Skogestad, and Ian Postlethwaite
Multivariable Feedback Control
Analysis and Design (2005)

Short term requirements:

1. Enroll for the course on BS
2. Get book

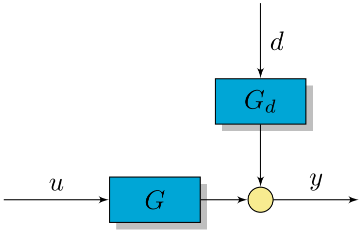
Lectures

1. Preliminaries (Chap. 1 & 2 (excluding 2.7))
2. Intro to MIMO control (Chap. 3)
3. Syst. Theory & Limit. (Chap. 4 & 5 (sect.5.2-5.4 & 5.6-5.9), Chap. 6 as ref. info)
4. Uncertainty & robustness SISO (Chap. 7)
5. Uncertainty & robustness MIMO (Chap. 8.1-8.7)
6. Structured Singular Value (Chap. 8.9-8.13+Chap. 12.1-12.2)
7. Overview +recap (Chap. 9 (sections 9.1-9.3))

The process of control system design

1. Study the system (process, plant)
2. Modeling, simplify, linearize
3. Scaling, and determine properties of the model
4. Select: Controllable inputs
5. Select: Sensors, which ones where to place?
6. Select: Control configuration
7. Select: Type of controller
8. Specify: Performance specs
9. Design a controller
10. Analyze the result
11. Simulate
12. Iterate (back to 6 or 8)
13. Choose hardware and software
14. Test and validate controller

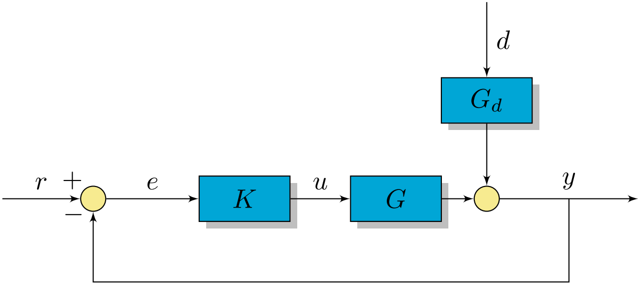
The control problem



- Manipulate u to counteract the effect of the disturbance d (Regulator problem)
- Manipulate u to track a reference r (Servo problem)

Minimize control error (e)!!

The control problem (cont'd)



Design K with a priori information of d and/or r, G, and G_d

What about uncertainties?? What about MIMO??

The control problem (cont'dd)

We define a class of models: $G_p = G + \Delta$ with $\Delta \leq 1$

- Nominal Stability (**NS**):
  G is stable
- Nominal Performance (**NP**):
  G satisfies certain performance bounds
- Robust Stability (**RS**):
  G_p is stable for all systems within the class
- Robust Performance (**RP**):
  G_p satisfies certain performance bounds for systems within the class

Transfer functions

We consider: $G(s) = \frac{\beta_{n_z} s^{n_z} + \ldots + \beta_1 s + \beta_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$

- Rational transfer function
- Linear Time Invariant (LTI)
- MIMO $\Rightarrow$ Matrix transfer function
  e.g. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
- If proper $\Rightarrow$ state space realization

Scaling

Why?

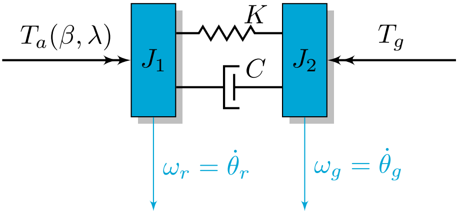
- For numerical stability
- For weighting purposes (MIMO)
- For norm bounded control

How?

- Determine max. d, u change (scaling $\frac{d}{d_{max}}$ and $\frac{u}{u_{max}}$)
- Same for e, r, or y (note same units so pick one)

Modeling: WT example, drive-train

Vestas V164-8MW (D=164m)



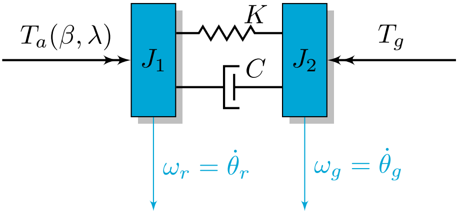
$$J_1\ddot{\theta}_r = T_a\left(\beta, \lambda\right) + K\left(\theta_g - \theta_r\right) + C\left(\dot{\theta}_g - \dot{\theta}_r\right)$$

$$J_2\ddot{\theta}_g = -T_g - K\left(\theta_g - \theta_r\right) - C\left(\dot{\theta}_g - \dot{\theta}_r\right)$$

Modeling: WT example, drive-train (cont'd)

The aerodynamic torque:
$T_a = \frac{1}{2}\rho R^3 C_Q(\beta,\lambda)V^2$

with:
$\lambda = \frac{\omega_r R}{V}$ - TSR
β -pitch angle
T_g -generator torque
C_Q - Torque coefficient



Taylor expansion:
$T_a(\beta,\lambda) \approx \frac{\partial T_a(\beta_o,\lambda_0)}{\partial \omega_r}\dot{\theta}_r$
Control ω_r or ω_g?

$$J_1\ddot{\theta}_r = T_a(\beta,\lambda) + K(\theta_g - \theta_r) + C\left(\dot{\theta}_g - \dot{\theta}_r\right)$$
$$J_2\ddot{\theta}_g = -T_g - K(\theta_g - \theta_r) - C\left(\dot{\theta}_g - \dot{\theta}_r\right)$$

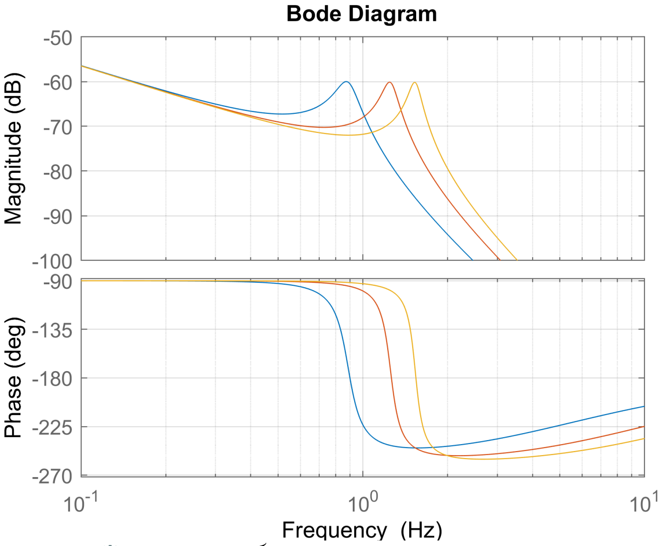
Modeling: WT example, drive-train (cont'dd)

$$\frac{\omega_r}{T_g}$$

Workpoint 1

Workpoint 2

Workpoint 3



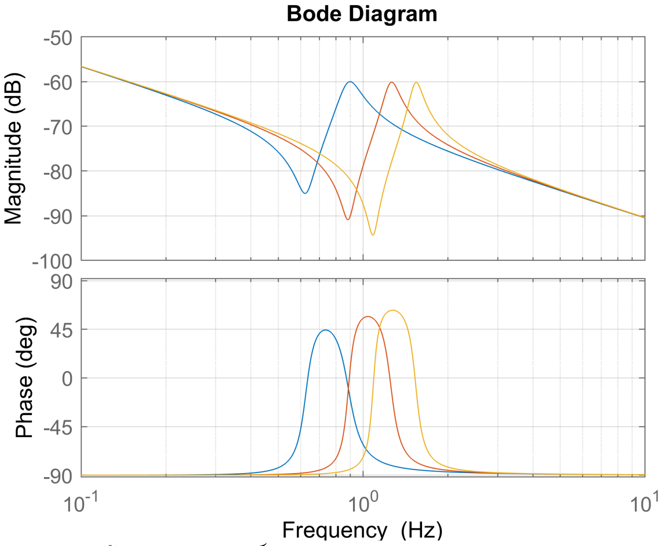
Modeling: WT example, drive-train (cont'ddd)

$$\frac{\omega_g}{T_g}$$

Workpoint 1

Workpoint 2

Workpoint 3



Modeling: WT example, drive-train (cont'ddd)

$$\frac{\omega_g}{T_g}$$

Workpoint 1

Workpoint 2

Workpoint 3

1. Actuator/Sensor placement plays a role
2. Linearization/Working point will play a role
3. Uncertainty in parameters will play a role

Frequency Response Function

FRF: $G(s) \Rightarrow G(j\omega)$

- A systems response to sinusoidal frequencies
- Frequency content of a deterministic signal (Fourier)
- Frequency distribution of a stochastic signal (PSD)

Frequency Response Function (cont'd)

- $u = u_0 \sin(\omega t + \alpha)$ (persistent)
- $y = y_0 \sin(\omega t + \beta)$
- $\|G(j\omega)\| = \frac{y_0}{u_0}$
- $\angle G(j\omega) = \beta - \alpha$

Example:

$G(s) = \frac{ke^{-\theta s}}{\tau s+1}, \qquad k = 5, \quad \theta = 2, \tau = 10;$

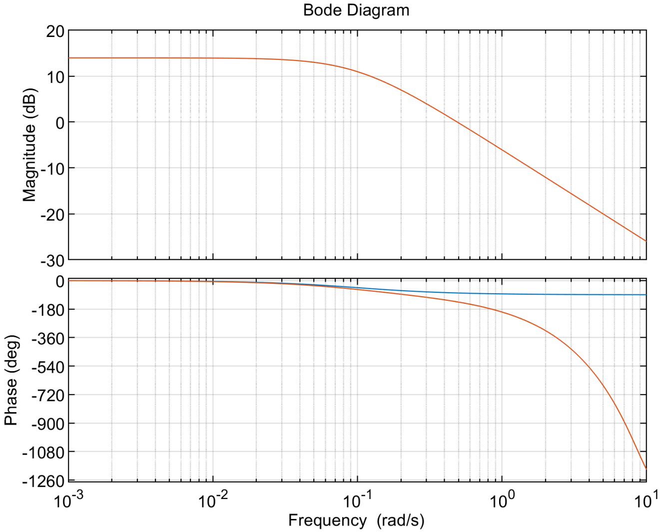
$\|G(j\omega)\| = \frac{k}{\sqrt{(\tau\omega)^2+1}} = \frac{5}{\sqrt{(10\omega)^2+1}}$

$\angle G(j\omega) = -\omega\theta - \arctan(\tau\omega) = -\omega\theta - \arctan(10\omega)$

Frequency Response Function (cont'dd)

$$G(s) = \frac{5}{10s+1}$$

$$G(s) = \frac{5e^{-2s}}{10s+1}$$



Minimum Phase System (MPS)

MPS: There exists an unique relation between the gain and phase of a frequency response function (for stable systems)

Problems: Right Half Plane (RHP) zeros, and time delays

Bode Gain-Phase relationship:

$$\angle G(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{d\ln|G(j\omega)|}{d\ln\omega}}_{N(\omega)} \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right| \frac{d\omega}{\omega}$$

$$\approx \frac{\pi}{2} N(\omega_0)[rad/s] = 90^o N(\omega_0)$$

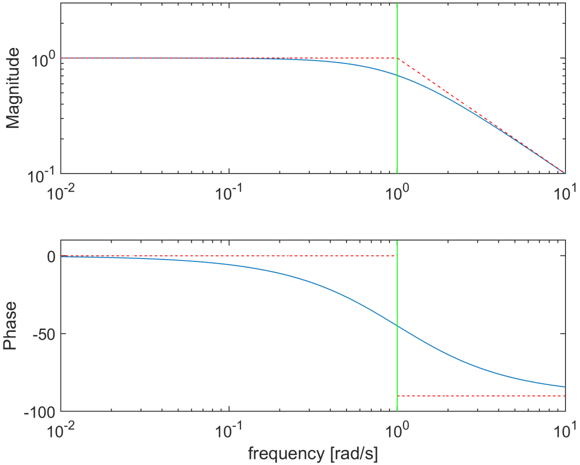
$$\text{Since :} \quad \left(\int_{-\infty}^{\infty} \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right| \frac{d\omega}{\omega} = \frac{\pi^2}{2}\right)$$

Asymptotes: single pole

$G(s) = \frac{1}{\tau s+1}$
(example: $\tau = 1$)

$\omega \ll \frac{1}{\tau}$
$N(\omega) = 0$
$\angle G(s) = 0$

$\omega \gg \frac{1}{\tau}$
$N(\omega) = -1$
$\angle G(s) = -90^o$

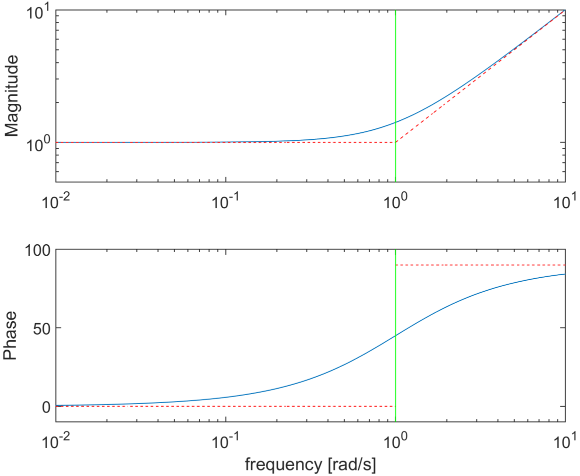


Asymptotes: single zero

$G(s) = \tau s + 1$
(example: $\tau = 1$)

$\omega \ll \frac{1}{\tau}$
$N(\omega) = 0$
$\angle G(s) = 0$

$\omega \gg \frac{1}{\tau}$
$N(\omega) = 1$
$\angle G(s) = 90^o$

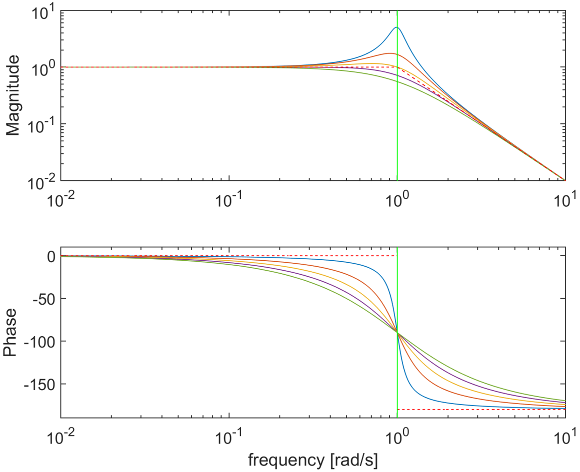


Asymptotes: double pole

$G(s) = \frac{\omega^2}{s^2+2\omega\beta s+\omega^2}$
(example: $\omega = 1$)

$\omega \ll \frac{1}{\tau}$
$N(\omega) = 0$
$\angle G(s) = 0$

$\omega \gg \frac{1}{\tau}$
$N(\omega) = -2$
$\angle G(s) = -180^o$

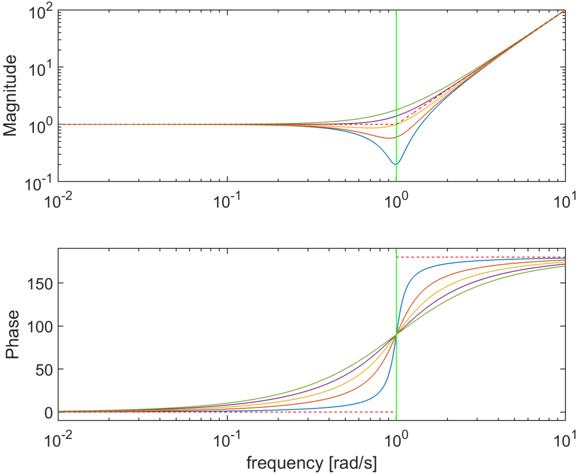


Asymptotes: double zero

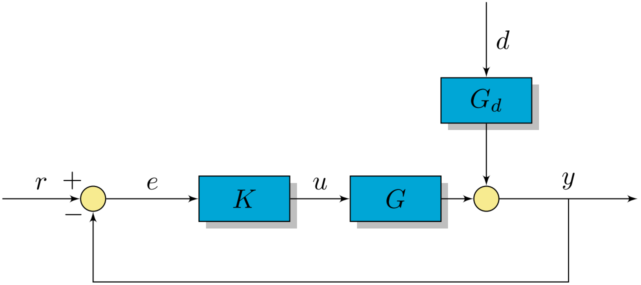
$G(s) =$
$s^2 + 2\omega\beta s + \omega^2$
(example: $\omega = 1$)

$\omega \ll \frac{1}{\tau}$
$N(\omega) = 0$
$\angle G(s) = 0$

$\omega \gg \frac{1}{\tau}$
$N(\omega) = 2$
$\angle G(s) = 180^o$



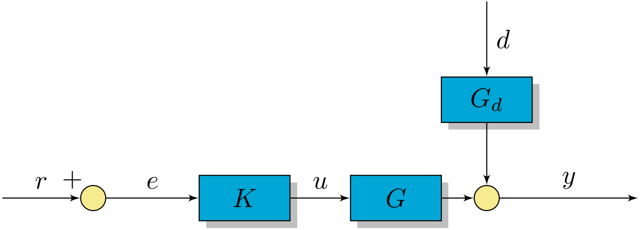
Feedback



- $y = Gu + G_d d$
- $y = GK(r - y) + G_d d$
- $y = \underbrace{(I + GK)^{-1}GK}_{T}\, r + \underbrace{(I + GK)^{-1}}_{S}\, G_d d$

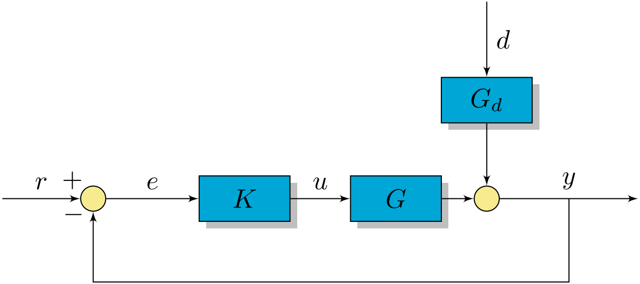
T=complementary sensitivity
S=sensitivity with $T + S = I$

Why Feedback



- Feedforward: $u = G^{-1}r - G^{-1}G_d d$
- G^{-1} can be hard to compute (RHP-zeros, delays)
- Signal uncertainty
- Model uncertainty Δ
- Unstable plant

High Gain Feedback



- $L = GK$ = loop transfer
- Consider $L \gg I$
- $S \approx 0$
- Since: $T + S = I$ we know $T \approx I$
- $u = KS(r - G_d d)$ and we know $KS = G^{-1}T$

High gain feedback: $u = G^{-1}r - G^{-1}G_d d$ (Similar as model inversion)

Stability

Problem with high gain feedback: Stability

How to check stability?

- Stable iff all the poles are in the LHP
- Nyquist: equating the number of encirclements and the number of RHP-poles
- For OL stable minimum phase systems: Bode's stability condition
  Stability $\Leftrightarrow |L(j\omega_{180})| < 1$
  ω_{180} is corresponding to $\angle L(j\omega_{180}) = -180^o$

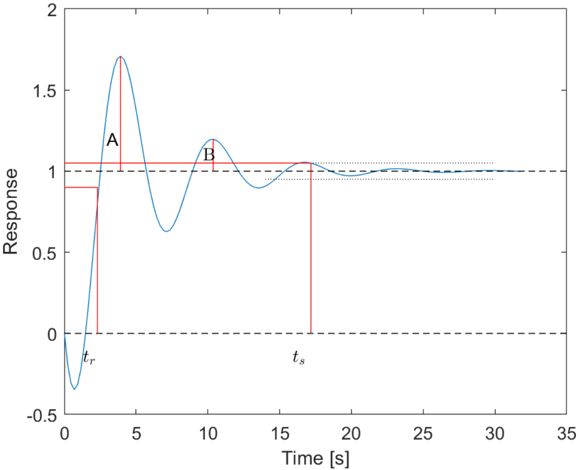
Performance: Time domain

Rise time (t_r)
90% steady state

Overshoot (A)
Max. peak

Damping ratio
(B/A)

Setting time
(within 5%)

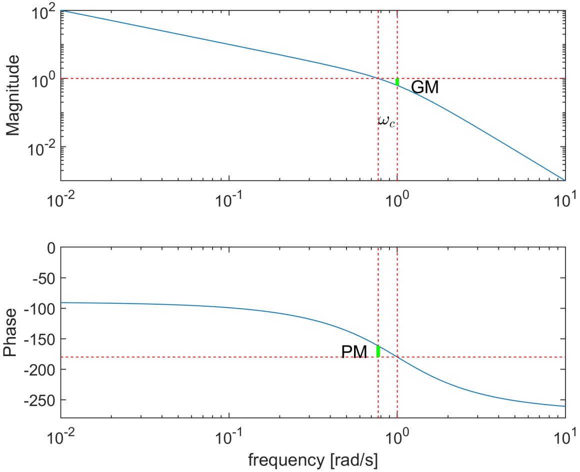


Performance: Frequency domain

Gain Margin (GM)

Phase Margin (PM)

Max. $\theta_{td} = \frac{\mathrm{PM}}{\omega_c}$

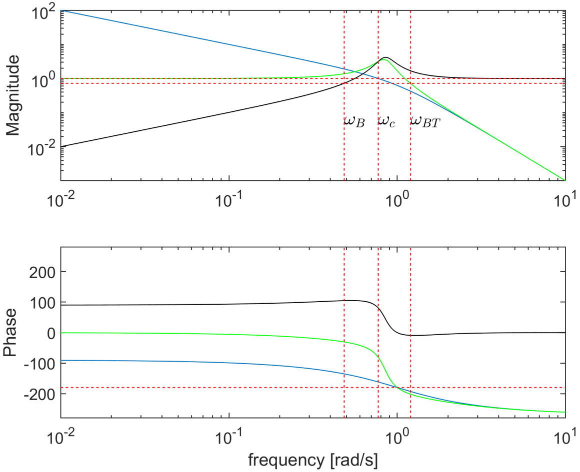


Performance: Frequency domain

L (ω_C 0dB cross-over)

T (ω_{BT} -3dB bandwidth T)

S (ω_B -3dB bandwidth S)



Performance: Frequency domain

Bandwidth: The closed-loop bandwidth, ω_B, is the frequency where $|S(j\omega)|$ first crosses $\frac{1}{\sqrt{2}} = 0.707(\approx -3dB)$ from below

Performance: Frequency domain

Typically: PM$> 30^o$ and GM> 2

Define: $M_S = \max_\omega |S(j\omega)|$ and $M_T = \max_\omega |T(j\omega)|$

Important for later: $M_S = ||S||_\infty$ and $M_T = ||T||_\infty$

Since: $T + S = I$ and $||S| - |T|| \leq |S + T| = 1$

So: M_S and M_T differ at most by one

Performance: Frequency domain $\Leftrightarrow$ Time domain

Given $M_S \Rightarrow \mathrm{GM} \geq \frac{\mathrm{M_S}}{\mathrm{M_S}-1}$ and $PM \geq 2\arcsin\left(\frac{1}{2M_S}\right) \geq \frac{1}{M_S}$

$M_S = 2$ we are guaranteed to have $\mathrm{GM} \geq 2$ and $\mathrm{PM} \geq 29^{\mathrm{o}}$

