A discussion of autobiographical numbers

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Summary

In this paper, I take a look at autobiographical numbers. Even though the results discussed in this paper are widely known and mentioned in for example [1] and [2], the paper itself isn't based on these sources. Only the name 'autobiographical numbers' is used because of them.

1 Introduction

First, let's introduce the concept of an autobiographical number. Because it is easier to think of them as subset of $\mathbb{N}^k$ for $k \in \mathbb{N}_0$, I will define them that way.

Definition 1. *Let $a = (a_0, a_1, ..., a_k)$ with $k \in \mathbb{N}$. Then a is an autobiographical number if and only if there exists a $b = (b_0, b_1, ..., b_k)$ with $b_i \neq b_j$ if $i \neq j$ such that:*

$$\forall i \in \{0, 1, 2, ..., k\} : \#\{a_j | j \in \{0, 1, 2, ..., k\} \wedge a_j = b_i\} = a_i$$

We call b the describing number of a.

Note that the given definition is a bit more general than the one discussed in [1], as they always $b = (0, 1, 2, 3, ..., k)$.

Before looking at a few examples, we first define what it means for two autobiographical numbers to be equivalent.

Definition 2. *Two autobiographical numbers a and b are equivalent when a is a permutation of b.*

We no give some examples. Later on, we will show that there are no other autobiographical numbers apart from the examples given here.

1. The number (1) is autobiographical. After all, there exists a describing number $b = (1)$ such that the number of ones in (1) is equal to $b_0 = 1$.

2. The number $a = (2, 0, 2, 0)$ is also autobiographical. In order to see this, look at the describing number $b = (0, 1, 2, 3)$. The number of zeros in a is equal to $a_0 = 2$. The number of ones is equal to $a_1 = 0$, the number of twos is equal to $a_2 = 2$ and the number of threes is equal to $a_3 = 0$. It helps visually to write the numbers in a matrix with b in the first row and a in the second:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

   Here we can read the matrix as follows: the number of times the i-th element of the first row occurs in the second row is equal to the i-th element of the second row. Later in this section we will continue to use this notation to write both a and b directly. Also note that there is a self-descriptive sequence $c_n = (2, 2, 0, 0)$ that is equivalent to a.

3. $(1, 2, 1, 0)$ is also autobiographical, as we can see by looking at the following matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

4. $(2, 1, 2, 0, 0)$ is autobiographical, as we can see by looking ath the following matrix:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 0 & 0 \end{bmatrix}$$

5. In the last example, we consider each number that can be written as: $(n, 2, 1, 1, 0, 0, 0, ..., 0)$ $(n\ in \mathbb{N} \wedge n \geqslant 3)$ where there are exactly n zeros in the number. These numbers are all autobiographical as the following matrix shows:

$$\begin{bmatrix} 0 & 1 & 2 & n & n+1 & n+2 & ... & 2n-2 & 2n-1 \\ n & 2 & 1 & 1 & 0 & 0 & ... & 0 & 0 \end{bmatrix}$$

2 Solution

Lemma 1. *Let $(a_0, a_1, ..., a_k)$ be an autobiographical number, then $\sum_{i=0}^{k} a_i = k + 1$*

Proof. Let b be a describing number of a. Then a_i is the number of times that b_i occurs in the number a. Since each element of a must exist in b and since the elements of b are different from each other, the sum of all elements in a must be equal to the number of elements in a, namely $k + 1$. ☐

Lemma 2. *Let $(a_0, a_1, ..., a_k)$ be an autobiographical number not equal to (1), then a must contain a 0.*

Proof. There is only one autobiographical number of length 1, namely (1). So $k > 1$. Suppose a does not contain any 0, then each element must be equal to 1. After all, as soon as one element is greater than 1, the number cannot be an autobiographical number because of lemma 1. So there are exactly $k + 1$ ones in the number. However, this is impossible. After all, suppose there is a descriptive sequence b. Then there must be an i such that $b_i = 1$. Therefore, a_i must indicate the number of ones in the number. However, it holds that $a_i = 1$ for every i and that the number of ones is strictly greater than 1. This is a contradiction. ☐

Lemma 3. *Let $(a_0, a_1, ..., a_k)$ be an autobiographical number with a describing number b. Then there is at most one i with $b_i \neq 0$ and $a_i > 1$. More specifically, it applies for this i, if it exists, that $a_i = 2$.*

Proof. Note that the theorem holds for $a = (1)$. Every other row a contains a number of zeros, call this number j. This means that there are $k + 1 - j$ elements belonging to a that are greater than 0. One of these elements is j, then $k - j$ elements remain. Suppose one of these elements is greater than or equal to 3 or there are two elements that are greater or equal to 2. Then the sum of these non-zero elements is greater than or equal to $k - j + 2$. Then it holds that the sum of all elements in a is greater than or equal to $k + 2$. However, this is impossible because of lemma 1. ☐

Lemma 4. *Let $(a_0, a_1, ..., a_k)$ be an autobiographical number with a describing number b. Then the following statements are true:*

1. *There is at most one i where $b_i \geqslant 3$ and $a_i > 0$.*
2. *There is no i where $b_i \geqslant 3$ and $a_i > 1$.*

Proof. Both statements follow fairly quickly from lemma 3. We first prove the first statement. Suppose there exist an i and j with $i \neq j$ and $b_i \geqslant 3$ and $a_i > 0$ and $b_j \geqslant 3$ and $a_j > 0$. Then there must exist at least one index l such that $b_l > 0$ and $a_l = b_i$ or $a_l = b_j$. Otherwise, the number wouldn't be autobiographical. However, this goes directly against lemma 3, since $a_l > 2$.

We also prove the second statement. Suppose there exists an i such that $b_i \geqslant 3$ and $a_i > 1$. Then there must exist at least one index l such that $b_l > 0$ and $a_l = b_i$. Otherwise the number wouldn't be autobiographical. However, this goes against lemma 3 for the same reason. ☐

Theorem 1. *Each autobiographical number a is equivalent to one of the autobiographical number as given in examples 1 to 5.*

Proof. If no zeros belong to a, then it follows from lemma 2 that $a = (1)$. Suppose there are only one or two zeros in (a_n). Then $k < 6$. After all, suppose that $k > 5$ and that b is a describing number of a. Since all elements of b must be different from one another, there must be at least three indices i so that $b_i > 2$. Since the number of zeros is less than three, then for one of these indices, say j, it must hold that $a_j = 1$. However, this is impossible since there are less than three zeros