Given $M_T \Rightarrow \mathrm{GM} \geq 1 + \frac{1}{\mathrm{M_T}}$ and $PM \geq 2\arcsin\left(\frac{1}{2M_T}\right) \geq \frac{1}{M_T}$

$M_T = 2$ we are guaranteed to have $\mathrm{GM} \geq 1.5$ and $\mathrm{PM} \geq 29^{\mathrm{o}}$

Controller design

Approaches:

1. Shaping of transfer functions
   1. Shaping $L(j\omega)$ (Manual loop shaping B.Sc.)
   2. Shaping S, T, and KS
2. Signal based approach (LQG, $\mathcal{H}_2$, $\mathcal{H}_\infty$)
3. Direct numerical optimization (MPC)

Loop Shaping

Key Idea: Shape $L(s)$ using Nyquist/Bode for closed-loop performance and stability

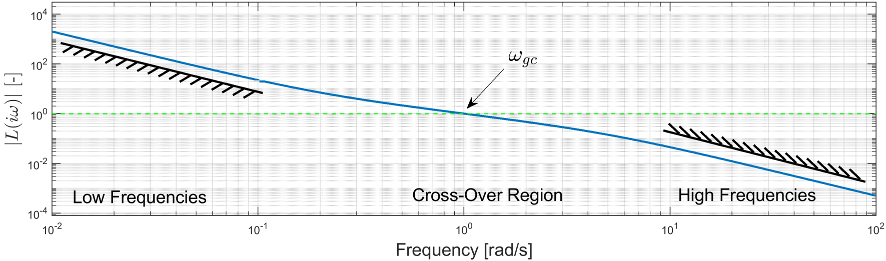
How (I): $C(s) = \frac{L_d(s)}{P(s)}$ where L_d is desired loop transfer function

How (II): place poles and zeros using Bode/Nyquist to get desired loop transfer function

Three key areas:

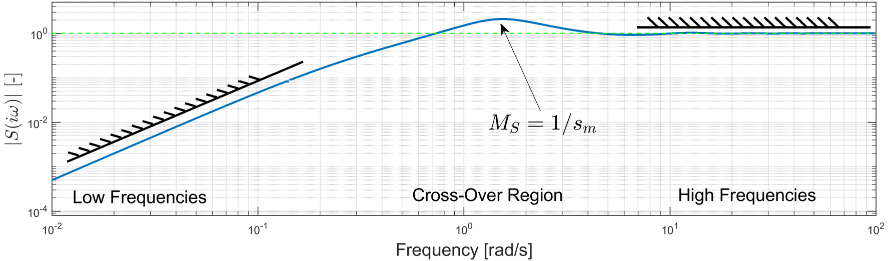
1. Low Frequencies, Load disturbance attenuation (High gain)
2. Cross-Over region, Robustness (Take care of margins)
3. High Frequencies, High frequency measurement noise (Low gain)

Loop Shaping: Loop Transfer Function



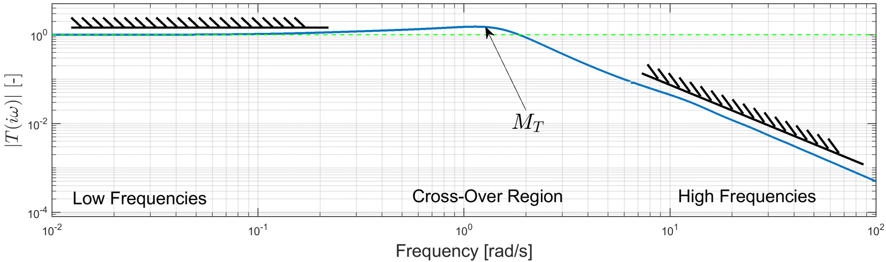
- Low Frequencies, High gain
- High Frequencies, Low gain
- Cross-Over region, Stability Margins, (PM=30^o, 45^o, 60^o equals Slope $-5/3$, $-3/2$, $-4/3$)

Loop Shaping: Sensitivity



- Low Frequencies, Low gain, Disturbance rejection
- High Frequencies, Gain=1, No difference between OL or CL
- Cross-Over region, Inevitable Peak M_S, OL will outperform CL at some freq.

Loop Shaping: Compl. Sensitivity



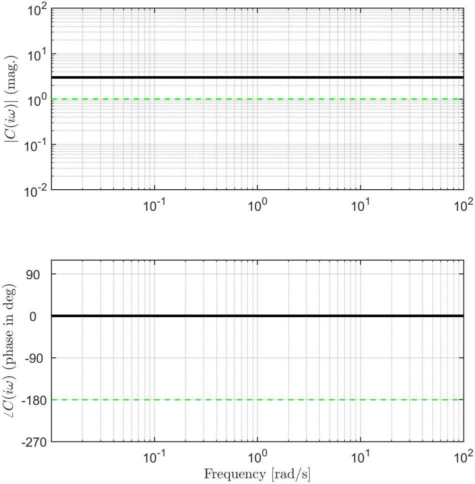
- Low Frequencies, Gain=1, Tracking of the reference
- High Frequencies, Low gain, No tracking of the reference
- Cross-Over region, Inevitable Peak M_T, Remember the equality $S + T = I$, Peak in S results in peak in T

Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$$C(s) = K_p$$

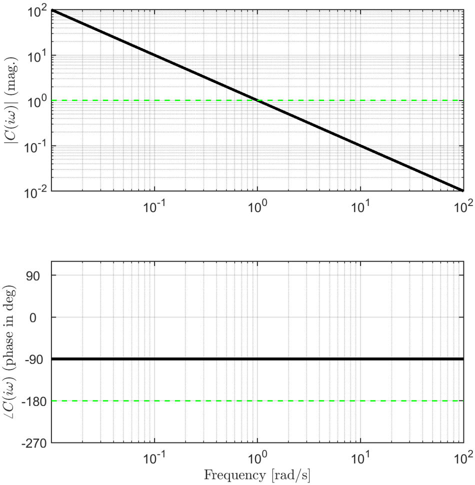


Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$$C(s) = \frac{K_i}{s} \quad \text{or} \quad \left(\frac{T_i}{s}\right)$$



Control elements

We have e.g.:

- P-action
- I-action
- D-action
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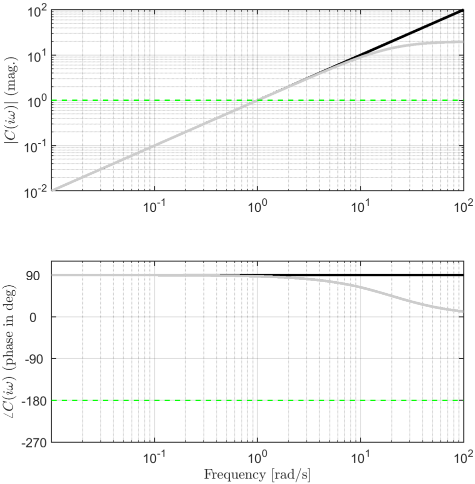
- Lead-lag

$C(s) = K_d s$ or $\left(\frac{K_d s}{T_f s+1}\right)$

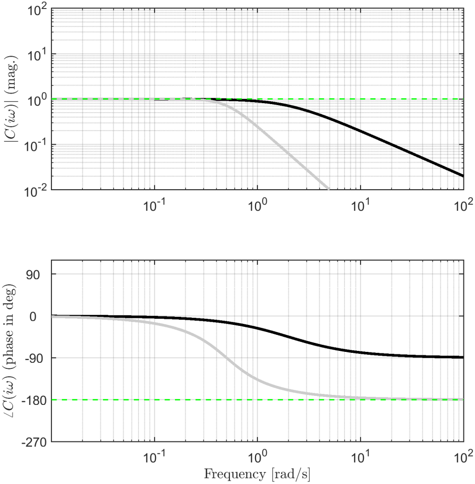


Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$C(s) = \frac{1}{\tau s+1}$ or $\frac{\omega^2}{s^2+2\omega\beta s+\omega^2}$

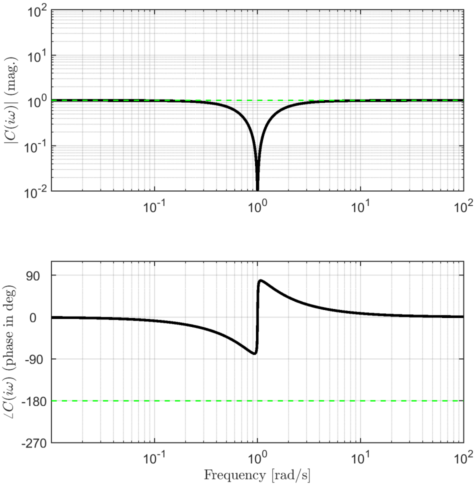


Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$$C(s) = \frac{s^2+2\omega\beta_1 s+\omega^2}{s^2+2\omega\beta_2 s+\omega^2}$$

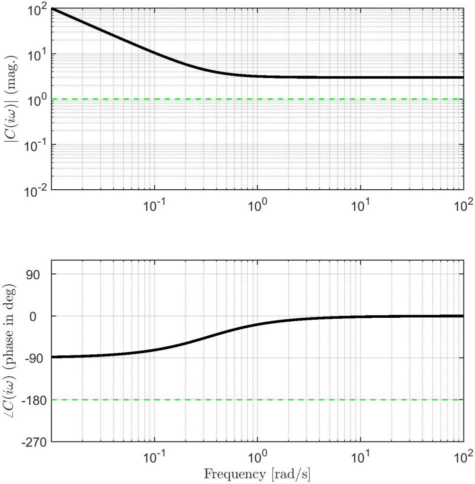


Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$$C(s) = \quad K_p\left(1+\frac{T_I}{s}\right)$$

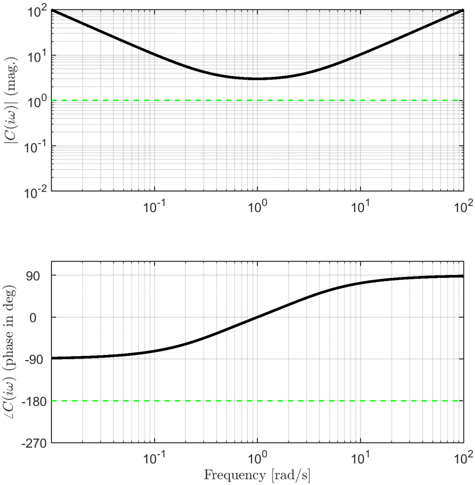


Control elements

We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

$$C(s) = \qquad K_p + \frac{K_i}{s} + K_d s$$

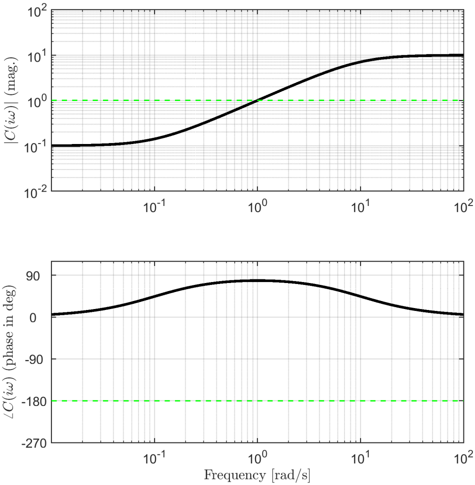


Control elements

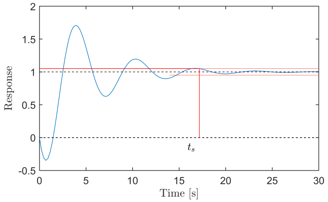
We have e.g.:

- P-action
- I-action
- D-action
- Low pass filter
- Notch
- PI
- PID
- Lead-lag

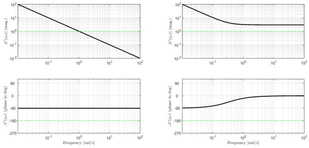
$$C(s) = \frac{\tau_1 s+1}{\tau_2 s+1}$$

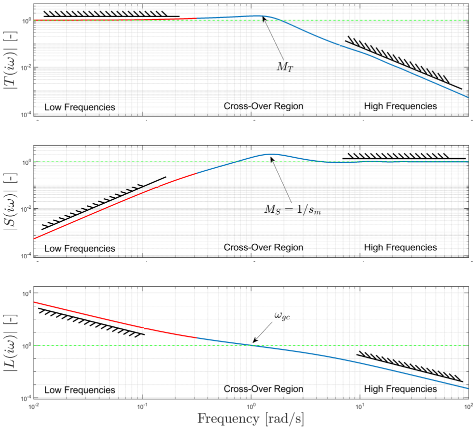


Design process (Tracking): Low Frequencies



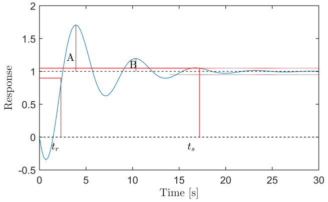
$e.g.$ I or PI



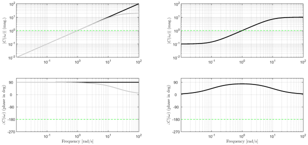


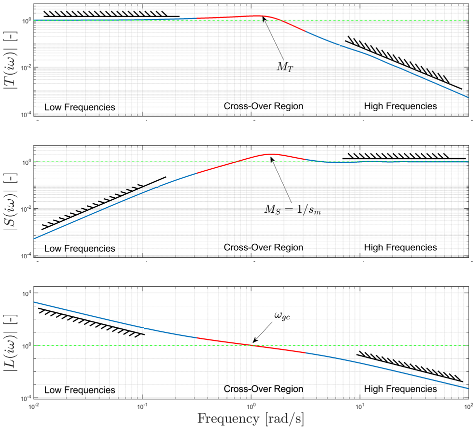
Reduce steady state errors by increasing the loop gain

Design process (Tracking): Cross-Over



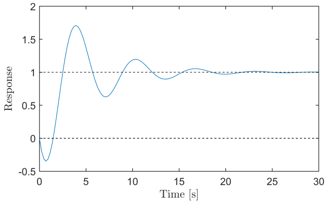
$e.g.$ D or Lead-Lag



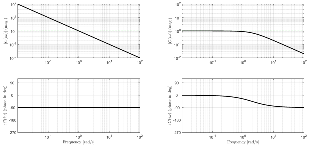


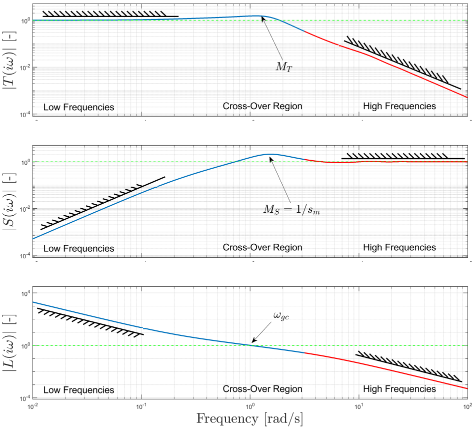
Increase the phase to satisfy stability margins

Design process (Tracking): High frequencies



$e.g.$ I or Low-Pass



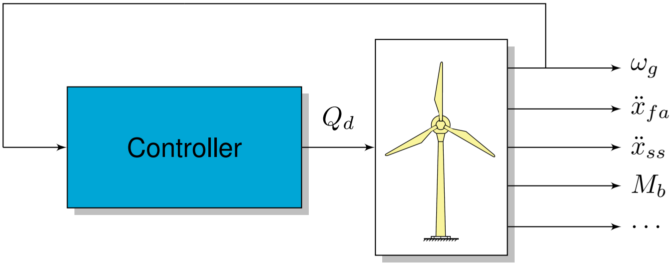


Reduce the gain to mitigate the effect of measurement noise

Design process (Tracking)

- Determine gain cross-over frequency ($|L(i\omega_{gc})|$ crosses 1)
- Add low pass filter to suppress the effect of high frequent dynamics (integrator, low-pass, notch)
- Add phase (D-action, lead-lag)
- Set cross-over frequency (P-action)
- Reference tracking (I-action)

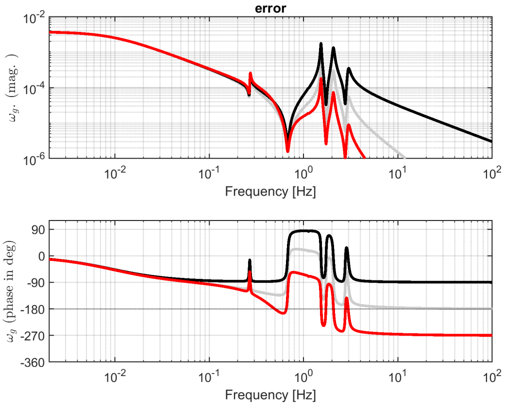
Torque control



Torque control: PI+

Reference tracking:

- $L(s)$
- Add LP $({}_{1^{st}}, {}_{2^{nd}})$



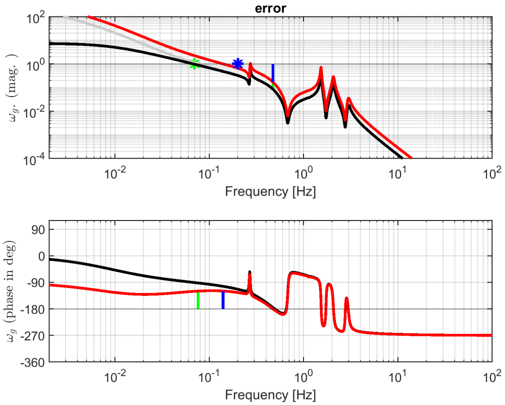
L(s):

- $P(s)LP(s)$

Torque control: PI+

Reference tracking:

- $L(s)$
- Add LP $(1^{st}, 2^{nd})$
- Tune Gain
- Add I-action
- Add more gain



L(s):

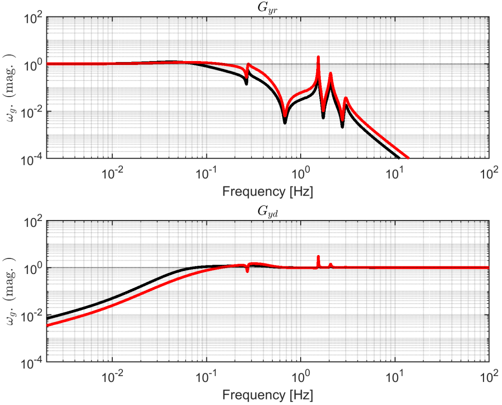
$$-2e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)$$

$$-5e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)$$

Torque control: PI+

Reference tracking:

- $L(s)$
- Add LP $({\color{gray}1^{st}}, {\color{red}2^{nd}})$
- Tune Gain
- Add I-action
- Add more gain
- $\frac{L(s)}{1+L(s)}$



L(s):

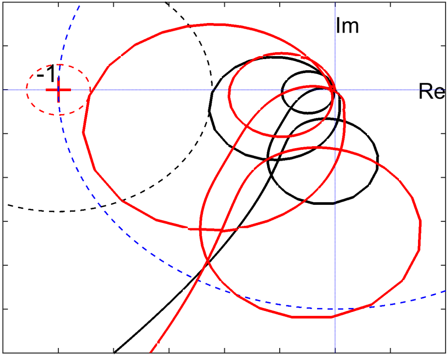
$$-2e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)$$

$${\color{red}-5e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)}$$

Torque control: PI+

Reference tracking:

- $L(s)$
- Add LP $(1^{st}, 2^{nd})$
- Tune Gain
- Add I-action
- Add more gain
- Nyquist



L(s):

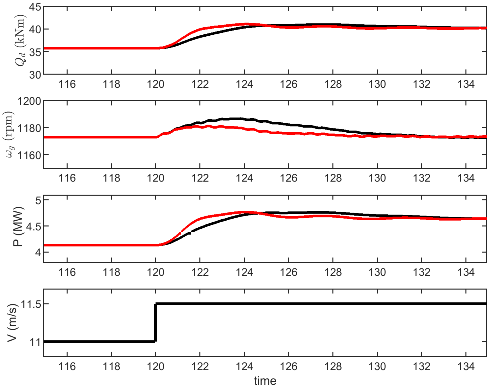
$$-2e3\left(1 + \frac{0.25}{s}\right)P(s)LP(s)$$
$$-5e3\left(1 + \frac{0.25}{s}\right)P(s)LP(s)$$

Torque control: PI+

Reference tracking:

- $L(s)$
- Add LP $(1^{st}, 2^{nd})$
- Tune Gain
- Add I-action
- Add more gain

- Time Domain



L(s):

$-2e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)$

$-5e3\left(1+\frac{0.25}{s}\right)P(s)LP(s)$

Shaping closed-loop transfer functions

We have seen a relation between GM/PM and M_S/M_T

$\mathcal{H}_\infty$ norm of a SCALAR and STABLE transfer function f(s) is simply the peak value of $|f(j\omega)|$ as a function of frequency, i.e.

$$||f(s)||_\infty \triangleq \max_\omega |f(j\omega)|$$

Other norms:

$$||f(s)||_p \triangleq \max_{p\to\infty} \left(\int_{-\infty}^{\infty} |f(j\omega)|^p d\omega \right)^{\frac{1}{p}}$$

Note: $\mathcal{H}$ represents a ”Hardy Space” which is a space that contains

all the stable and proper transfer functions.

Sensitivity

Advantages:

1. Good performance indicator
2. Need only amplitude information

We can specify:

1. Minimum Bandwidth (ω_B)
2. Maximum tracking error at selected frequencies
3. Shape of S
4. Max peak of S

Weighted Sens. design: $|S(j\omega)| < 1/|w_P(j\omega)|, \forall\omega$ or $||w_P S||_\infty < 1$

Sensitivity

w_P sets a lower bound on bandwidth, doesn't allow us to specify roll-off.

Solution: bound other closed-loop transfer functions (Mixed sensitivity)

$$||N||_\infty = \max_\omega \overline{\sigma}(N(j\omega)) < 1; \qquad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u K S \end{bmatrix}$$

$\mathcal{H}_\infty$ optimal controller is obtained by solving the problem

$$\min_K ||N(K)||_\infty$$

How to set-up synthesis? How to deal with MIMO?

Introduction to multi-variable control

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SC42145, 2021/22

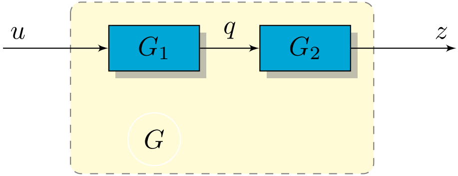
Introduction

We consider:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1m} \\ G_{21} & G_{22} & \dots & G_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{\ell 1} & G_{\ell 2} & \dots & G_{\ell m} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

- Interaction, one input can affect multiple outputs
- Main difference with SISO, input and output have a direction
- Main tool: Singular Value Decomposition (SVD)
- $GK \neq KG$
- No generalization of Bode's stability condition

Transfer functions for MIMO systems

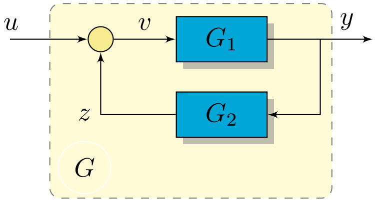


Cascade rule:

$$z = G_2 q \qquad \text{and} \qquad q = G_1 u$$

$$z = \underbrace{G_2 G_1}_{G} u$$

Transfer functions for MIMO systems

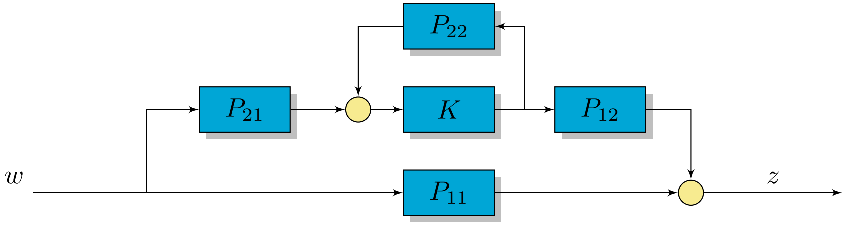


Feedback Rule: $v = (I - L)^{-1} u$ with $L = G_2 G_1$

The plant: $y = \underbrace{G_1 (I - L)^{-1}}_{G} u$

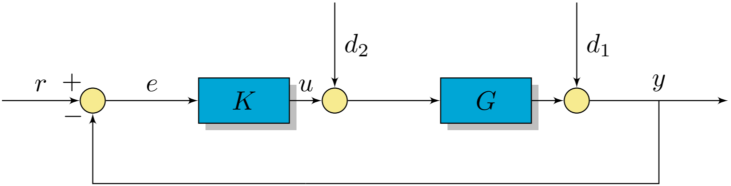
Push through rule: $G_1 (I - G_2 G_1)^{-1} = (I - G_1 G_2)^{-1} G_1$

Transfer functions for MIMO systems



Quiz: can you derive the transfer function?

Transfer functions for MIMO systems



For the output side:

$$\mathrm{S_O}\colon \frac{y}{d_1} = (I+GK)^{-1} = S = \frac{e}{r} \quad \text{and} \quad \mathrm{T_O}\colon \frac{y}{r} = GK(I+GK)^{-1} = T$$

For the input side:

$$\mathrm{S_I}\colon \frac{u+d_2}{d_2} = (I+KG)^{-1} \quad \text{and} \quad \mathrm{T_I}\colon -\frac{u}{d_2} = KG(I+KG)^{-1}$$

Gain of a system

SISO (independent of magnitude):

$$\frac{|y(\omega)|}{|d(\omega)|} = \frac{|G(j\omega)d(\omega)|}{|d(\omega)|} = |G(j\omega)|$$

How can we do this for MIMO? Sum up the magnitudes?

$$\frac{||y(\omega)||_2}{||d(\omega)||_2} = \frac{\sqrt{\sum_j |y_j(\omega)|^2}}{\sqrt{\sum_j |d_j(\omega)|^2}} = \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2}$$

This gain measure depends on direction of input

Max. gain given by:

$$\max_{d\neq 0} \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2} = \max_{||d||_2=1} ||G(j\omega)d(\omega)||_2 = \overline{\sigma}(G(j\omega))$$

Min. gain given by:

$$\min_{d\neq 0} \frac{||G(j\omega)d(\omega)||_2}{||d(\omega)||_2} = \min_{||d||_2=1} ||G(j\omega)d(\omega)||_2 = \underline{\sigma}(G(j\omega))$$

Gain of a system: Example

Define $d = \begin{bmatrix} d_{10} \\ d_{20} \end{bmatrix}$ with the following 5 inputs:

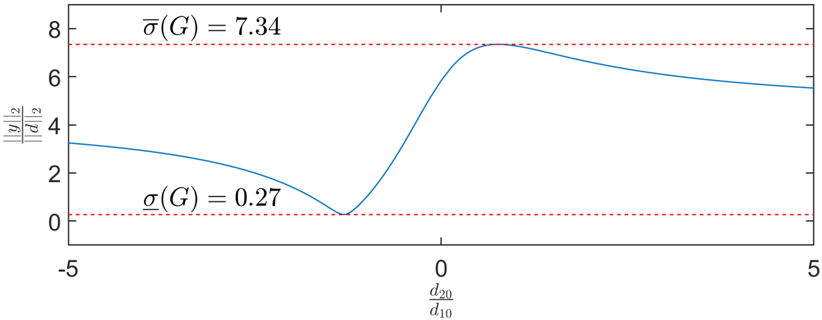
$$d_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \quad d_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, \quad d_5 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix},$$

We consider: $G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$

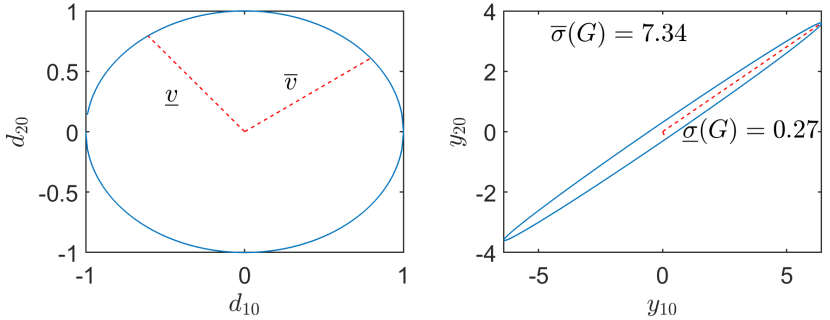
$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 6.36 \\ 3.54 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, \quad y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix},$$

Note that $||d||_2 = 1$ and we have:
$||y_1||_2 = 5.83$, $||y_2||_2 = 4.47$, $||y_3||_2 = 7.30$, $||y_4||_2 = 1.00$, $||y_5||_2 = 0.28$

Gain of a system: Example (cont'd)



Gain of a system: Example (cont'dd)



We can use the SVD

Eigenvalues as measure?

- Can only be computed for square matrices
- Can be really misleading

Let's consider:

$$G = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}$$

From eigenvalues one might conclude that the gain is zero

Note that: $d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ results in $y = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

Eigenvalues are a poor measure and can not be captured in a matrix norm

The Singular Value Decomposition

Let's consider a fixed frequency ω_o. Then, every matrix can be decomposed in:

$$G(\omega_o) = U\Sigma V^H$$

where:

- $\Sigma \in \ell \times m$
- $U \in \ell \times \ell$ Unitary matrix Output singular vectors
- $V \in m \times m$ Unitary matrix Input singular vectors

Structure for real-valued 2×2 matrix:

$$G = \underbrace{\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\theta_2 & \pm\sin\theta_2 \\ -\sin\theta_2 & \pm\cos\theta_2 \end{bmatrix}^T}_{V^T}$$

The Singular Value Decomposition (cont'd)

Note that since V is unitary we have $GV = U\Sigma$ or:

$$G\begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & & 0 \\ 0 & \sigma_2 & & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix}$$

where $k = \min(m, \ell)$.

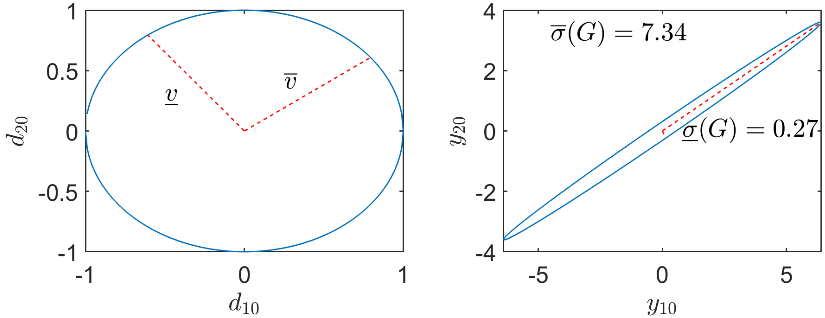
Max. gain given by:

$$\overline{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_1||_2}{||v_1||_2}$$

Min. gain given by:

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_k||_2}{||v_k||_2}$$

Gain of a system: Example (cont'ddd)



We can use the SVD (note that this also works for non-square plants)

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.872 & 0.490 \\ 0.490 & -0.872 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.794 & -0.608 \\ 0.608 & 0.794 \end{bmatrix}^T}_{V^T}$$

Singular values for performance

Sensitivity S provides useful information regarding the effectiveness of control.
For SISO systems this is defined as $\left|\frac{e}{r}\right|$

For MIMO we define $\frac{||e||_2}{||r||_2}$ where $||\cdot||_2$ represents the vector 2-norm

This gain depends on direction

We can bound it:

$$\underline{\sigma}(S(j\omega)) \leq \frac{||e||_2}{||r||_2} \leq \overline{\sigma}(S(j\omega))$$

In terms of performance it is reasonable to require that the gain is small for all directions (so look at $\overline{\sigma}(\omega)$)

Singular values for performance (cont'd)

Weighted Sensitivity design:

$\overline{\sigma}(S(j\omega)) < 1/|w_P(j\omega)|, \forall\omega$ or $||w_P S||_\infty < 1$

Bandwidth, ω_B: Frequency where $\overline{\sigma}(S)$ crosses 0.7 from below

Note: the bandwidth is at least ω_B

Measure for directionality

1. Condition Number
   Defined as: $\gamma \triangleq \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)}$
   - If large then ill-conditioned
   - High $\overline{\sigma}(G)$ no issue but low $\underline{\sigma}(G)$ can be an issue
   - Large condition number *may indicate* control issues
2. Relative gain array (RGA)
   See next section

Relative Gain Array (RGA)

The Relative Gain Array is defined as:

$$\mathrm{RGA(G)} = \Lambda(\mathrm{G}) \triangleq \mathrm{G} \times \left(\mathrm{G}^{-1}\right)^{\mathrm{T}}$$

here $\times$ represents the Hadamard product.

Example for a 2×2 matrix with g_{ij}:

$$\Lambda(G) = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 1 - \lambda_{11} \\ 1 - \lambda_{11} & \lambda_{11} \end{bmatrix}$$

with $\lambda_{11} = \frac{1}{1 - \frac{g_{12} g_{21}}{g_{11} g_{22}}}$

Note that the RGA is frequency dependent.

Interpretation: RGA as an interaction measure

Let's consider: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}}_{G} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\hat{g}_{11}} & \frac{1}{\hat{g}_{12}} \\ \frac{1}{\hat{g}_{21}} & \frac{1}{\hat{g}_{22}} \end{bmatrix}}_{G^{-1}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Look at interaction between u_1 and y_1:

1. Other loops open $u_2 = 0$ and compute $\frac{\partial y_1}{\partial u_1}$.
   g_{11}
2. Other loops closed with perfect control $y_2 = 0$ and compute $\frac{\partial y_1}{\partial u_1}$.
   $\hat{g}_{11}$

RGA$= \begin{bmatrix} \frac{g_{11}}{\hat{g}_{11}} & \frac{g_{12}}{\hat{g}_{21}} \\ \frac{g_{21}}{\hat{g}_{12}} & \frac{g_{22}}{\hat{g}_{22}} \end{bmatrix} = G \times \left(G^{-1}\right)^T$

We prefer to pair variables with an RGA of 1 (unaffected by other loops).

Pairing rules:

1. Prefer pairings such that the rearranged system, with the selected pairings along the diagonal, has an RGA matrix close to identity at the frequencies around the bandwidth
2. Avoid (if possible) pairing on negative steady-state RGA elements.

Other properties:

- RGA independent of scaling
- Rows and columns sum up to 1
- Use pseudo-inverse for non-square plants
- Plants with large RGA elements are ill-conditioned (> 10 difficult to control)

Two step procedure

The easiest way to control MIMO systems is by using a diagonal controller (Decentralized control).

The easiest way to do this is to decouple the system in a diagonal system $\hat{G}(s) = G(s)W_1$. How?

- Dynamic decoupling, $\hat{G}(s) = G(s)G(s)^{-1}$ where the controller is given by $\frac{k}{s}I$ (inversed based control)
- Steady state decoupling, $\hat{G}(s) = G(s)G(0)^{-1}$ which is a constant pre-compensator.
- Approximate decoupling, $\hat{G}(s) = G(s)G(j\omega_o)^{-1}$ typically ω_o is chosen close to the bandwidth
- Approximate decoupling with post-compensator, $\hat{G}(s) = \underbrace{U(j\omega_o)^T}_{W_2} G(s) \underbrace{V(j\omega_o)}_{W_1}$ typically ω_o is chosen close to the bandwidth

Direct synthesis of MIMO controller

Typical approach:

$$||N||_\infty = \max_\omega \overline{\sigma}(N(j\omega)) < 1; \qquad N = \begin{bmatrix} W_p S \\ W_u K S \end{bmatrix}$$

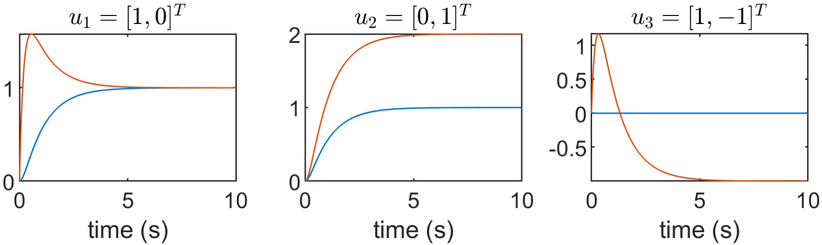
- Design W_p, common choice $W_p = diag(w_{p,i})$ with $w_{p,i} = \frac{s/M_i + \omega_{Bi}}{s + \omega_{Bi} A_i}$ with $A_i \ll 1$
- Design W_u, common I or $W_u = s/(s+\omega_l)$ (so low penalty at low frequencies)
- To find suitable values, design a controller by hand (decoupling)
- Add more transfer functions to the objective function

Example

Let's consider:

$$G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

Step response (y_1 and y_2):

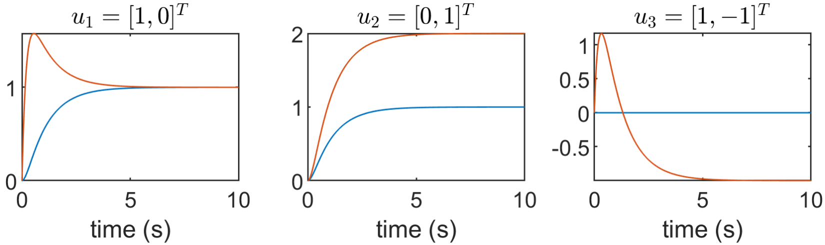


Looking at the TF's there is no reason to assume that we have a RHP-zero. However, there is one at $z = 0.5$

Example (cont'd)

$$G(0.5) = \frac{1}{1.65}\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix} = \underbrace{\begin{bmatrix}0.45 & 0.89\\ 0.89 & -0.45\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}1.92 & 0\\ 0 & 0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}0.71 & 0.71\\ 0.71 & -0.71\end{bmatrix}^T}_{V^T}$$

The $blue$ elements represent the input and output direction corresponding to a RHP-zero



MIMO zeros

MIMO zeros:z_i is a zero of G(s) if the rank of $G(z_i)$ is less than the normal rank of $G(z_i)$

Compute MIMO zeros:

1. $$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$
2. $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix}$$
3. $$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \\ \frac{s-1}{s+2} & \frac{s-2}{s+1} \end{bmatrix}$$

What is the effect of a MIMO RHP-zero?

Example (cont'dd)

We solve:

$$||N||_\infty = \max_\omega \overline{\sigma}(N(j\omega)) < 1; \qquad N = \begin{bmatrix} W_P S \\ W_u K S \end{bmatrix}$$

with $W_u = I$, $W_p = diag(w_{Pi})$ with $w_{Pi} = \frac{s/M_i + \omega_{Bi}}{s + \omega_{Bi} A_i}$ with $A_i = 10^{-4}$

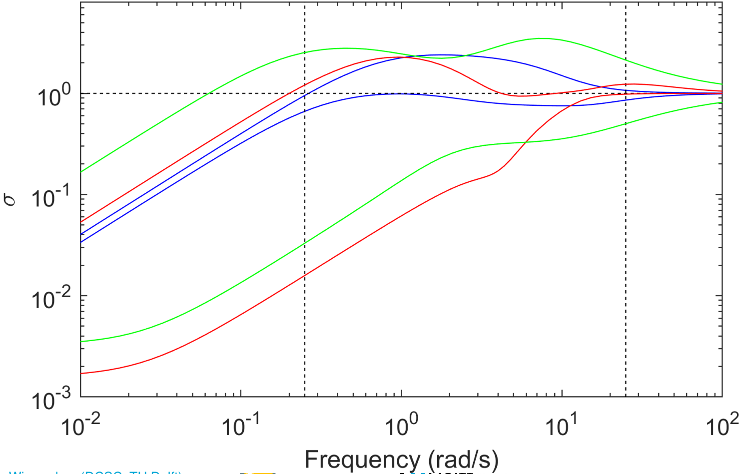
```
>>s=tf('s');
>>G=1/(0.2*s+1)/(s+1)*[ 1 1; 1+2*s 2];
>>wB1=0.25; % desired closed-loop bandwidth
>>wB2=0.25; % desired closed-loop bandwidth
>>A=1/1000; % desired disturbance attenuation inside bandwidth
>>M=1.5 ; % desired bound on hinfnorm(S)
>>Wp=[(s/M+wB1)/(s+wB1*A) 0; 0 (s/M+wB2)/(s+wB2*A)]; %
Sensitivity weight
>>Wu=eye(2); % Control weight
>>Wt=[] % Empty weight
>>[K,CL,GAM,INFO]=mixsyn(G,Wp,Wu,Wt);
```

Example (cont'ddd)

Design 1: $M_1 = M_2 = 1.5, \omega_{B1} = \omega_{B2} = 0.25.$ $||N||_\infty = 2.80$

Design 2: $M_1 = M_2 = 1.5, \omega_{B1} = 0.25, \omega_{B2} = 25.$ $||N||_\infty = 2.92$

Design 3: $M_1 = M_2 = 1.5, \omega_{B1} = 25, \omega_{B2} = 0.25.$ $||N||_\infty = 6.70$



Example (cont'dddd)

Design 1: $M_1 = M_2 = 1.5, \omega_{B1} = \omega_{B2} = 0.25.$ $||N||_\infty = 2.80$
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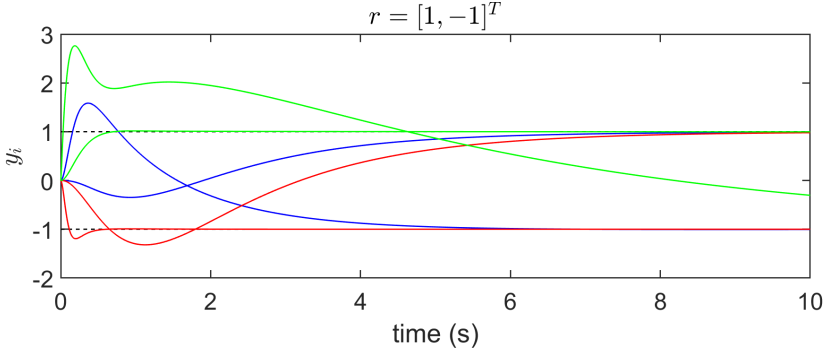
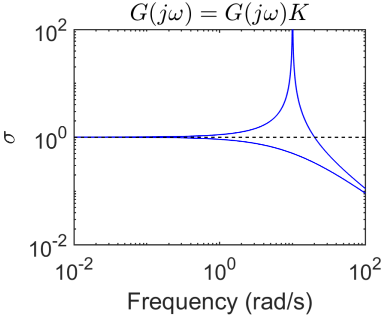


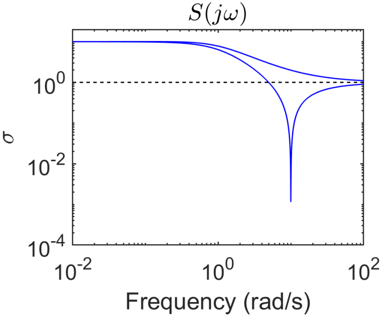
Example 1: Spinning satellite

$G(s) = \frac{1}{s^2+100}\begin{bmatrix} s-100 & 10s+10 \\ -10s-10 & s-100 \end{bmatrix}$ system has poles at $s = \pm j10$

Apply negative unity, I, feedback.

- **NS:** The closed loop system has two poles at $s = -1$
- **NP:**



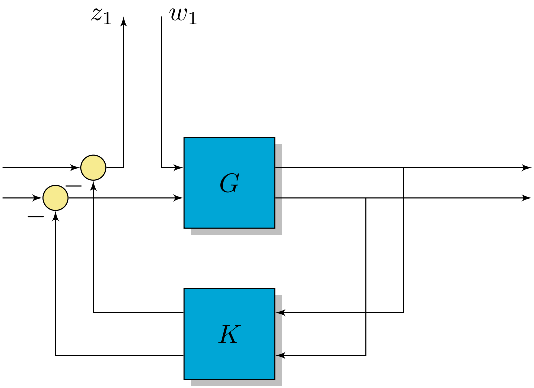


- **RS:** See next slide

Example 1: Spinning satellite (cont'd)

RS: We will consider diag. input uncertainty (present in every plant)

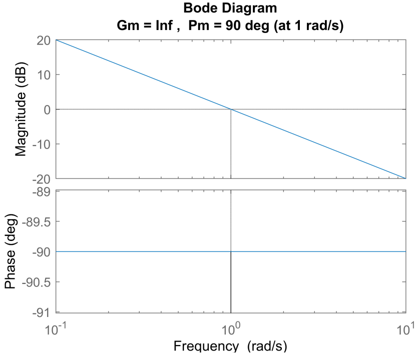
Consider opening one loop:



Example 1: Spinning satellite (cont'd)

RS: We will consider diag. input uncertainty (present in every plant)

Always stable (for all input perturbations)

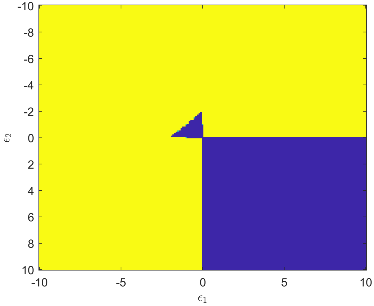


Similar for other loop

Example 1: Spinning satellite (cont'dd)

Uncertainty in input is given by: $u_1' = (1+\epsilon_1)\,u_1, \quad u_2' = (1+\epsilon_2)\,u_2$

It is easy to show that the system is stable for $-1 < \epsilon_1 < \infty, \epsilon_2 = 0$ and $\epsilon_1 = 0, -1 < \epsilon_2 < \infty$



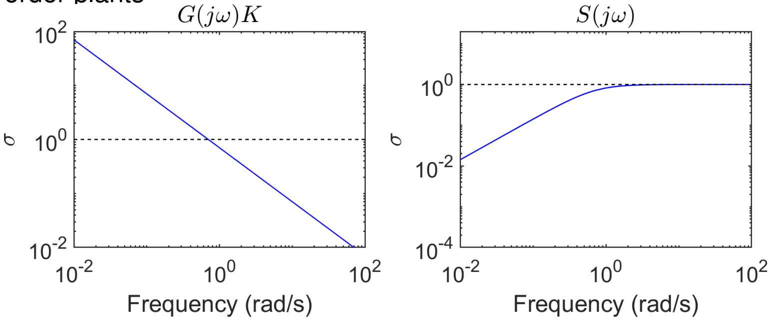
RS: For MIMO GM and PM do not provide RS information. Large $\overline{\sigma}(S)$ indicate robustness issues.

Example 2: Distillation column

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & 86.4 \\ 108.2 & -109.6 \end{bmatrix} \text{ with RGA } \forall\omega \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix}$$

Due to large elements in RGA difficult to control

Controller: inverse with integral action $K_{inv} = \frac{0.7}{s}G(s)^{-1}$

- **NS:** With inverse control you end up with decoupled two first order plants
- 

- **RS:** No high $\overline{\sigma}(S)$ but high RGA values cause for concern ($\Rightarrow$)

Example 2: Distillation column (cont'd)

RS: We will consider diag. input uncertainty
(typically 20% for process applications)

Uncertainty in input is given by: $u_1' = (1+\epsilon_1)\,u_1, \quad u_2' = (1+\epsilon_2)\,u_2$

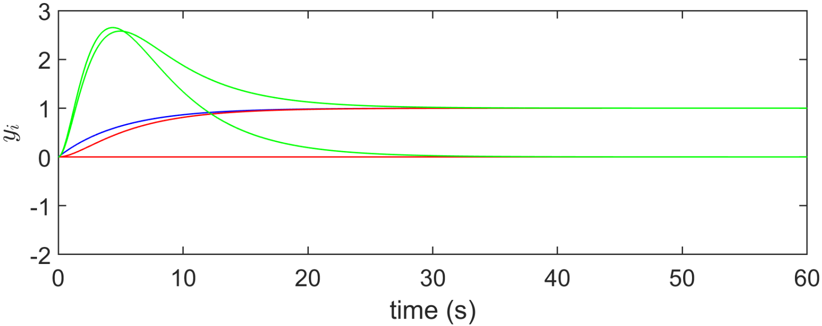
We have: $L(s) = \frac{0.7}{s}\begin{bmatrix} 1+\epsilon_1 & 0 \\ 0 & 1+\epsilon_2 \end{bmatrix}$

Compute poles: $\det(I+L) = (s+0.7\,(1+\epsilon_1))\,(s+0.7\,(1+\epsilon_2))$. We can have up to 100% error in all the input channels

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

Example 2: Distillation column (cont'dd)

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

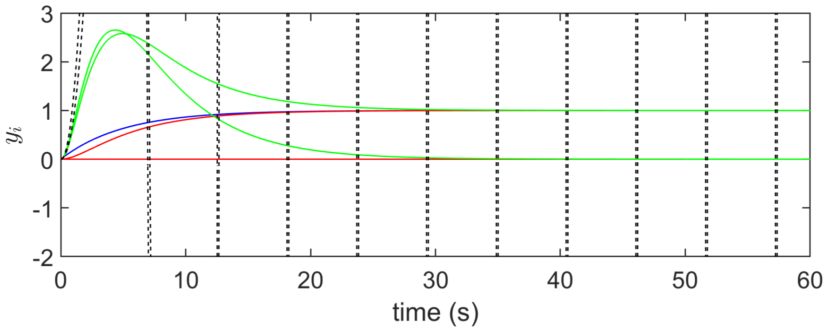


Reference, Nominal control, Uncertain Input, Uncertain I & O

RP From the response we can conclude that we don't have RP

Example 2: Distillation column (cont'ddd)

RS: Consider $u_1' = 1.15u_1$, $u_2' = 0.85u_2$ and $y_1' = 1.15y_1$, $y_2' = 0.85y_2$



Reference, Nominal control, Uncertain Input, Uncertain I & O

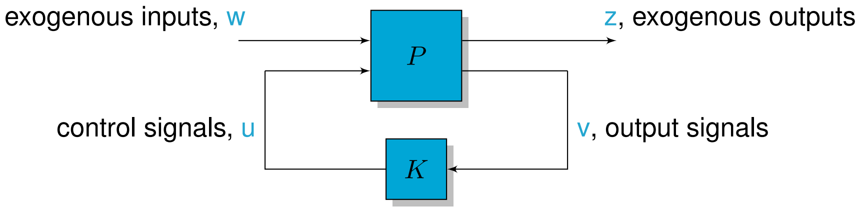
RS With additional output uncertainty also no RS

Conclusions from examples

1. Example 1, excellent PM, GM for single loops but no RS
   Might have been expected from high $\overline{\sigma}(S)$
2. Example 2, good $\overline{\sigma}(S)$ good RS for input uncertainty but no RP
   The RGA can be seen as an indicator for bad diagonal control
3. Example 2, bad RS if we also add output uncertainty

RGA and $\overline{\sigma}(S)$ are indicators for robustness but we need better tools for analysis and synthesis. Later we will develop tools for this within the $\mathcal{H}_\infty$ robust control framework using the structured singular value

Generalized Plant: Definition

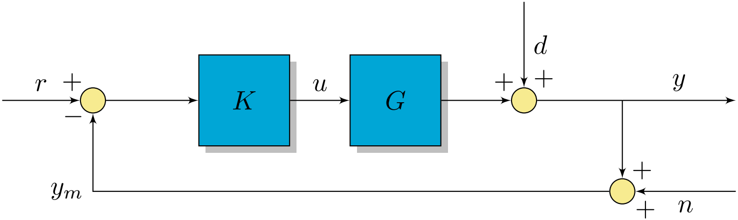


(Note positive feedback)

Find a controller, K, which counteracts the influence of w on z (minimizing a certain norm *e.g.* $\mathcal{H}_\infty$, $\mathcal{H}_2$)

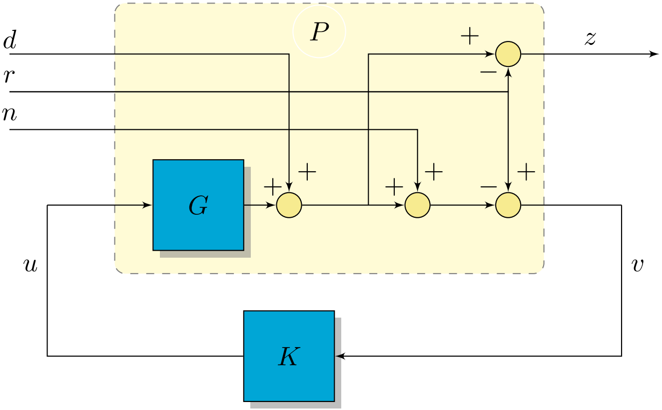
Generalized Plant: Example

Consider the following feedback structure:



$$w = \begin{bmatrix} d \\ r \\ n \end{bmatrix}, \; v = r - y_m \, , \; z = y - r$$

Generalized Plant: Example (cont'd)

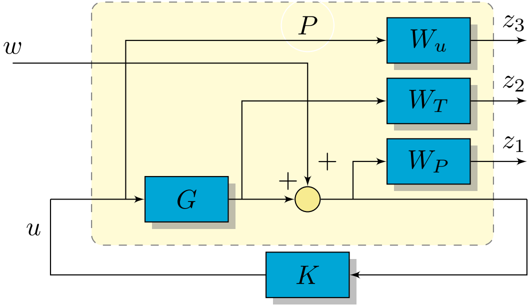


$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix}$$

Generalized Plant: Mixed sensitivity

Can we also embed the Mixed sensitivity problem in this framework?

$$\min_K ||N||_\infty \quad N = \begin{bmatrix} W_P S \\ W_T T \\ W_u K S \end{bmatrix} \qquad P = \left[\begin{array}{c|c} W_P & W_P G \\ 0 & W_T G \\ 0 & W_u \\ \hline I & G \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$



Generalized Plant: Mixed sensitivity (cont'd)

Code: (you need code from Section MIMO Design to run)

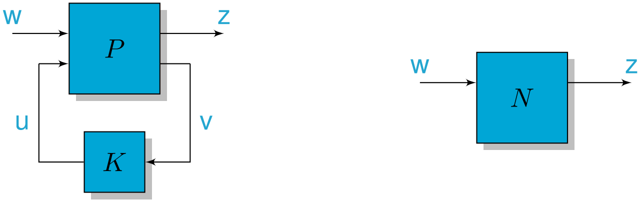
```
 Uses the robust control toolbox
>>systemnames ='G Wp Wu Wt'; % Define systems
>>inputvar ='[w(2); u(2)]'; % Input generalized plant
>>input_to_G= '[u]';
>>input_to_Wu= '[u]';
>>input_to_Wt= '[G]';
>>input_to_Wp= '[w+G]';
>>outputvar= '[Wp; Wt; Wu; G+w]'; % Output generalized plant
>>sysoutname='P';
>>sysic;
>>[K2,CL2,GAM2,INFO2] = hinfsyn(P,2,2); % Hinf design
```

Generalized Plant: Closing the loop

We have:

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$

We have $z = Nw$ if we close the loop $u = Kv$:

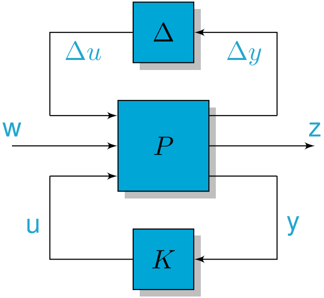


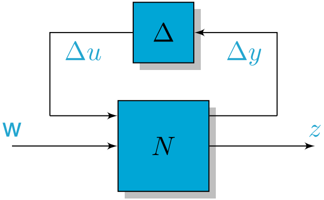
$$N = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21} \triangleq F_l(P, K)$$

$F_l(P, K)$ is called the lower Linear Fractional Transformation (LFT)

Generalized Plant: Robust control

What we are going to do later: Synthesis for robust control (Direct synthesis for RS and RP).





where Δ is a block diagonal matrix containing all the perturbations (typically $||\Delta||_\infty \leq 1$)

Elements of linear system theory & Performance limitations

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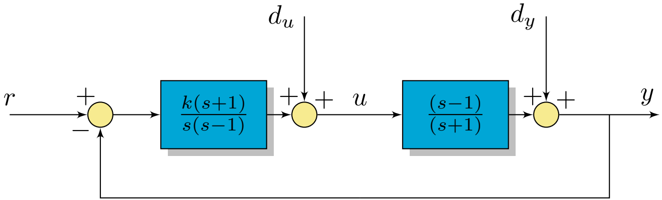
SC42145, 2021/22

Basic Elements (Section 4.1-4.6)

- Super Position $f(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 f(u_1) + \alpha_2 f(u_2)$
- System representations (Transfer Functions, State-Space, Impulse Response, FRF)
- Controllability and Observability
- Minimal Realization (smallest possible dimension of the state space realization)
- Stability, bounded input results in bounded output
- Stabilizable, if all unstable modes are controllable
- Detectable, if all unstable modes are observable
- Poles and zeros, note that they have associated direction for MIMO systems

Internal stability of feedback systems: Example



Q: What happens if we look at the loop transfer function L?

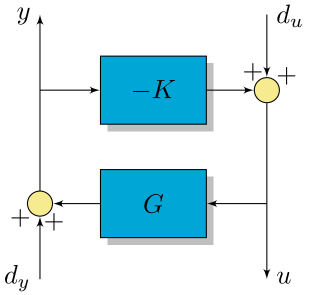
A: $L = \frac{k}{s}$ and $S = \frac{y}{d_y} = \frac{s}{s+k}$. So, stable for $k > 0$

Q: What happens if we look at $\frac{u}{d_y} = -K\left(I + GK\right)^{-1}$?

A: $\frac{u}{d_y} = \frac{-k(s+1)}{(s-1)(s+k)}$. Unstable $\forall k$

The system is internally unstable!!

Internal stability of feedback systems



$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y$$
$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y$$

Internal stability I: Assume G and K contain no unstable hidden modes. Internal stable iff all the above transfer functions are stable

Internal stability II: All RHP-poles are contained in the min. realization of GK and KG. The system is internally stable iff one of the above transfer functions is stable

Internal stability of feedback systems: Implications

If we have internal stability. The following holds:

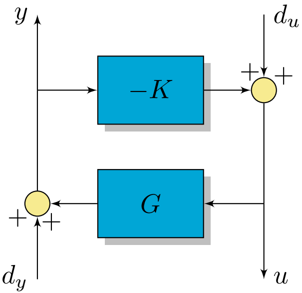
1. If $G(s)$ has a RHP-zero at z. The following TF's also have a RHP-zero at z: L, L_I, T, SG, T_I

   Sketch of proof: (1) there are no RHP pole-zero cancelations (2) S is stable $\Rightarrow T = GKS$ there is not RHP-pole so no cancelation, etc.

2. If $G(s)$ has a RHP-pole at p. The following TF's have a RHP-pole at p: L, L_I. While the following TF's have a RHP-zero at p. S, KS, S_I

   Sketch of proof: (1) part 1 is obvious (2) From LS it follows that S should contain RHP-zero at p to cancel RHP-pole in L

Internal Model Control (IMC)



$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y$$
$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y$$

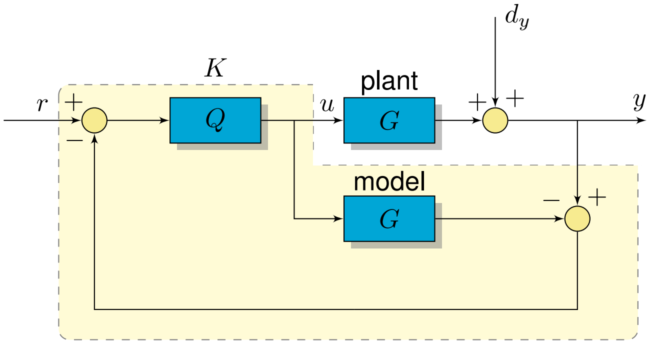
$$u = (I - QG) d_u - Q d_y$$
$$y = G(I - QG) d_u + (I - GQ) d_y$$

(use $S + T = I$ and push through rule)

Given $Q = K\left(I + GK\right)^{-1}$. If G is **stable** the system above is internally stable iff Q is stable

The set of all stabilizing controllers: $K = \left(I - QG\right)^{-1} Q = Q\left(I - GQ\right)^{-1}$ where Q is *any stable TF.*

Internal Model Control (IMC) /Youla parameterization



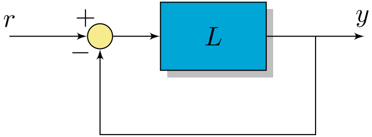
$K = (I - QG)^{-1} Q = Q (I - GQ)^{-1}$ where Q is *any stable TF.*

What is the order of the controller?

Why useful? For optimization, we only have to search over stable Q and all the other TF's are affine in Q (T or S in the form: $H_1 + H_2 Q H_3$)

Stability analysis in the frequency domain

- Stability: NO RHP-poles of the closed loop
- Bode stability criteria (only information $L(j\omega)$) enables design. Only SISO!
- Nyquist can be extended to MIMO (only information $L(j\omega) = GK(j\omega)$)



Stability analysis in the frequency domain

Stability of the OL is determined by the poles of L.

$L(s) = \left[\begin{array}{c|c} A_{ol} & B_{ol} \\ \hline C_{ol} & D_{ol} \end{array}\right]$ or in TF: $L(s) = C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol}$

The OL Poles are given by: $\phi_{ol}(s) = \det\left(sI - A_{ol}\right)$

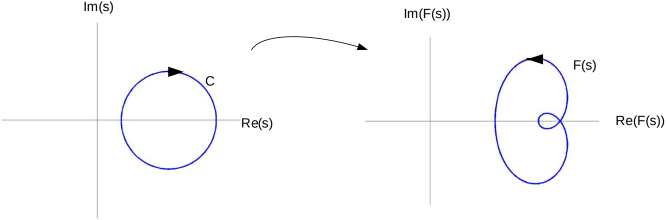
Internal stability $\Rightarrow$ stability of $(I + L(s))^{-1}$. The state matrix of S:
$A_{cl} = A_{ol} - B_{ol}(I + D_{ol})^{-1}C_{ol}$

The CL Poles are given by:
$\phi_{cl}(s) = \det\left(sI - A_{ol} + B_{ol}(I + D_{ol})^{-1}C_{ol}\right)$

Determinant return difference operator:
$\det(I + L(s)) = \det(I + C_{ol}(sI - A_{ol})^{-1}B_{ol} + D_{ol}) = c\frac{\phi_{cl}(s)}{\phi_{ol}(s)}$ (using Schur's formula)

Cauchy’s argument principle

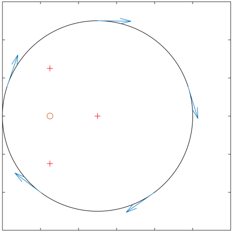
Evaluate a transfer function $F(s)$ along a clockwise contour C



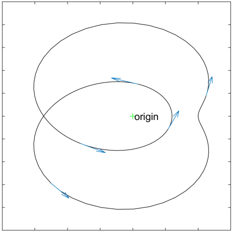
Contour Map encircles origin $N = Z - P$ times clock wise

- Z: number of zeros $F(s)$ inside C
- P: number of poles $F(s)$ inside C

Example 1

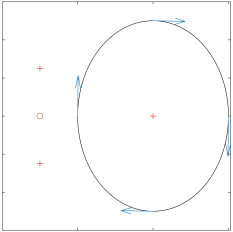


$\xrightarrow{F(s)}$

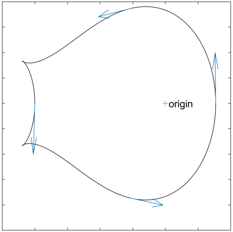


- $F(s)$ has 1 zero in $C \Rightarrow Z = 1$
- $F(s)$ has 3 poles in $C \Rightarrow P = 3$
- Hence: $N = Z - P = -2$

Example 2

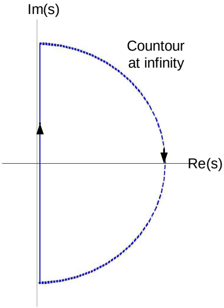


$\overrightarrow{F(s)}$



- $F(s)$ has 0 zeros in $C \Rightarrow Z = 0$
- $F(s)$ has 1 pole in $C \Rightarrow P = 1$
- Hence: $N = Z - P = -1$

Nyquist: Let C contain RHP



Apply the argument principle on $\det(I + L(s))$

$$\det(I+L(s)) = c \cdot \frac{\text{Closed Lp characteristic polynomial}}{\text{Open Lp characteristic polynomial}}$$

- Z (zeros in C = RHP) : unstable closed loop poles
- P (poles in C = RHP) : unstable open loop poles

Generalized Nyquist: N is the number clock wise encirclements of origin by Contour Map $\det(I + L(s))$

Small gain theorem

The small-gain theorem is a very general result.

Small-gain theorem: Consider a system with a stable $L(s)$. Then the CL is stable if

$$||L(j\omega)|| < 1 \forall \omega,$$

where $|| \cdot ||$ represents a matrix norm.

Says: if the system gain is less than 1 in all directions and for all frequencies, then all signal deviations will die out, system is stable

Conservative: consider SISO, $||L(j\omega)||_F = |L(j\omega)|$ we know from Bode stability condition we only require $|L(j\omega)| < 1$ for $\angle L(j\omega) = 180^o + n \cdot 360^o$

Theorem doesn't consider phase information.

Extremely useful for RS and RP synthesis (comes later)

Norms

Norm gives a single number, of a: vector, matrix, signal or system.

Norm: A norm of e is a real number, denoted by $||e||$, that satisfies:

1. Non-negative: $||e|| \geq 0$
2. Positive: $||e|| = 0 \Leftrightarrow e = 0$
3. Homogeneous: $||\alpha e|| = |\alpha| \cdot ||e|| \quad \forall \alpha \in \mathcal{C}$
4. Triangle inequality: $||e_1 + e_2|| \leq ||e_1|| + ||e_2||$
5. (for matrix norms) Multiplicative property: $||AB|| \leq ||A|| \cdot ||B||$ where A and B are matrices.

Vector: $||a||_p = (\sum_i |a_i|^p)^{\frac{1}{p}}$

$||a||_1 = \sum_i |a_i|$, $||a||_2 = \sqrt{(\sum_i |a_i|^2)}$, $||a||_\infty = \max_i |a_i|$

Matrix:

$$||A||_F = \sqrt{\left(\sum_{i,j} |a_{ij}|^2\right)} = \sqrt{\mathrm{tr}(A^H A)}$$

For control (remember generalized plant): $\min_K ||N||$. Which norm?

System norms

where: w is an input signal, z is an output signal and G stable TF with impulse response $g(t)$.

Given w how big can z become?

For the output signal we use: $||z(t)||_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 d\tau}$.

Three types of inputs:

1. $w(t)$ is a series of unit impulses. $\mathcal{H}_2$-norm
2. $w(t)$ any signal $||w(t)||_2 = 1$. $\mathcal{H}_\infty$-norm
3. $w(t)$ any signal $||w(t)||_2 = 1$, but with $w(t) = 0$ for $t \geq 0$ measure z for $t \geq 0$. Hankel-norm

System norms: $\mathcal{H}_2$-norm

(assume strictly proper, $D = 0$) For $\mathcal{H}_2$-norm: $w(t)$ is a series of unit impulses.

$$w \longrightarrow \boxed{G} \longrightarrow z$$

Frequency domain interpretation:

$$||G(s)||_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\operatorname{tr}\left(G(j\omega)^H G(j\omega)\right)}_{||G(j\omega)||_F^2 = \sum_i \sigma_i^2(G(j\omega))} d\omega}$$

Time domain interpretation:

$$||G(s)||_2 = ||g(t)||_2 \triangleq \sqrt{\int_0^{\infty} \underbrace{\operatorname{tr}\left(g(\tau)^H g(\tau)\right)}_{||g(\tau)||_F^2} d\tau}$$

System norms: $\mathcal{H}_\infty$-norm

(assume proper, $D \neq 0$) For $\mathcal{H}_\infty$-norm: $||w(t)||_2 = 1$.

Frequency domain interpretation (peak of TF):

$$||G(s)||_\infty \triangleq \max_\omega \overline{\sigma}\left(G(j\omega)\right)$$

Time domain interpretation (worst case gain):

$$||G(s)||_\infty = \max_{w(t)\neq 0} \frac{||z(t)||_2}{||w(t)||_2} = \max_{||w(t)||_2=1} ||z(t)||_2$$

How to compute? (assume $D = 0$) Smallest γ for which $H = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}$ has no eigenvalues on imaginary axis.

Difference between $\mathcal{H}_2$ and $\mathcal{H}_\infty$

For $\mathcal{H}_\infty$-norm: Minimizing the peak, maximum singular value (worst case direction, frequency)
Push down the peak of largest singular value

For $\mathcal{H}_2$-norm: Minimizing all the singular values (all frequencies, directions)
Push down the whole thing

Why do we like the $\mathcal{H}_\infty$ norm?

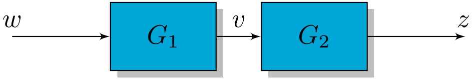
Convenient to represent uncertainties, and it is an induced norm that satisfies the multiplicative property: $||AB|| \leq ||A|| \cdot ||B||$ (useful for cascade interconnections)

What is wrong with the $\mathcal{H}_2$ norm?

Nothing, but no induced norm, no multiplicative property

Why do we like the multiplicative property

Consider two systems in serie:



Multiplicative property: $||AB|| \leq ||A|| \cdot ||B||$ holds for $\mathcal{H}_\infty$ norm

We have $||G_2G_1||_\infty = \max_{w(t)\neq 0} \frac{||z(t)||_2}{||w(t)||_2} = \max_{w\neq 0} \frac{||G_2G_1w||_2}{||w||_2}$.

Multiply with $\frac{||v||_2}{||v||_2}$ we get:

$$\max_{w(t)\neq 0} \frac{||G_2v||_2}{||v||_2} \cdot \frac{||G_1w||_2}{||w||_2} \leq \max_{v(t)\neq 0} \frac{||G_2v||_2}{||v||_2} \cdot \max_{w(t)\neq 0} \frac{||G_1w||_2}{||w||_2}$$

Maximize of v and w independently

Example: $G = G_1 = G_2 = \frac{1}{s+a}$, we have $||G||_\infty = \frac{1}{a}$, $||G||_2 = \sqrt{\frac{1}{2a}}$, $||GG||_\infty = \frac{1}{a^2}$, $||GG||_2 = \sqrt{\frac{1}{a}\frac{1}{2a}}$ so for $a < 1$ we have a problem for the $\mathcal{H}_2$-norm.

Fundamental limitations on sensitivity

Three limitations:

1. We know: $S + T = I$ with $S = (I + L)^{-1}$ and $T = L(I + L)^{-1}$

   Implies: For every ω either $|T| > 0.5$ or $|S| > 0.5$ and $||T| - |S|| < 1$

2. Interpolation constraint I: If p is a RHP-pole of L then: $T(p) = 1$, $S(p) = 0$

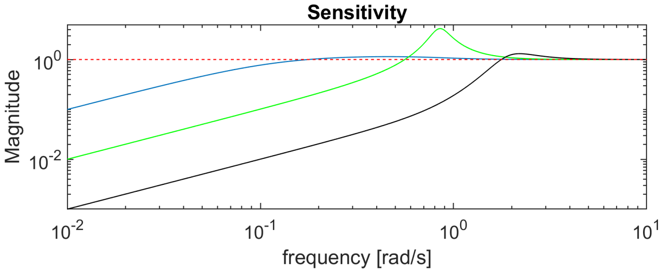
   Interpolation constraint II: If z is a RHP-zero of L then: $T(z) = 0$, $S(z) = 1$

   Follows from internal stability requirement

3. Waterbed effect $\rightarrow$

The waterbed effects

Consider: $G(s) = \frac{1}{s^3+1.6s^2+s}$. Now $0.1 \cdot G(s)$, $1 \cdot G(s)$, $10 \cdot G(s)$



Assume L has a relative degree ≥ 2, and N_p RHP-poles p_i. We have:

$$\int_0^\infty \ln |S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} Re(p_i)$$

So, if there are no RHP-poles surface below and below gain 1 should be equal.

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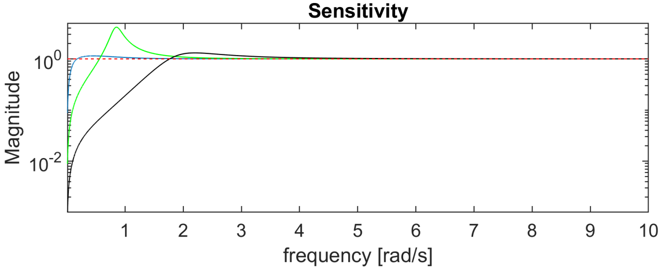
Consider: $G(s) = \frac{1}{s^3+1.6s^2+s}$. Now $0.1 \cdot G(s)$, $1 \cdot G(s)$, $10 \cdot G(s)$



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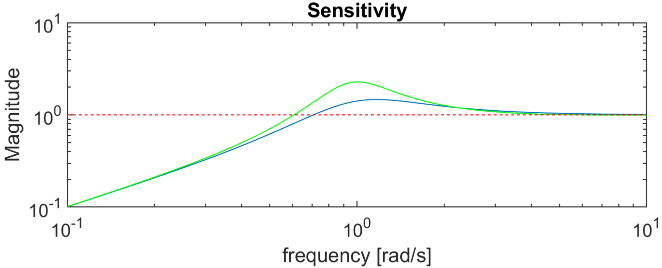
So, if there are no RHP-poles surface below and below gain 1 should be equal.

The waterbed effects, effect of RHP-zero

Consider: $G(s) = \frac{2}{s^2+s}$. Now $G(s)$, $\frac{-s+5}{s+5}G(s)$



Assume L has a single RHP-zero at z, and N_p RHP-poles p_i ($\overline{p}_i$ is complex conjugate). We have:

$$\int_0^{\infty} \ln|S(j\omega)| w(z,\omega) d\omega = \pi \cdot \ln \prod_{i=1}^{N_p} \left| \frac{p_i + z}{\overline{p}_i - z} \right|$$

with $w(z,\omega) = \frac{2z}{z^2+\omega^2}$

The waterbed effects, effect of RHP-zero

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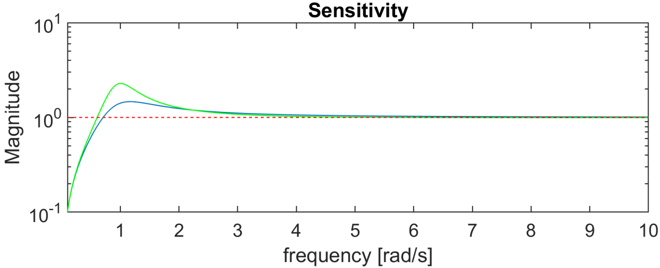

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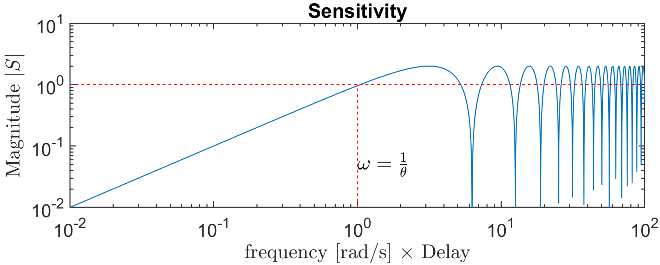
with $w(z,\omega) = \frac{2z}{z^2+\omega^2}$

Limitations imposed by time delays

It should be clear that time delays $(e^{-\theta s})$ can form a significant limitation.

The bandwidth is limited to be less than $\frac{1}{\theta}$.

Consider the ideal T which is given by $T(s) = e^{-\theta s}$ then $S = 1 - e^{-\theta s}$



Pade approximation: $e^{-\theta s} \approx \frac{\left(1-\frac{\theta}{2n}s\right)^n}{\left(1+\frac{\theta}{2n}s\right)^n}$, where n is the order.

Limitations imposed by RHP-zero

Remember, system zeros are invariant under feedback!

From high gain feedback we know that poles will go to zeros

From interpolation constraint we know $S(z) = 1$. When designing w_P we have to realize this.

Example: We consider $w_P(s) = \frac{s/M+\omega_B}{s+\omega_B A}$.

Example: $|w_P(z)| \leq ||W_p S||_\infty < 1$ and therefore $\left|\frac{z/M+\omega_B}{z+\omega_B A}\right| < 1$

Example: If z is real: $\omega_B < z\frac{1-1/M}{1-A}$

MIMO: Fundamental limitations on sensitivity

Again we have to think about directions!!

Same effects are present but expressions are far more complicated (See book).

Drive train dynamics

Let's consider again our drive-train example:

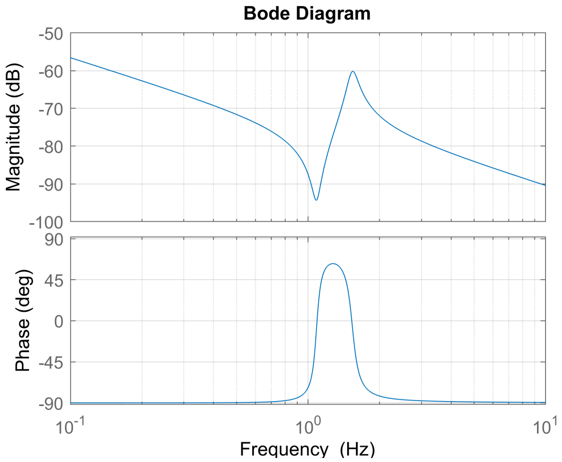
$$\frac{\omega_g}{T_g}$$

Workpoint 1

Workpoint 2

Workpoint 3

Let's design a controller for this WP and TF



Mixed-sensitivity code

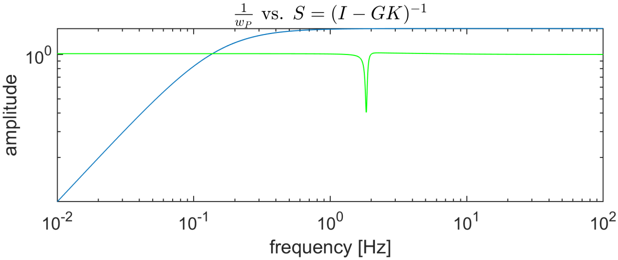
G is given

```
>>wB1=0.1*2*pi; % desired closed-loop bandwidth of 0.1Hz
>>A=1/100; % desired disturbance attenuation inside bandwidth
>>M=1.5 ; % desired bound on hinfnorm(S)
>>Wp=[(s/M+wB1)/(s+wB1*A)]; % Sensitivity weight
>>Wu=1; % Control weight
>>systemnames ='G Wp Wu '; % Define systems
>>inputvar ='[w(1); u(1)]'; % Input generalized plant
>>input_to_G= '[u]';
>>input_to_Wu= '[u]';
>>input_to_Wp= '[w+G]';
>>outputvar= '[Wp; Wu; G+w]'; % Output generalized plant
>>sysoutname='P';
>>sysic;
>>[K,CL,GAM,INFO] = hinfsyn(P,1,1); % Hinf design
```

Design results

Results:
$$\frac{\omega_g}{T_g}$$

Sensitivity



$$||N||_\infty = 498!!$$

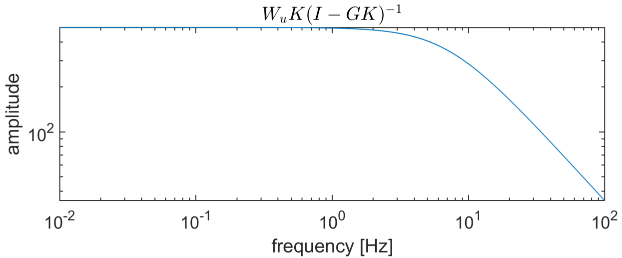
Design results

Results:

$$\frac{\omega_g}{T_g}$$

Weight on input

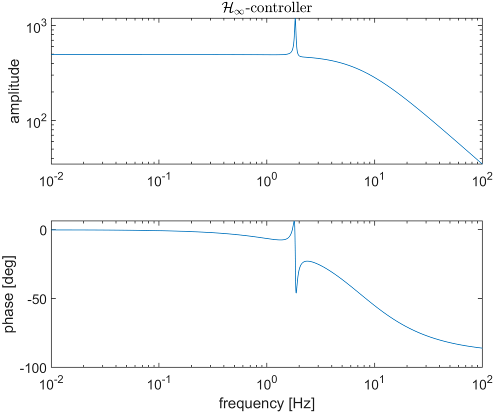
$$||N||_\infty = 498!!$$



Design results

Results:

$$\frac{\omega_g}{T_g}$$



Controller

$$||N||_\infty = 498!!$$

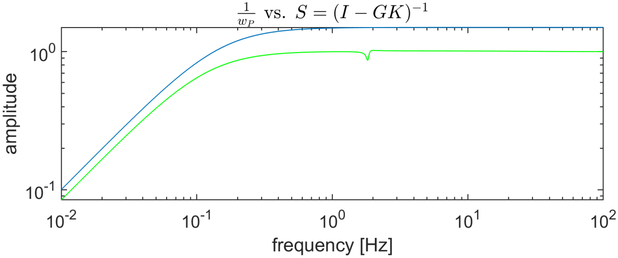
We didn't apply scaling!!!

Design results (cont'd)

Results with scaling of w_U:

$$\frac{\omega_g}{T_g}$$

Sensitivity



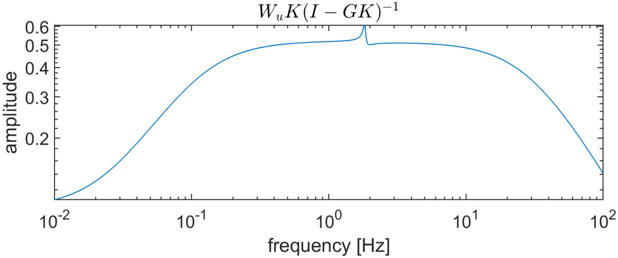
$$||N||_\infty = 0.85$$

Design results (cont'd)

Results with scaling of w_U:

$$\frac{\omega_g}{T_g}$$

Weight on input

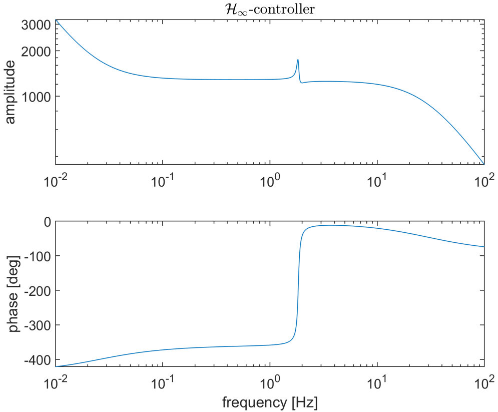


$$||N||_\infty = 0.85$$

Design results (cont'd)

Results with scaling of w_U:

$$\frac{\omega_g}{T_g}$$



Controller

$$||N||_\infty = 0.85$$

Controller unstable but S stable with RHP-zeros that cancel RHP-poles of K!!!

Uncertainty and robustness for SISO systems

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SC42145, 2021/22

How do we define robustness?

A control system is robust if it is insensitive to differences between the actual system and the model used for controller synthesis

The difference between the model and the real system is called: model/plant mismatch or model uncertainty

In the $\mathcal{H}_\infty$ robust control paradigm we check if the system is **RS** and **RP** for the worst-case scenario

In this lecture we introduce the tools/framework to check **RS** and **RP** for SISO systems (for $N = W_p S$)

Approach:

1. Determine the uncertainty set (structured and unstructured uncertainty)
2. Check **RS** for all the systems in the uncertainty set
3. If **RS** we check **RP** for all the systems in the uncertainty set

Comments:

a. Will not always achieve optimal performance
b. We will not consider faulty sensors/actuators, or robustness of the optimization algorithm
c. We assume a fixed linear controller

How do we define an uncertainty set?

Π - set of all possible perturbed models (Uncertainty set)

$G(s) \in \Pi$ - Nominal plant

$G_p(s) \in \Pi$ or $G'(s) \in \Pi$- Particular plant models

In this course we use a norm-bounded uncertainty description. For example, this will look like $\Pi = G + Gw_I\Delta$ where w_I is used for scaling and $||\Delta||_\infty < 1$ is the set of all normalized perturbations.

Why not add the uncertainty as additional fictitious disturbances

Suppose we have a nominal plant: $y = Gu + G_d d$

And let $G_p = G + E$ (where E is an additive uncertainty)

If we design a controller, K, for: $y = Gu + \underbrace{Eu}_{d_1} + \underbrace{G_d d}_{d_2}$

For the closed-loop we get: $S = (I + (G + E)\,K)^{-1}$ can be unstable for some $E \neq 0$ so no **RS**

Where is the uncertainty coming from?

- There are always unknown parameters in the model
- Parameter may enter the system in a nonlinear way
- Imperfections measurement device
- At high frequencies the model is not accurate
- It can be preferable to have a low order model

Model uncertainty can be grouped in two classes:

1. **Parametric (real) uncertainty**: uncertain parameters in a known model structure
2. **Dynamic (frequency-dependent) uncertainty**: error due to missing dynamics

Parametric uncertainty vs Dynamic uncertainty

Parametric uncertainty:

We assume a parameter is bounded within a certain interval $[k_{min}, k_{max}]$

This can be written as: $k_p = \overline{k}\,(1 + r_k\Delta)$. Where $\overline{k}$ is the mean value, $r_k = (k_{max} - k_{min})\,/\,(k_{max} + k_{min})$ is the relative uncertainty, and Δ is any *real scalar* $|\Delta| \leq 1$

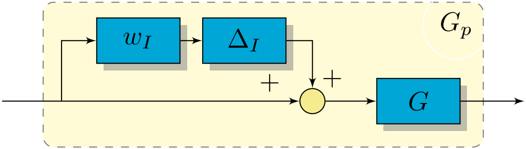
Dynamic uncertainty:

Little bit more difficult to grasp but typically we will use the frequency domain to quantify this uncertainty

Typically this leads to normalize *complex perturbations*: $||\Delta||_\infty \leq 1$

Main topic of this lecture

Multiplicative uncertainty



Often lump dynamic uncertainties into multiplicative uncertainty of the form:

$$\Pi_I : \quad G_p(s) = G(s)\left(1 + w_I(s)\Delta_I(s)\right); \qquad \underbrace{|\Delta_I(j\omega)| \leq 1 \forall\omega}_{||\Delta_I||_\infty \leq 1}$$

Remark 1: Δ_I can be any stable transfer function which is norm bounded by 1

Remark 2: I stands for input but for SISO systems we can also rewrite an output uncertainty as an input uncertainty

Remark 3: You also have the inverse multiplicative uncertainty:

$$\Pi_I : \quad G_p(s) = G(s)\left(1 + w_{iI}(s)\Delta_{iI}(s)\right)^{-1}; \qquad ||\Delta_{iI}||_\infty \leq 1$$

From Parametric to multiplicative uncertainty

Some parametric uncertainties can be rewritten as multiplicative uncertainties

Gain uncertainty:

Suppose $G_p = k_p G_o$ with $k_{min} \leq k_p \leq k_{max}$ this can be rewritten as:

$$G_p = \underbrace{\overline{k}G_o}_{G}\left(1 + \underbrace{r_k}_{w_I}\Delta\right) \text{ with } |\Delta| \leq 1$$

Time constant uncertainty:

Suppose $G_p = \frac{1}{\tau_p s + 1} G_o$ with $\tau_{min} \leq \tau_p \leq \tau_{max}$. By rewriting τ_p as $\overline{\tau}\left(1 + r_\tau \Delta\right)$ we get: $G_p = \frac{G_o}{1+\overline{\tau}s + r_\tau \overline{\tau} s \Delta} = \underbrace{\frac{G_o}{1+\overline{\tau}s}}_{G} \frac{1}{1+w_{i1}\Delta}$ with

$$w_{i1} = \frac{r_\tau \overline{\tau} s}{1+\overline{\tau}s}$$

Pole uncertainty:

See time constant uncertainty

Example

Lets consider the following uncertain plant:

$$\dot{x} = \begin{bmatrix} -(1+k) & 0 \\ 1 & -(1+k) \end{bmatrix} x + \begin{bmatrix} \frac{1-k}{k} \\ -1 \end{bmatrix} u$$

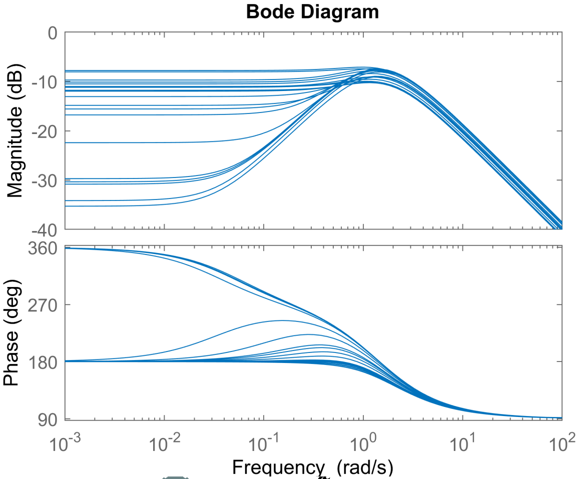
$$y = \begin{bmatrix} 1 & \alpha \end{bmatrix} x$$

where $k = 0.5 + 0.1\delta_1$, $|\delta_1| \leq 1$ and $\alpha = 1 + 0.2\delta_2$ with $|\delta_2| \leq 1$
Code:

```
 Making use of the robust control toolbox
>>k = ureal('k', 0.5, 'Range',[0.4 0.6]); % uncertain parameter
>>alpha = ureal('alpha', 1, 'Range', [0.8 1.2]);
>>A = [-(1+k) 0; 1 -(1+k)];
>>B = [(1/k -1), -1]';
>>C = [0 alpha];
>>Gp = ss(A,B,C,0);
```

Example (cont'd)

```
Making use of the robust control toolbox
>>bode(Gp);
```



Uncertainty in frequency domain

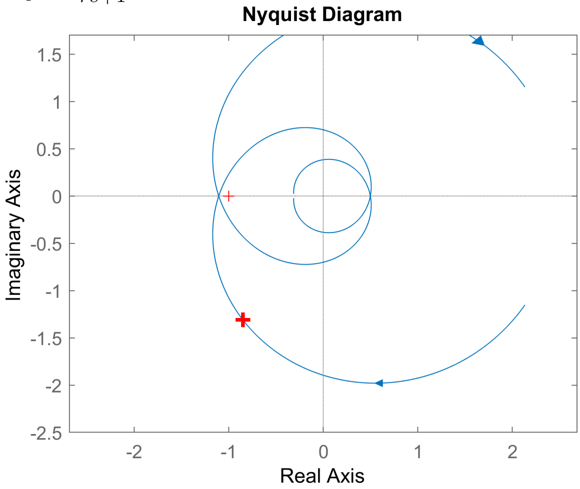
The design of feedback controllers in the presence of non-parametric and unstructured uncertainty ... is the *raison d'etre* for $\mathcal{H}_\infty$ feedback optimization, for parametric uncertainty there are other methods

Parametric uncertainty is often replaced by complex uncertainty:
$|\Delta| \leq 1 \quad \Rightarrow |\Delta(j\omega)| \leq 1$

This becomes beneficial when there are multiple uncertain parameters and then we can lump them in one single complex uncertainty

Example

$$G_p = \frac{k}{\tau s+1}e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \omega = 0.5$$



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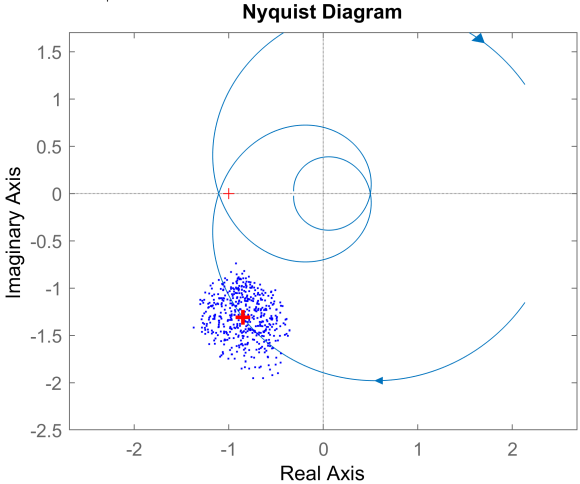

Example

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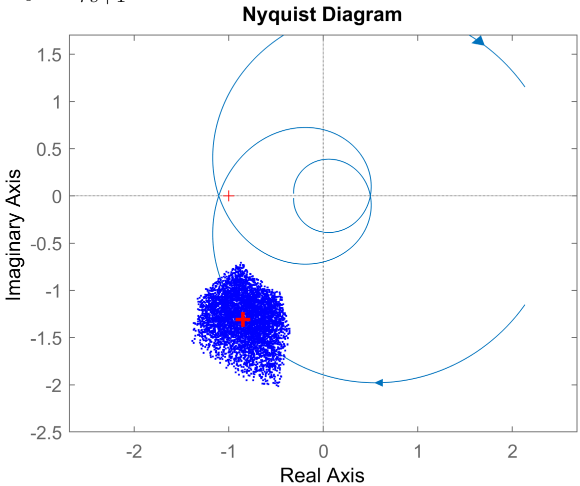
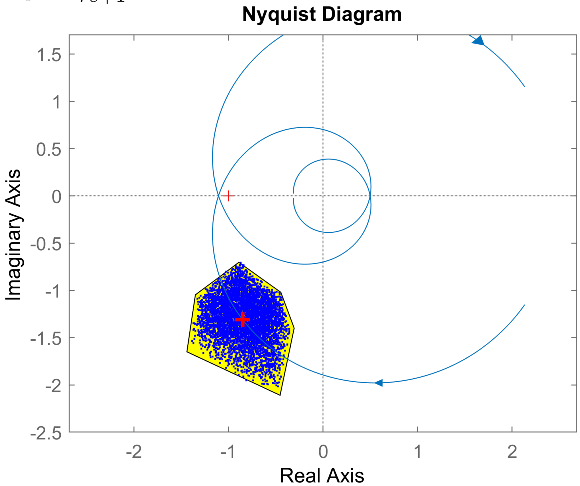


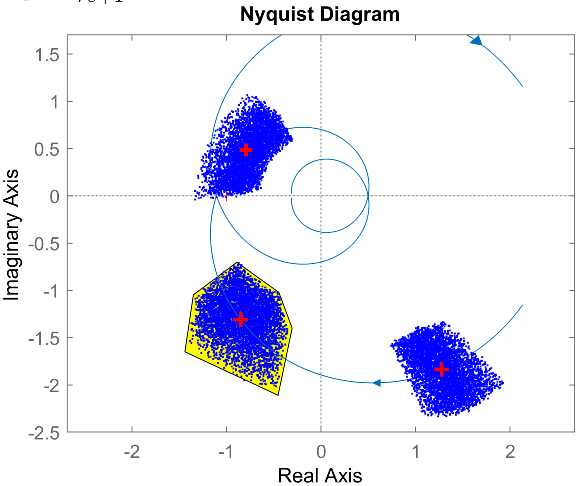
Example

$$G_p = \frac{k}{\tau s+1}e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \omega = 0.5$$



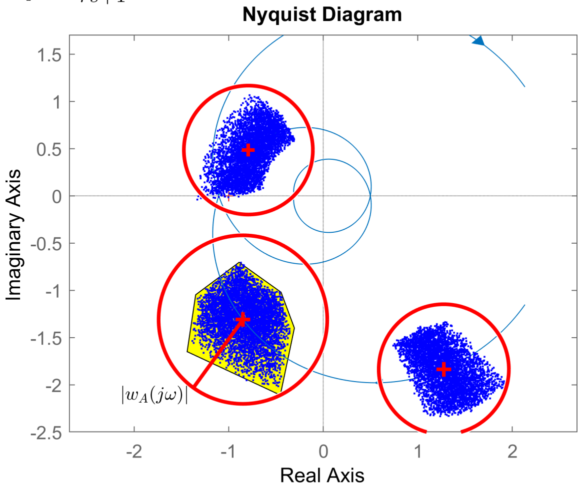
Example

$G_p = \frac{k}{\tau s+1} e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3, \omega = 0.5$, and $\omega = 1$ and $\omega = 0.2$



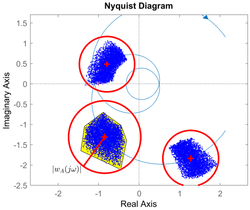
Example

$G_p = \frac{k}{\tau s+1}e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3$, $\omega = 0.5$, and $\omega = 1$ and $\omega = 0.2$



Covered by a disc (complex (additive) uncertainty)

Additive uncertainty and Multiplicative uncertainty



Additive uncertainty: Π_A : $G_p = G + w_A\Delta_A$ with $|\Delta_A(j\omega)| \leq 1 \forall\omega$

Multiplicative uncertainty: Π_I : $G_p = G\,(1 + w_I\Delta_I)$ with $\Delta_I(j\omega) \leq 1 \forall\omega$

With $|w_I(j\omega)| = \frac{|w_A(j\omega)|}{|G(j\omega)|}$ these two representations are equivalent

Remember: The Δ's are **any** stable transfer function

How can we construct Δ_A or Δ_I?

1. Select nominal model G
2. Additive uncertainty find the smallest radius that includes all possible plants: $l_A(\omega)$ ($\forall\omega$) and then fit a transfer function $|w_A(j\omega)| \geq l_A(\omega) \quad \forall\omega$
3. Multiplicative uncertainty find the smallest l_I that satisfies:

$$l_I(\omega) = \max_{G_p \in \Pi} \left| \frac{G_p(j\omega) - G(j\omega)}{G(j\omega)} \right|$$

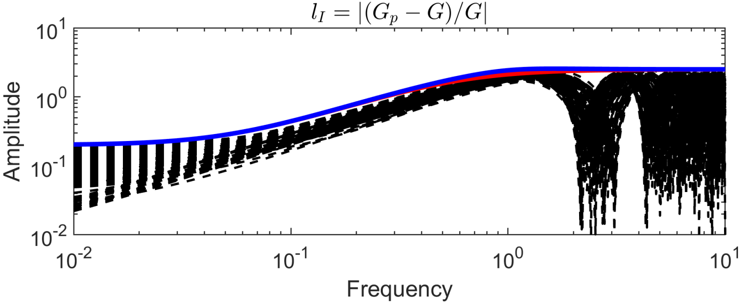
and then fit a transfer function $|w_I(j\omega)| \geq l_I(\omega) \quad \forall\omega$

Example: How can we construct Δ_I?

Again we consider: $G_p = \frac{k}{\tau s+1}e^{-\theta s}, \quad 2 \leq k, \theta, \tau \leq 3$

We can use *usample* from robust control toolbox but can not deal with delay

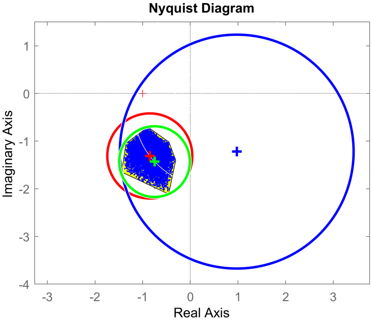
We just randomly sample k, τ, θ around nominal value 2.5, 2.5 and 0 respectively



$w_I = \frac{4s+0.2}{\frac{4}{2.5}s+1} \frac{s^2+1.6s+1}{s^2+1.4s+1}$

Choice of nominal model

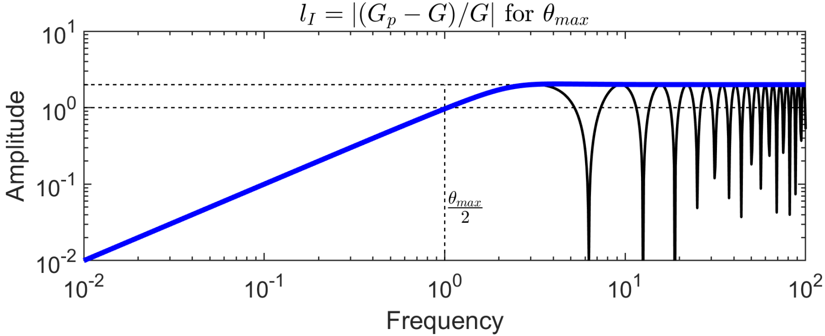
1. Mean parameter values
2. A simplified model (delay free)
3. Smallest discs



Neglected dynamics represented

We consider $G_p = G_o f$ where f represents the neglected dynamics

Neglected delay: $f = e^{-\theta_p s}$ where $0 \leq \theta_p \leq \theta_{max}$



$$w_I = \frac{\left(1+\frac{r_k}{2}\right)\theta_{max} s + r_k}{\frac{\theta_{max}}{2}s+1} \cdot \frac{\left(\frac{\theta_{max}}{2.363}\right)^2 s^2 + 2\cdot 0.838 \cdot \frac{\theta_{max}}{2.363}s+1}{\left(\frac{\theta_{max}}{2.363}\right)^2 s^2 + 2\cdot 0.685 \cdot \frac{\theta_{max}}{2.363}s+1}$$

With relative gain uncertainty r_k

Unmodelled dynamics uncertainty

With unmodelled we refer to dynamics we don't know (different from neglected dynamics)

We typically use the following multiplicative uncertainty:

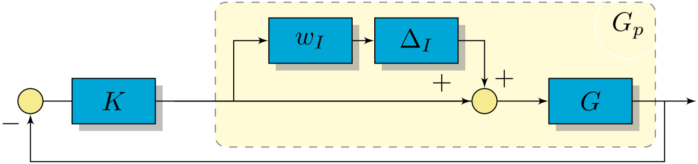
$$w_I = \frac{\tau s + r_0}{\frac{\tau}{r_\infty} s + 1},$$

where:

- r_0 is relative uncertainty at steady state,
- $\frac{1}{\tau}$ the frequency where you have 100% uncertainty,
- r_∞ the weight at high frequencies (> 2)

Selection of these parameters based on application

Robust Stability (RS)



$$L_p = G_p K = GK\left(1 + w_I \Delta_I\right) = L + w_I L \Delta_I,\ \text{where } |\Delta_I(j\omega)| \leq 1 \quad \forall \omega$$

How can we check for **RS**??

Robust Stability (RS): nyquist

$L_p = L + w_I L \Delta_I$,
where $|\Delta_I(j\omega)| < 1 \quad \forall\omega$

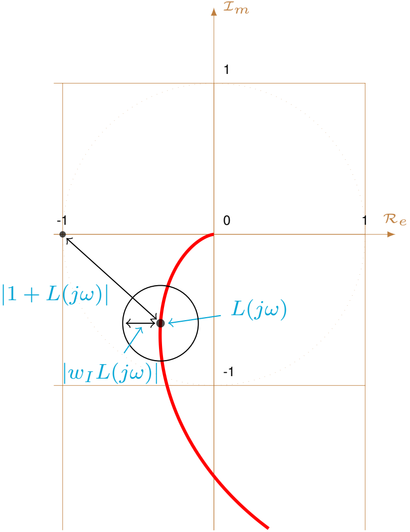
RS $\Leftrightarrow$ stable $\forall L_p$

$\Leftrightarrow$ L_p should not encircle -1

RS $\Leftrightarrow |1 + L(j\omega)| > |w_I L(j\omega)|$

RS $\Leftrightarrow \left|\frac{w_I L(j\omega)}{1+L(j\omega)}\right| < 1, \forall\omega$

RS $\Leftrightarrow ||w_I T||_\infty < 1$



Robust Stability (RS): Algebraic derivation

Assume: Stable L_p and **NS** (nyquist doesn't encircle -1)

Line of reasoning: If one of the circles contains the point -1 we don't have **RS**

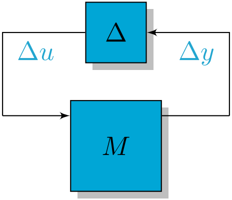
$$\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1+L_p| \neq 0, \qquad && \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1+L_p| > 0, \qquad && \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1+L+w_I L\Delta_I| > 0, \qquad && \forall|\Delta_I| < 1, \forall\omega
\end{aligned}$$

Last condition is most easily violated if $\Delta_I = 1$ and if $1+L$ and $w_I L\Delta_I$ have opposite signs

$$|1+L| - |w_I L| > 0, \quad \forall\omega$$

RS $\Leftrightarrow \left|\frac{w_I L(j\omega)}{1+L(j\omega)}\right| = |w_I T| < 1, \forall\omega$

Robust Stability (RS): $M\Delta$-structure derivation



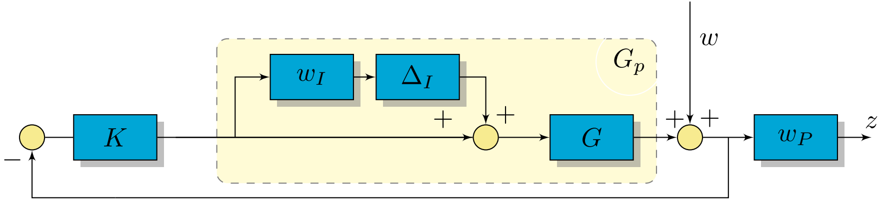
Apply nyquist to this new feedback structure and assume M stable

It should hold that $|1 + M\Delta| > 0 \quad \forall\omega, \quad \forall|\Delta| \leq 1$

$$\begin{aligned} \text{RS} \quad &\Leftrightarrow \quad 1 - |M(j\omega)| > 0, \quad \forall\omega \\ &\Leftrightarrow \quad |M(j\omega)| < 1, \quad \forall\omega \end{aligned}$$

In the next lecture we will use the small-gain theorem for this structure

Robust Performance (RP)



We assume **NS**

We can check for **RS**

How can we check for **RP**

Robust Performance (RP): nyquist

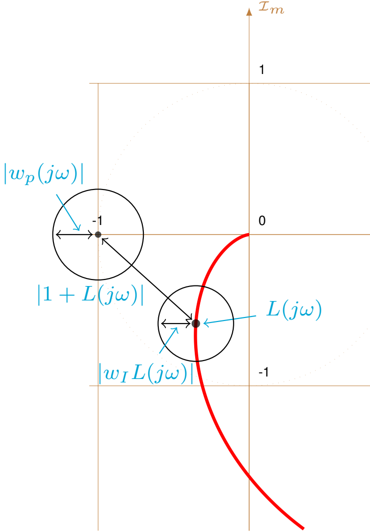
NP: $|w_P S| < 1 \; \forall\omega \Leftrightarrow |w_P| < |1 + L| \; \forall\omega$

RP: $|w_P S_p| < 1 \; \forall\omega \Leftrightarrow |w_P| < |1 + L_p| \; \forall\omega, L_p$

RP: $\Leftrightarrow |w_P| < |1 + L + w_I L \Delta_I| \; \forall\omega, |\Delta_I| < 1$

$$\begin{aligned} \text{RP} \quad &\Leftrightarrow \quad |w_P| + |w_I L| < |1 + L|, \quad \forall\omega \\ &\Leftrightarrow \quad |w_P S| + |w_I T| < 1, \quad \forall\omega \end{aligned}$$

RP: $\Leftrightarrow \max_\omega |w_P S| + |w_I T| < 1$



Summary:

NP: $\Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega$

RS: $\Leftrightarrow \quad |w_I T| < 1 \quad \forall\omega$

RP: $\Leftrightarrow \quad |w_P S| + |w_I T| < 1 \quad \forall\omega$

Robust Stability for MIMO systems

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Delft University of Technology
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SC42145, 2021/22

Uncertainty

Previous lecture we introduced an uncertainty block Δ.

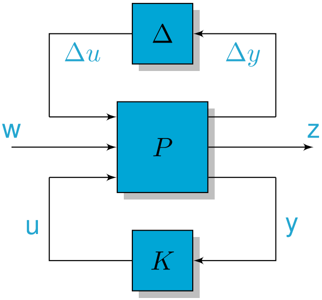
For MIMO systems it is of interest to consider several Δ_i's

Approach: We collect all the uncertainties in a big block diagonal uncertainty block:

$$\Delta = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix}$$

If we now pull out all the uncertainties and controller:

$\Downarrow$



Useful for controller synthesis

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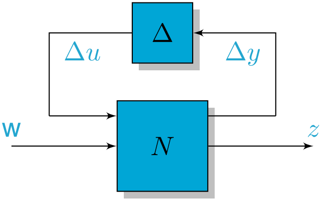

Useful for **RP**

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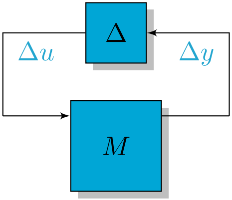


Useful for **RS**

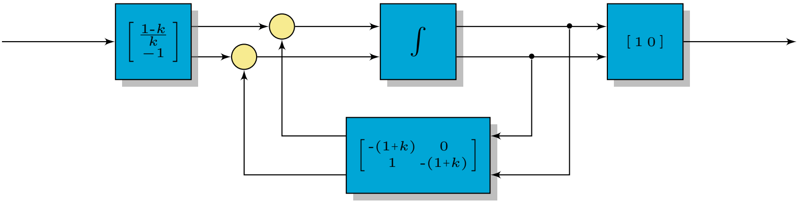
Example

Suppose we have the following SISO system:

$$\dot{x} = \underbrace{\begin{bmatrix} -(1+k) & 0 \\ 1 & -(1+k) \end{bmatrix}}_{A_p} x + \underbrace{\begin{bmatrix} \frac{1-k}{k} \\ -1 \end{bmatrix}}_{B_p} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

where $k = 0.5 + 0.1\delta$ where $|\delta| < 1$.



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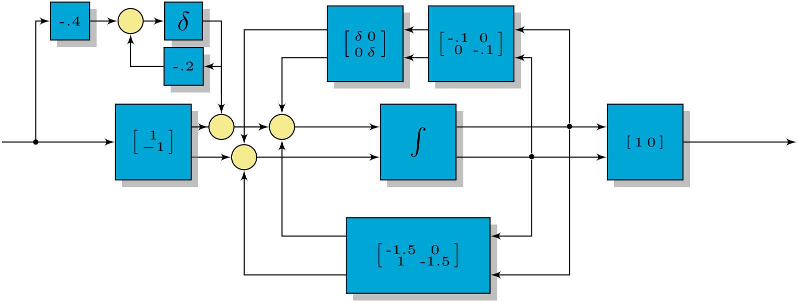
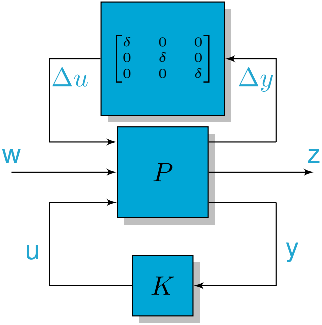


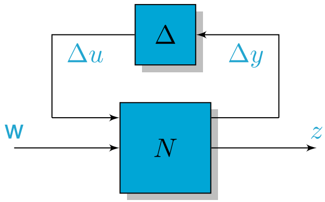
Example (cont'd)



Observe:

- We have a diagonal block uncertainty
- We have considered a SISO system
- There is more structure all the $\delta's$ are the same

Generalization

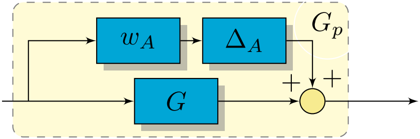


- For every Δ_i we have $\overline{\sigma}(\Delta_i(j\omega)) \leq 1 \forall \omega$
- Due to diagonal structure $||\Delta||_\infty \leq 1$
- Remember that Δ has structure, so we don't allow all Δ's
- Therefore we will introduce in this lecture the structured singular value, μ
- μ can directly be used for analysis
- μ can be used for synthesis (solving a couple of scaled H_∞ problems)

SISO vs MIMO uncertainty descriptions

The representation of parametric uncertainty carries straight over to MIMO systems

Unstructured perturbations are often used to get a simple uncertainty model. One uncertainty source is considered to be full. For example:



Additive uncertainty: Π_A : $\quad G_p = G + w_A \Delta_A$

For SISO lump uncertainties in a single complex uncertainty.
For MIMO?

SISO vs MIMO uncertainty descriptions

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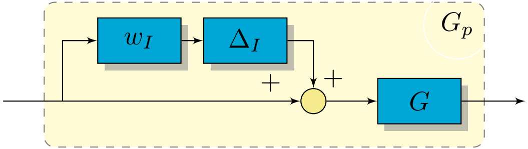

Multiplicative input uncertainty: Π_I : $\quad G_p = G(I + w_I \Delta_I)$

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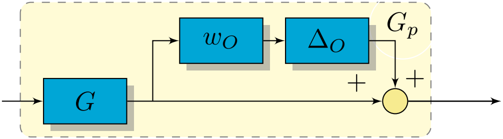

Multiplicative output uncertainty: Π_O : $\quad G_p = (I + w_O \Delta_O)G$

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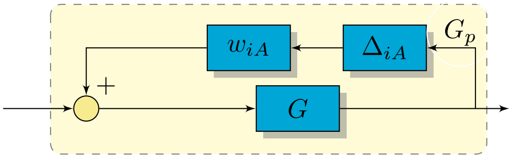

Inv. additive uncertainty: $\Pi_{iA}: \quad G_p = G\left(I - w_{iA}\Delta_{iA}G\right)^{-1}$

For SISO lump uncertainties in a single complex uncertainty.
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SISO vs MIMO uncertainty descriptions

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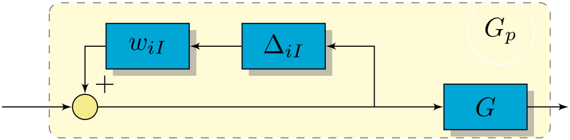

Inv. multiplicative input uncertainty: Π_{iI} : $\quad G_p = G\left(I - w_{iI}\Delta_{iI}\right)^{-1}$

For SISO lump uncertainties in a single complex uncertainty.
For MIMO?

SISO vs MIMO uncertainty descriptions

The representation of parametric uncertainty carries straight over to MIMO systems

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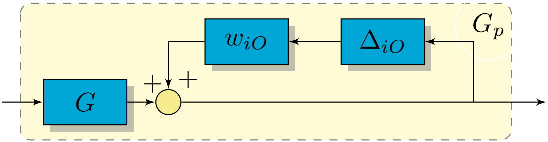


Inv. multiplicative output uncertainty: $\Pi_{iO}: \quad G_p = (I - w_{iO}\Delta_{iO})^{-1}G$
For SISO lump uncertainties in a single complex uncertainty.

For MIMO?

SISO vs MIMO uncertainty descriptions (cont'd)

You can but, uncertainty set becomes **bigger**

Example: We consider the following unstructured input uncertainty:
$G_p = G(I + E_I)$

We can simply apply the SISO approach to find a multiplicative input uncertainty description:
$l_I(\omega) = \max_{E_I} \overline{\sigma}\left(G^{-1}\left(G_p - G\right)\right) = \max_{E_I} \overline{\sigma}\left(E_I\right)$

We can simply apply the SISO approach to find a multiplicative output uncertainty description:
$l_O(\omega) = \max_{E_I} \overline{\sigma}\left(\left(G_p - G\right) G^{-1}\right) = \max_{E_I} \overline{\sigma}\left(G E_I G^{-1}\right)$

For SISO $l_O = l_I$ for MIMO $l_O \sim \gamma l_I$ where γ is the condition number.

Note that if $l_I(\omega)$ or $l_O(\omega)$ is above 1 the system can basically not be controlled at that frequency.

Input uncertainty

Input uncertainty comes from: amplifier dynamics, signal converter

The physical input to the system is $m_i = h_i(s)u_i$ (typically decoupled for every channel i)

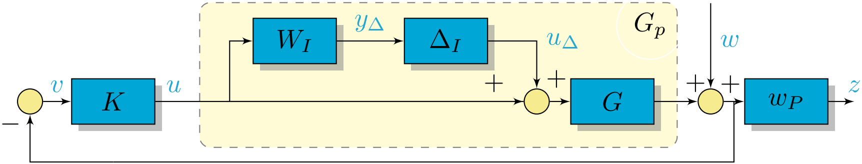
The known dynamics is typically absorbed in the plant model but the uncertainty can easily be presented by a multiplicative uncertainty:
$h_{pi}(s) = h_i(s)(1 + w_{Ii}(s)\delta_s(s)), \quad |\delta_i(j\omega)| \leq 1, \forall\omega$

Combining this leads to: $G_p = G(I + W_I\Delta_I)$ where both W_I and Δ_I are diagonal matrices (Typically w_{Ii} is given by $\frac{\tau s + r_o}{\frac{\tau}{r_\infty}s+1}$)

Diagonal input uncertainty should always be considered:

1. It is always present in a real system
2. It often restricts the performance with MIMO control

Obtaining P, N, and M



$$P: \quad \begin{bmatrix} y_\Delta \\ z \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_I \\ W_P G & W_p & W_P G \\ -G & -I & -G \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}$$

$$N: \quad \begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} -W_I KG(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}$$

$$M: \quad \begin{bmatrix} y_\Delta \end{bmatrix} = \begin{bmatrix} -W_I KG(I+KG)^{-1} \end{bmatrix} \begin{bmatrix} u_\Delta \end{bmatrix}$$

Use `N=lft(P,K)` and `M=lft(Δ,N)` to generate systems

Question

Given: $\frac{y}{u} = \frac{s+a}{s+b} = G(s)$

Find P such that $F_l\left(P, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = G(s)$

Definition robust stability and performance

We have the $N\Delta$ structure (unstructured Δ block):

$$N: \quad \begin{bmatrix} y_\Delta \\ z \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} u_\Delta \\ w \end{bmatrix}$$

$$F_u(N,\Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \quad \text{(e.g. } F_u(N,\Delta) = W_P S_p\text{)}$$

NS: N is internally stable

NP: $||N_{22}||_\infty < 1$ and **NS**

RS: $F_u(N,\Delta)$ is stable for all Δ, $||\Delta||_\infty \leq 1$ and **NS**

RP: $||F_u(N,\Delta)||_\infty < 1$ for all Δ, $||\Delta||_\infty \leq 1$ and **NS**

Robust stability of the $M\Delta$ structure

$F_u(N,\Delta) \triangleq N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$ we assume **NS** (N_{11}, N_{21}, N_{12}, and N_{22} stable)

We also assume Δ to be stable

We have **RS** if $(I - M\Delta)^{-1}$ is stable with $M = N_{11}$

Remember generalized Nyquist theorem:
- The $\det(\mathrm{I} - \mathrm{M}\Delta)$ will not encircle the point 0, $\forall\Delta$
- $\det(\mathrm{I} - \mathrm{M}\Delta) \neq 0, \forall\Delta, \forall\omega$
- $\lambda_i(M\Delta) \neq 1, \forall i, \Delta, \forall\omega$

Remember: $\det(I\lambda_i - M\Delta) = 0$ characteristic polynominal $M\Delta$ and λ_i eigenvalues of $M\Delta$

RS: $\rho(M\Delta(j\omega)) < 1, \forall\omega, \forall\Delta$ (or $\max_\Delta \rho(M\Delta(j\omega)) < 1, \quad \forall\omega$)

Robust stability of the $M\Delta$ structure: complex unstructured uncertainty

Now we assume that Δ is allowed to be any (full) complex transfer function with $||\Delta||_\infty \leq 1$ (Unstructured uncertainty). For this case the following equality holds:

$$\max_{\Delta} \rho(M\Delta) = \max_{\Delta} \overline{\sigma}(M\Delta) = \max_{\Delta} \overline{\sigma}(\Delta)\overline{\sigma}(M) = \overline{\sigma}(M)$$

Sketch of proof: We know that:

$\max_\Delta \rho(M\Delta) \leq \max_\Delta \overline{\sigma}(M\Delta) \leq \max_\Delta \overline{\sigma}(\Delta)\overline{\sigma}(M) = \overline{\sigma}(M)$

$\Rightarrow$For every M there exists a Δ' such that $\rho(M\Delta') = \overline{\sigma}(M)$

$\Rightarrow$$\Delta' = VU^H$ where $M = U\Sigma V^H$

$\Rightarrow \max_\Delta \rho(U\Sigma U^H) = \max_\Delta \overline{\sigma}(U\Sigma U^H) = 1 \times \overline{\sigma}(U\Sigma V^H) = \overline{\sigma}(U\Sigma V^H)$

RS: $\overline{\sigma}(M) < 1, \forall\omega$ (or $||M||_\infty < 1$)

Application of the unstructured RS condition

Remember all the six uncertainty structures and redefine $E = W_2 \Delta W_1$, with $||\Delta||_\infty \leq 1$.

We now isolate the perturbations to obtain $M = W_1 M_o W_2$

We now have:

- $G_p = G + E_A$: $\quad M_o = K(I+GK)^{-1} = KS$
- $G_p = G(I + E_I)$: $\quad M_o = K(I+GK)^{-1}G = T_I$
- $G_p = (I + E_O)G$: $\quad M_o = GK(I+GK)^{-1} = T$

RS: $||W_1 M_o W_2||_\infty < 1$

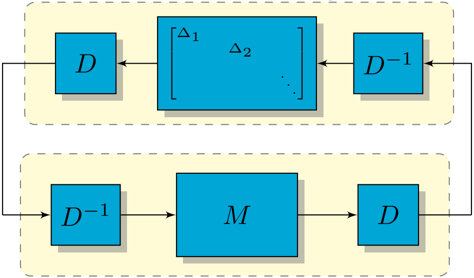
- Multiplicative Input uncertainty (scalar weight): $||w_I T_I||_\infty < 1$

Robust stability with structured uncertainty

Structured uncertainty: **RS** if $\overline{\sigma}(M) < 1, \forall\omega$ (note if and not iff)

We introduce scaling: $D = \mathrm{diag}\{\mathrm{d_i I_i}\}$

Same uncertainty



New $M : DMD^{-1}$

The following should also hold: **RS** if $\overline{\sigma}(DMD^{-1}) < 1, \forall\omega$

To obtain the least conservative **RS** condition:
RS if $\min_{D(\omega)\in\mathcal{D}} \overline{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1}) < 1, \forall\omega$

μ-The structured singular value

The structured singular value is a generalization of the maximum singular value (also μ, Mu, mu, SSV)

Find the smallest structured Δ (measured in terms of $\overline{\sigma}(\Delta)$) which makes the matrix $I - M\Delta$ singular; then $\mu(M) = 1/\overline{\sigma}(\Delta)$

Or: $\dfrac{1}{\mu(M)} \triangleq \min_{\Delta}\{\overline{\sigma}(\Delta)|\det(\mathrm{I}-\mathrm{M}\Delta)=0 \quad \mathrm{for} \quad \mathrm{structured} \quad \Delta\}$

Or:

$$\frac{1}{\mu(M)} \triangleq \min\{k_m|\det(\mathrm{I}-\mathrm{k_m M}\Delta)=0 \quad \mathrm{for} \quad \mathrm{structured} \quad \overline{\sigma}(\Delta)\leq 1\}$$

Small μ is good, large μ bad

The scalar μ gives a measure instead of yes/no condition.

The scalar μ depends on M and the structure of Δ.

μ-The structured singular value (cont'd)

Remember: **RS** if $\det(I - M\Delta(j\omega)) \neq 0$, $\forall\omega$, $\forall\Delta$, $\overline{\sigma}(\Delta(j\omega)) \leq 1$

Note that this is a yes or no condition

Find the smallest k_m such that $\det(I - k_m M\Delta(j\omega)) = 0$

From the definition of μ we have $\mu = \frac{1}{k_m}$ and allowing structured uncertainty

RS iff $\mu(M(j\omega)) < 1, \quad \forall\omega$

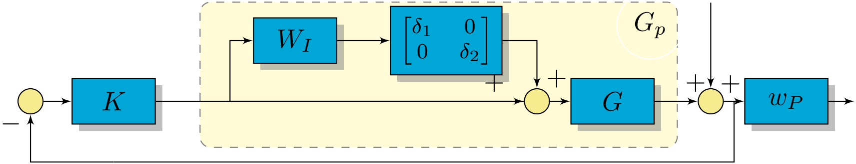
μ-The structured singular value (cont'dd)

Some properties (assuming complex perturbations):

1. $\mu(M) = \max_{\Delta, \overline{\sigma}(\Delta) \leq 1} \rho(M\Delta)$
2. For a scalar α it holds that $\alpha\mu(M) = \mu(\alpha M)$
3. For $\Delta = I\delta$ (δ a complex scalar) it holds that $\mu(M) = \rho(M)$
4. For a full complex Δ it holds that $\mu(M) = \overline{\sigma}(M)$
5. In general it holds that $\rho(M) \leq \mu(M) \leq \overline{\sigma}(M)$
6. Consider $\Delta D = D\Delta$ it holds that $\mu(DMD^{-1}) = \mu(M)$
7. Upper bound: $\mu(M) \leq \overline{\sigma}(DMD^{-1})$ (using 5 and 6)

Remarks: In practice a really tight bound and a convex problem

Example 8.9:

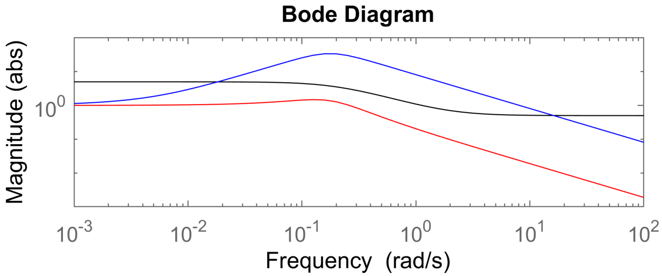


$$G = \frac{1}{\tau s+1}\begin{bmatrix} -87.8 & 1.4 \\ -108.2 & -1.4 \end{bmatrix},\ K = \frac{1+\tau s}{s}\begin{bmatrix} -0.0015 & 0 \\ 0 & -0.075 \end{bmatrix},\ w_I = \frac{s+0.2}{0.5s+1}$$

RS if $\sigma(M(j\omega)) < 1, \quad \forall\omega$ or for this example if $\overline{\sigma}(T_I) < \frac{1}{|w_I|} \quad \forall\omega$

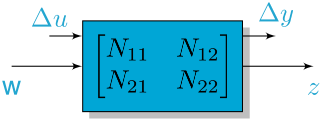
RS iff $\mu(M(j\omega)) < 1, \quad \forall\omega$ or for this example iff $\mu(T_I) < \frac{1}{|w_I|} \quad \forall\omega$

Example 8.9 (cont'd):



$\frac{1}{|w_I|}$ $\overline{\sigma}(T_I)$ $\mu(T_I)$

Summary:



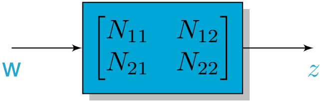
NS: N internally stable

NP: $\overline{\sigma}(N_{22}) < 1 \quad \forall\omega$ and **NS**

RS: $\mu_\Delta(N_{11}) < 1 \quad \forall\omega$ and **NS**

RP: Next lecture

Summary:



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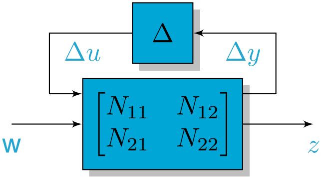

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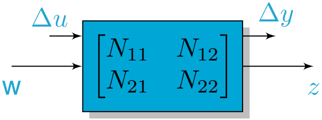

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RP: Next lecture

Robust Controller Synthesis

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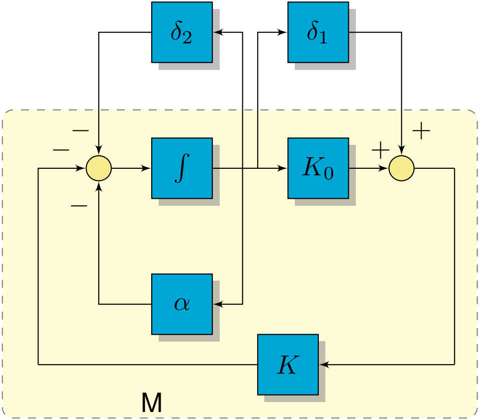
Example

Given:

- Real parametric uncertainty
- $|\delta_1| \leq 1$
- $|\delta_2| \leq 1$
- stable nominal system
- static feedback $K > 0$

Questions:

- Compute $\max_\omega \mu(M(j\omega))$?
- Compute $||M||_\infty$?
- What does it mean?
- What are the real conditions for stability?



$$M = \frac{1}{s+\alpha+K_0K}\begin{bmatrix} -K & -1 \\ -K & -1 \end{bmatrix}$$

example taken from: robust control (lecture notes) by Ad Damen and Siep Weiland

Example: compute μ

First compute $\mu(M(j\omega))$:

$$\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{\mu}\frac{1}{\mathrm{s}+\alpha+\mathrm{K_0K}}\begin{bmatrix}K & 1\\K & 1\end{bmatrix}\begin{bmatrix}\delta_1 & 0\\0 & \delta_2\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{1}{\mu}\frac{1}{\mathrm{s}+\alpha+\mathrm{K_0K}}\begin{bmatrix}K\delta_1 & \delta_2\\K\delta_1 & \delta_2\end{bmatrix}\right) = 0$$

$$1 + \frac{1}{\mu}\frac{K\delta_1+\delta_2}{s+\alpha+K_0K} = 0$$

Now we have to find the biggest μ for which the above equality holds:

$$\mu = \frac{-K\delta_1-\delta_2}{s+\alpha+K_0K} = \frac{|K|+1}{s+\alpha+K_0K}$$

Compute $\max_\omega \mu(M(j\omega))$ (note: low pass filter):

$$\max_\omega \mu(M(j\omega)) = \frac{|K|+1}{|\alpha+K_0K|}$$

Example: compute $||M||_\infty$

First remember that $||M||_\infty = \max_\omega \overline{\sigma}(M(\omega))$. So, first find $\overline{\sigma}(M(\omega))$:

$$
\begin{aligned}
M = & \frac{1}{s+\alpha+K_0K}\begin{bmatrix} -K & -1 \\ -K & -1 \end{bmatrix} \\
= & \underbrace{\frac{\begin{bmatrix} 1 & * \\ 1 & * \end{bmatrix}}{\sqrt{(2)}}}_{U} \underbrace{\begin{bmatrix} \sqrt{(2)}\frac{1}{|s+\alpha+K_0K|}\sqrt{(K^2+1)} & 0 \\ 0 & 0 \end{bmatrix}}_{S} \underbrace{\frac{\begin{bmatrix} -K & -1 \\ * & * \end{bmatrix}}{\sqrt{(K^2+1)}}}_{V^T}
\end{aligned}
$$

Now compute $||M||_\infty = \max_\omega \overline{\sigma}(M(\omega))$(note: low pass filter):

$$
||M||_\infty = \frac{\sqrt{2\,(K^2+1)}}{\sqrt{(\omega^2+(\alpha+K_0K)^2}} = \frac{\sqrt{2\,(K^2+1)}}{|\alpha+K_0K|}
$$

Example: What does it mean?

First observe that:

$$\begin{aligned}\mu(M(\omega)) &\leq \overline{\sigma}(M(\omega)) \\ \frac{|K|+1}{\sqrt{\omega^2+(\alpha+K_0K)^2}} &\leq \frac{\sqrt{2\left(K^2+1\right)}}{\sqrt{\omega^2+(\alpha+K_0K)^2}}\end{aligned}$$

It means that we know that the system is **RS** if $\max_\omega \mu(M(\omega)) < 1$ or $||M||_\infty < 1$.

Scaling:If $\delta_1 \leq \frac{1}{\gamma}$ and $\delta_2 \leq \frac{1}{\gamma}$ the **RS** turns out to be $\max_\omega \mu(M(\omega)) < \gamma$ or $||M||_\infty < \gamma$.

Example: What are the real conditions for stability?

Note that the closed loop pole is given by: $-(\alpha+\delta_2)-(K_0+\delta_1)K$
Or: $-\alpha-K_0K-\delta_2-\delta_1K$

The system is **NS** and consequently $\alpha+K_0K>0$

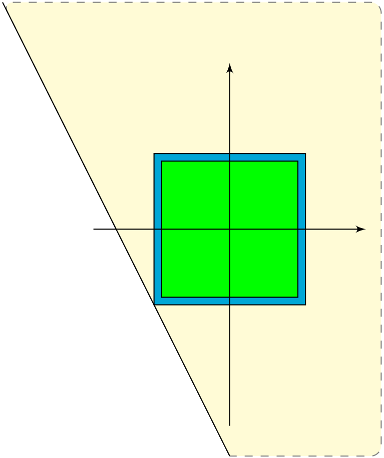
This directly implies for **RS** that $\alpha+K_0K>-\delta_2-\delta_1K$

Numerical example: $K_0=\alpha=1$ and $K=2$ for **RS** it should hold that:

- True: $2\delta_1+\delta_2>-3$
- According μ: $|\delta_1|<1$ and $|\delta_2|<1$
- According ∞: $|\delta_1|<\frac{1}{\sqrt{\frac{10}{9}}}$ and $|\delta_2|<\frac{1}{\sqrt{\frac{10}{9}}}$

Example: Graphical representation

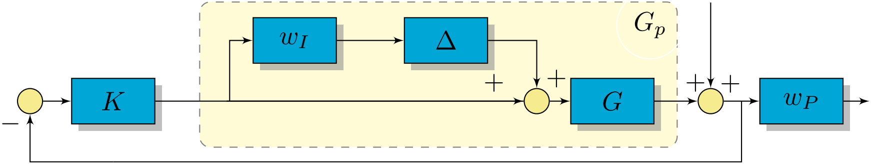
- True
- μ
- ∞



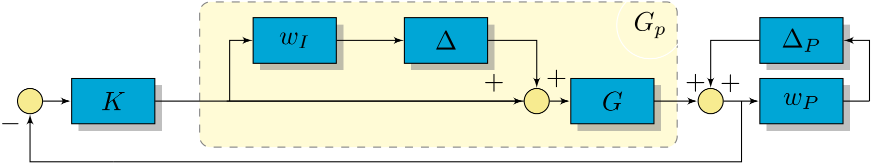
Why didn't we work with: $M = \frac{1}{s+\alpha+K_0 K} \begin{bmatrix} -K & -1 \end{bmatrix}$

The similarity between RS and RP (SISO example)

Consider:

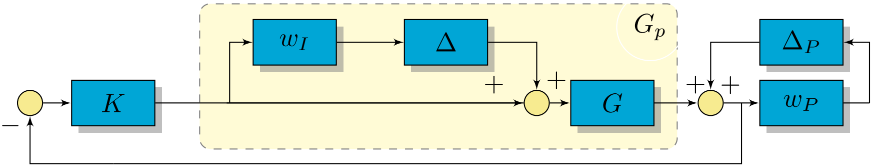


Remember: **RS:** $|w_I T| < 1\ \forall\omega$ and **RP:** $|w_I T| + |w_P S| < 1\ \forall\omega$



Main idea: Check **RS:** for the settings above

The similarity between RS and RP (SISO example)



Assume: Stable L_p and **NS** (nyquist doesn't encircle -1)

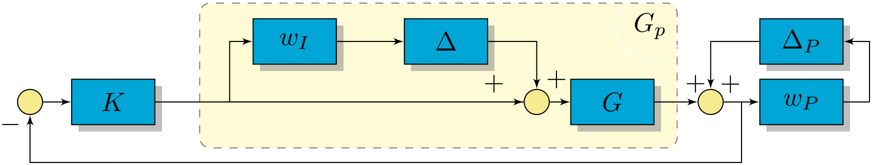
$$\begin{aligned}
\text{RS} \quad &\Leftrightarrow \quad |1 + L_p| > 0, \quad \forall L_p, \forall\omega \\
&\Leftrightarrow \quad |1 + L(1 + w_I\Delta)(1 - w_P\Delta_P)^{-1}| > 0, \forall\Delta, \forall\Delta_P, \forall\omega \\
&\Leftrightarrow \quad |1 + L + w_I L\Delta - w_P\Delta_P| > 0, \forall\Delta, \forall\Delta_P, \forall\omega
\end{aligned}$$

Last condition is most easily violated if $|\Delta| = |\Delta_P| = 1$ and if $1 + L$ and $w_I L\Delta$ and $w_P\Delta_P$ have opposite signs

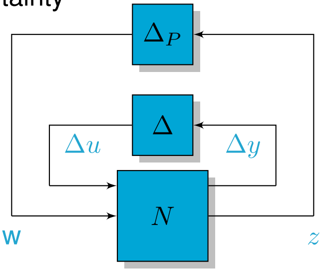
$$|1 + L| - |w_I L| - |w_P| > 0, \quad \forall\omega$$

RS $\Leftrightarrow |w_I T| + |w_P S| < 1, \forall\omega$

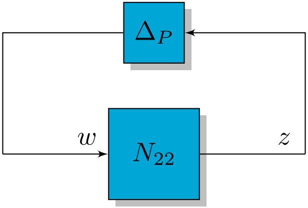
The similarity between RS and RP (SISO example)



Conclusion: We can reformulate the **RP** problem into an **RS** problem with structured uncertainty



The similarity between RS and NP (General)



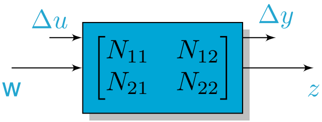
In the $\mathcal{H}_\infty$ framework we have NP if $||N_{22}||_\infty < 1$ (given NS).

Lets consider the following feedback loop with a full Δ_P block (for all stable $||\Delta_P||_\infty \leq 1$)

Apply Generalized Nyquist theorem: $\det(\mathrm{I} - \mathrm{N}_{22}(\mathrm{j}\omega)\Delta_\mathrm{P}(\mathrm{j}\omega))$ shouldn't encircle the origin

Same condition RS $M\Delta$-structure (see last lecture).
So, NP can be represented by a full uncertainty structure.

General conditions (still no controller synthesis!)



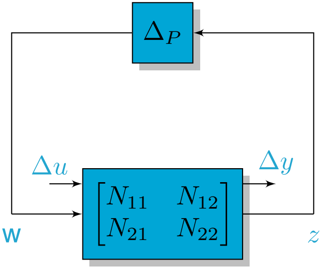
NS: N internally stable

NP: $\overline{\sigma}(N_{22}) < 1 \quad \forall\omega$ (or $\mu_{\Delta_P}(N_{22}) < 1$) and **NS**

RS: $\mu_{\Delta}(N_{11}) < 1 \quad \forall\omega$ and **NS**

RP: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall\omega$, $\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$ and **NS**

General conditions (still no controller synthesis!)



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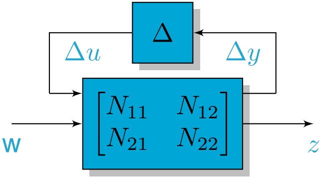
General conditions (still no controller synthesis!)



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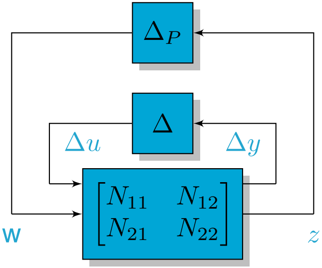
General conditions (still no controller synthesis!)



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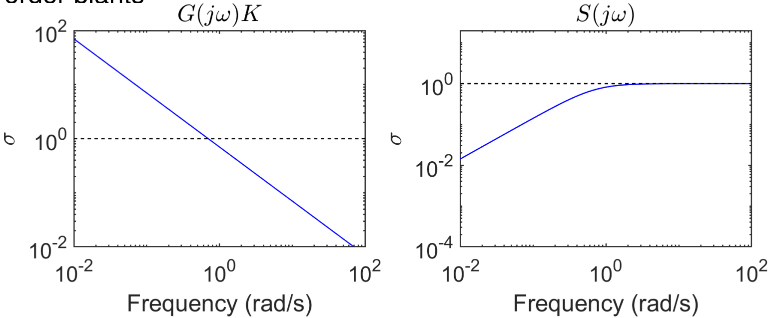
RP: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall\omega,\ \hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$ and **NS**

Old Example 2: Distillation column

$$G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \text{ with RGA } \forall\omega \begin{bmatrix} 35.1 & -34.1 \\ -34.1 & 35.1 \end{bmatrix}$$

Due to large elements in RGA difficult to control

Controller: inverse with integral action $K_{inv} = \frac{0.7}{s} G^{-1}$

- **NS:** With inverse control you end up with decoupled two first order plants
- 

- **RS:** No high $\overline{\sigma}(S)$ but high RGA values cause for concern ($\Rightarrow$)

Old Example 2: Distillation column (cont'd)

RS: We will consider diag. input uncertainty
(typically 20% for process applications)

Uncertainty in input is given by: $u_1' = (1 + \epsilon_1)\, u_1, \quad u_2' = (1 + \epsilon_2)\, u_2$

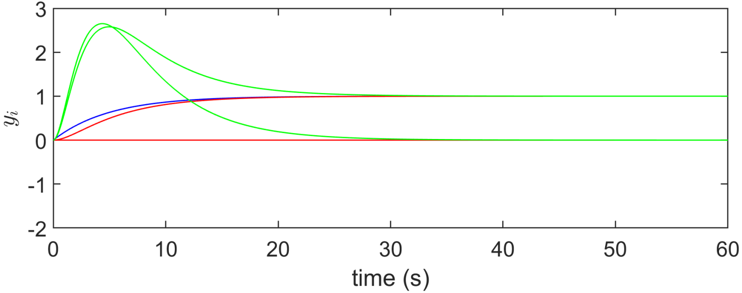
We have: $L(s) = \frac{0.7}{s} \begin{bmatrix} 1 + \epsilon_1 & 0 \\ 0 & 1 + \epsilon_2 \end{bmatrix}$

Compute poles: $\det(I + L) = (s + 0.7\,(1 + \epsilon_1))\,(s + 0.7\,(1 + \epsilon_2))$. We can have up to 100% error in all the input channels

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$

Old Example 2: Distillation column (cont'dd)

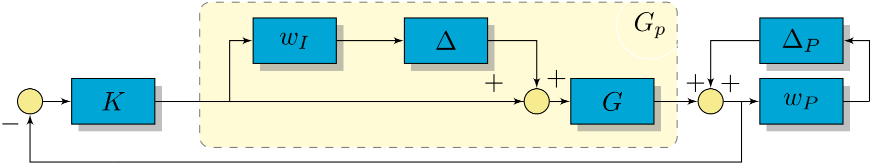
RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$



Reference, Nominal control, Uncertain Input

RP From the response we can conclude that we don't have RP

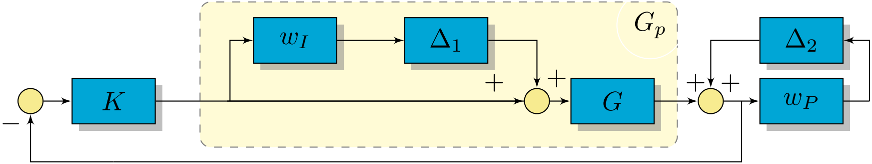
Old Example 2: **NS:** N internally stable



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>systemnames = 'G Wp Wi';
>>inputvar =' [udel(2); w(2); u(2)]';
>>outputvar=' [Wi; Wp; -G-w]';
>>input_to_G=' [u+udel]';
>>input_to_Wp=' [G+w]';
>>input_to_Wi=' [u]';
>>sysoutname='P'; cleanupsysic= 'yes'; sysic;
>>N=lft(P,Kinv); max(real(eig(N))); % -1E-6
```

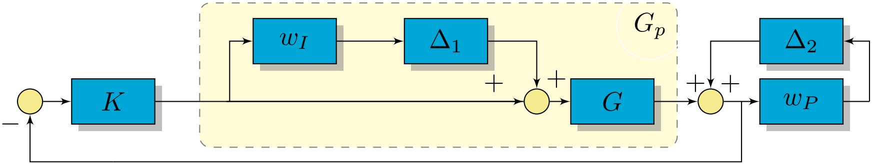
Old Example 2: **NP:** $\mu_{\Delta_P}(N_{22}) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 2 2]; % Full complex uncertainty block
>>[mubnds,muinfo]=mussv(Nf(3:4,3:4),blk,'c');
>>muNP=mubnds(:,1);
>>[muNPinf, muNPw]=norm(muNP,inf); % bound = 0.5
```

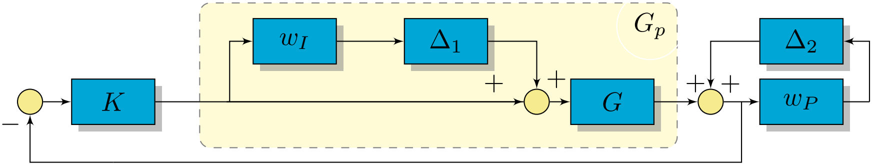
Old Example 2: **RS:** $\mu_\Delta(N_{11}) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 1 1; 1 1]; % structured uncertainty
>>[mubnds,muinfo]=mussv(Nf(1:2,1:2),blk,'c');
>>muRS=mubnds(:,1);
>>[muRSinf, muRSw]=norm(muRS,inf); % bound = 0.5242
```

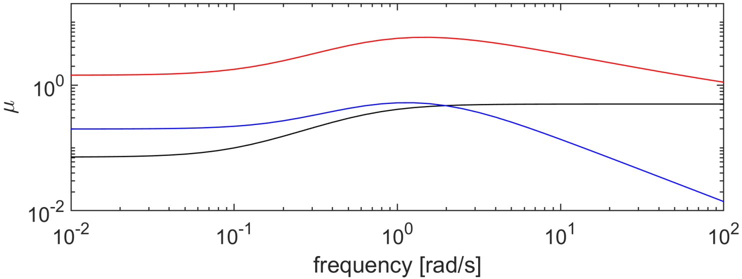
Old Example 2: **RP:** $\mu_{\hat{\Delta}}(N) < 1$



With: $w_I = \frac{s+0.2}{0.5s+1}$ and $w_P = \frac{\frac{s}{2}+0.05}{s}$

```
>>omega=logspace(-3,3,61);
>>Nf=frd(N,omega);
>>blk=[ 1 1; 1 1; 2 2]; % structured uncertainty and ΔP
>>[mubnds,muinfo]=mussv(Nf(1:4,1:4),blk,'c');
>>muRP=mubnds(:,1);
>>[muRPinf, muRPw]=norm(muRP,inf); % bound = 5.77
```

Old Example 2: The different SSV's



$$\underbrace{\mu_{\Delta_P}(N_{22}(j\omega))}_{NP} \quad \underbrace{\mu_{\Delta}(N_{11}(j\omega))}_{RS} \quad \underbrace{\mu_{\hat{\Delta}}(N(j\omega))}_{RP}$$

D-K iterations

For MIMO systems we know how to check for **NS**, **NP**, **RS**, **RP**:
μ-analysis

Seek a controller that minimizes a certain μ-condition:
the μ-synthesis problem

There is no direct method to synthesize a μ-optimal controller

However, we have an upperbound:
$\mu(N(K)) \leq \min_{D \in \mathcal{D}} \overline{\sigma}(DN(K)D^{-1})$

Seek a controller that: $\min_K (\min_{D \in \mathcal{D}} ||DN(K)D^{-1}||_\infty)$

Alternate, between minimizing $||DND^{-1}||_\infty$ using D or K.
(D-K iterations)

D-K iterations (cont'd)

The D-K iterations (start with $D = I$):

1. **K-step:** Synthesis a controller that minimizes the scaled problem: $\min_K ||DN(K)D^{-1}||_\infty$.
2. **D-step:** Find $D(j\omega)$ to minimize at each frequency $\overline{\sigma}(D(j\omega)N(j\omega)D^{-1}(j\omega))$ with fixed N (number of frequency points).
3. Fit a transfer function on top of $D(j\omega)$ and return to step-1.

Example (cont'dd)

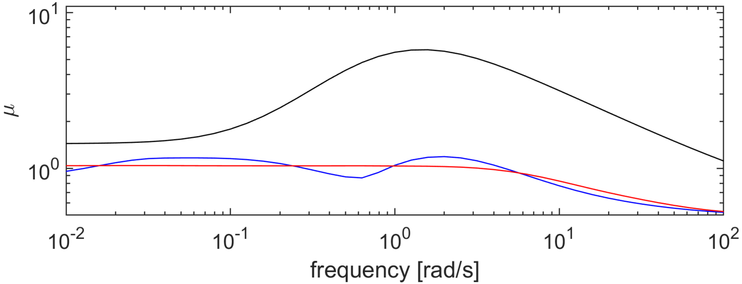
Back to the distillation column: $G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$

Automatic D-K iterations:

```
%% D-K iterations auto-tuning
>>Delta=[ultidyn('D_1',[1,1]) 0; 0 ultidyn('D_2',[1,1])];
>>Punc=lft(Delta,P);
>>opt=dkitopt('FrequencyVector', omega,'DisplayWhileAutoIter','on')
>>[K,clp,bnd,dkinfo]=dksyn(Punc,2,2,opt);
```

Example (cont'dd)

Back to the distillation column: $G(s) = \frac{1}{75s+1}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix}$

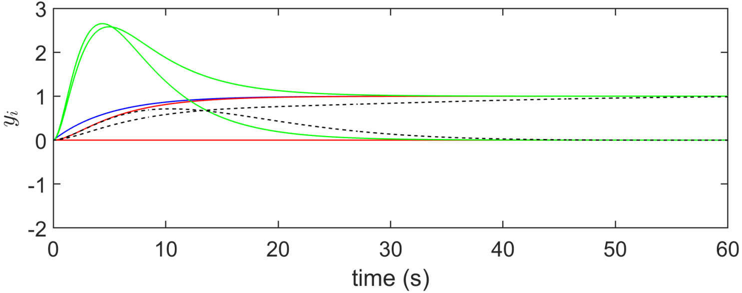


Decentralized design, μ-design iteration 1 , μ-design iteration 2

	K_{inv}	μ it. 1	μ it. 2
Peak μ-value	5.77	1.215	1.048
D-order	-	0	12
K-order	6	6	18

Example (cont'ddd)

RP: Consider $u_1' = 1.2u_1, \quad u_2' = 0.8u_2$



Reference, Nominal control, Uncertain Input, Robust

RP From μ we know we almost have **RP**.

D-scalings (example)

Given $M = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$ compute μ with $\overline{\sigma}(\Delta) \leq 1$:

1. where Δ is a full block (compute SVD)
2. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $(\det(..) = 1 + \frac{\delta_1}{\mu} - 3\frac{\delta_2}{\mu})$
3. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}$ $(\det(..) = 1 - 2\frac{\delta_1}{\mu})$

Answers: 1) $\sqrt{20}$, 2) 4 3) 2

D-scalings (example, cont'd)

Given $M = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$ compute D: not by hand

1. where Δ is a full block D=I
2. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$ $\left(D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\right)$
3. where $\Delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{bmatrix}$ (D full)

```
%% code to find D-scaling
>>M=[-1 -1; 3 3];
%% 1) full block
>>blk=[2 2];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
%% 2) Two diagonal element
>>blk=[1 0; 1 0];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
%% 3) Repeated block
>>blk=[2 0];
>>[mubnds,muinfo]=mussv(M,blk); mubnds(2)
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft
```

```
%% doing mu synthesis using hinfsyn
>> blk=[ 1 1; 1 1; 2 2];
>>omega=logspace(-3,3,61);
>>[K2,CL,GAM,INFO] = hinfsyn(P,2,2);
>>
>>i=1:1:10
>>Nf=frd(lft(P,K2),omega);
>>[mubnds,muinfo]=mussv(Nf(1:4,1:4),blk,'c');
>>muRP=mubnds(:,1); [muRPinf, muRPw]=norm(muRP,inf);
>>[VDelta,VSigma,VLmi] = mussvextract(muinfo);
>>D=VSigma.DLeft;
>>dd1 = fitmagfrd((D(1,1)/D(3,3)),6);
>>dd2 = fitmagfrd((D(2,2)/D(3,3)),6);
>>Dscale=minreal(append(dd1, dd2,tf(eye(4))));
>>[K2,CL,GAM2,INFO] = hinfsyn(minreal(Dscale*P*inv(Dscale)),2,2);
>>end
```

Robust Control overview

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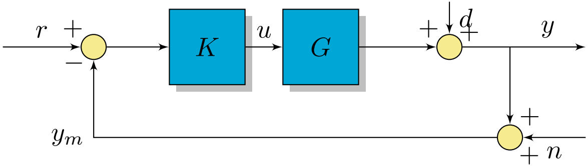
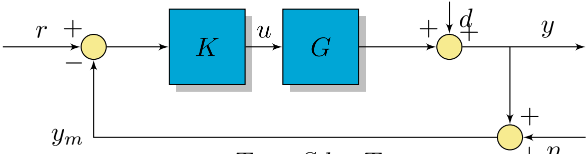
Specs. for closed loop transfer functions



$$y = Tr + Sd - Tn$$
$$u = KS(r - n - d)$$

1. For disturbance rejection make $\overline{\sigma}(S)$ small
2. For noise attenuation make $\overline{\sigma}(T)$ small
3. For reference tracking make $\overline{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$
4. For input usage reduction make $\overline{\sigma}(KS)$ small
5. For robust stability (additive uncertainty) make $\overline{\sigma}(KS)$ small
6. For robust stability (multipl. output uncertainty) make $\overline{\sigma}(T)$ small

Specs. for loop transfer



$$y = Tr + Sd - Tn$$

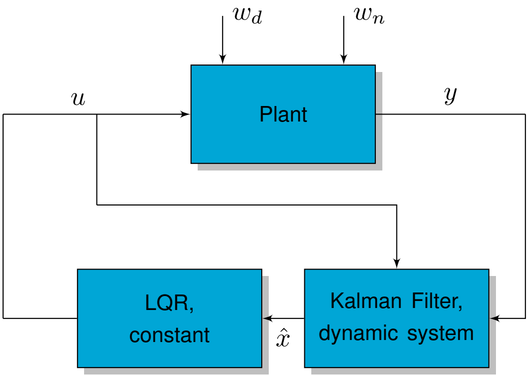
$$u = KS(r - n - d)$$

1. For disturbance rejection make $\underline{\sigma}(GK)$ large ($\underline{\sigma}(GK) \gg 1$)
2. For noise attenuation make $\overline{\sigma}(GK)$ small ($\overline{\sigma}(GK) \ll 1$)
3. For reference tracking make $\underline{\sigma}(GK)$ large ($\underline{\sigma}(GK) \gg 1$)
4. For input usage reduction make $\overline{\sigma}(K)$ small ($\overline{\sigma}(GK) \ll 1$)
5. For robust stability (additive uncertainty) make $\overline{\sigma}(K)$ small ($\overline{\sigma}(GK) \ll 1$)
6. For robust stability (multipl. output uncertainty) make $\overline{\sigma}(GK)$ small ($\overline{\sigma}(GK) \ll 1$)

LQG control

Optimal control theory reached maturity in *the sixties* (aerospace, etc).

For everyday industrial problems: no accurate plant model available and white noise disturbances not always realistic.



LQG control (cont'd)

The problem: Given the covariances W (of w_d) and V (of w_n) and weighting matrices Q and R minimize

$$E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \left[x^TQx+u^TRu\right]dt\right\}$$

LQG controller consists out of (separation principle):

1. Kalman Filter (solve one Riccati equation)
2. State feedback (solve one Riccati equation)

Trick required to include integral action

What about robustness margins?

Guaranteed Margins for LQG Regulators

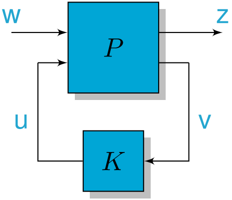
JOHN C. DOYLE

***Abstract*—There are none.**

$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control

Due to robustness issues people looked at $\mathcal{H}_\infty$ optimization for robust control in *the eighties*.

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right]$$



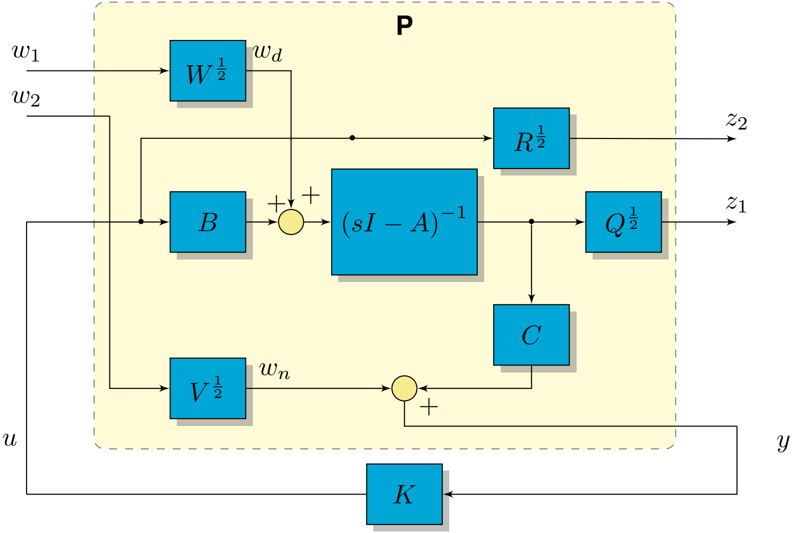
$$F_l(P,K) = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1} P_{21}$$

$\mathcal{H}_2$ control: $\sqrt{E\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T z^T z dt\right\}} = ||F_l(P,K)||_2$

$\mathcal{H}_\infty$ control: $\max_{w\neq 0} \frac{||z||_2}{||w||_2} = ||F_l(P,K)||_\infty$

+separation principle + we have to solve two riccati eqns

LQG: a special $\mathcal{H}_2$ controller ($||F_l(P,K)||_2$)



Trends

1. Formulate the $\mathcal{H}_\infty$ problem in the LMI framework
2. Research in the area of fixed-structure robust control
3. Research in the area of LPV control

$$\dot{x} = A(\mu)x + B(\mu)u + w_d$$
$$y = C(\mu)x + D(\mu)u + w_n$$

4. Identification for robust controller design

Purpose of the course

- Formulate control objectives in a mixed-sensitivity design
- Define stability and performance for MIMO LTI systems
- Construct a generalized plant for complex system interconnections
- Design MIMO controllers on the basis of the mixed-sensitivity
- Describe parametric and dynamic uncertainties
- Translate concrete controller synthesis problem into abstract framework of robust control
- Reproduce definition, properties and computation of the structured singular value
- Master application of structured singular value for robust stability and performance analysis
- Design robust controllers on the basis of the $\mathcal{H}_\infty$ control algorithm

The steps

1. Study the system (poles, zeros, dominant directions)
2. Explore decoupling possibilities
3. Define objectives of the controller and translate to open or closed loop properties
4. Design a nominal controller
5. Introduce uncertainty structures
6. Robustness analysis
7. If necessary design a robust controller

μ-The structured singular value

RS if $\det(I - M\Delta(j\omega)) \neq 0$, $\forall\omega$, $\forall\Delta$, $\overline{\sigma}(\Delta(j\omega)) \leq 1$

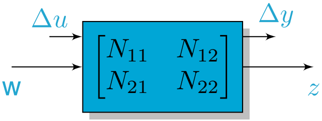
Note that this is a yes or no condition

Find the smallest k_m such that $\det(I - k_m M\Delta(j\omega)) = 0$

From the definition of μ we have $\mu = \frac{1}{k_m}$ and allowing only structured uncertainty

RS iff $\mu(M(j\omega)) < 1, \quad \forall\omega$

General conditions for analysis



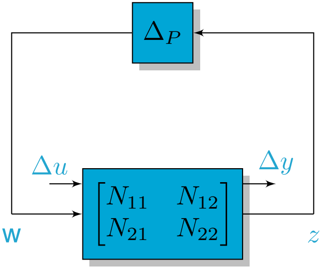
NS: N internally stable

NP: $\overline{\sigma}(N_{22}) < 1 \quad \forall\omega$ (or $\mu_{\Delta_P}(N_{22}) < 1$) and **NS**

RS: $\mu_{\Delta}(N_{11}) < 1 \quad \forall\omega$ and **NS**

RP: $\mu_{\hat{\Delta}}(N) < 1 \quad \forall\omega$, $\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$ and **NS**

General conditions for analysis



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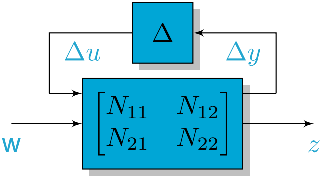

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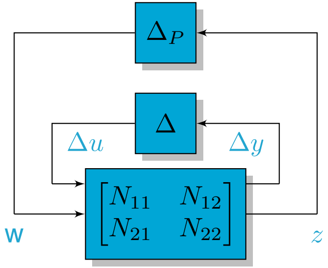

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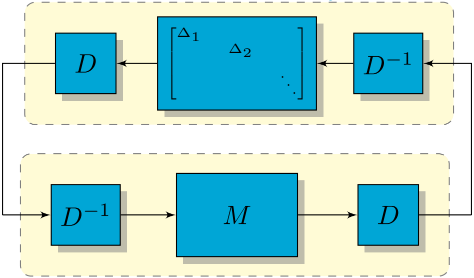
D-K iterations

Same uncertainty



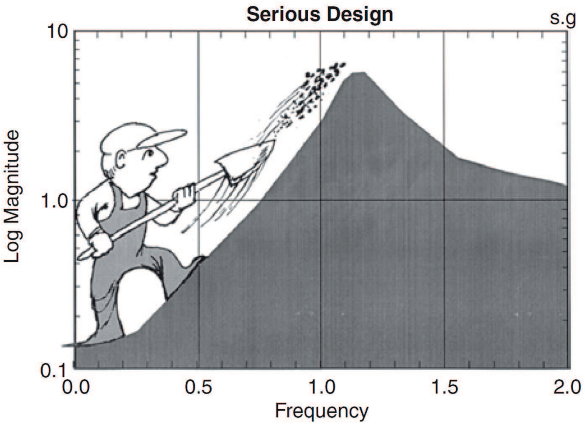
New M : DMD^{-1}

We have an upperbound: $\mu(N) \leq \min_{D \in \mathcal{D}} \overline{\sigma}(DND^{-1})$

Seek a controller that: $\min_K (\min_{D \in \mathcal{D}} ||DND^{-1}||_\infty)$

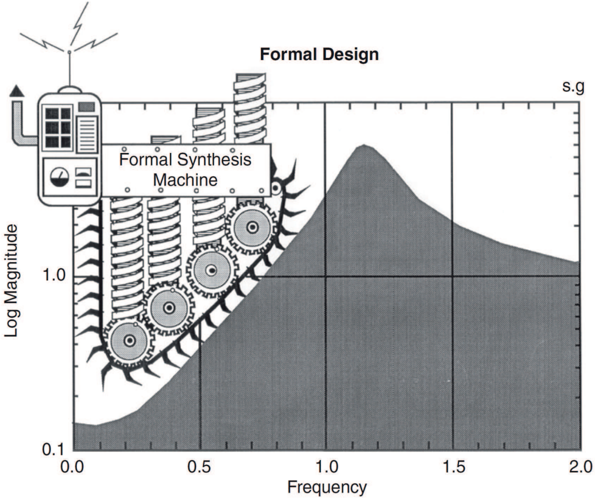
Alternate, between minimizing $||DND^{-1}||_\infty$ using D or K.

From manual loopshaping to modern tools



Pictures taken from: Respect the unstable, Gunter Stein

From manual loopshaping to modern tools



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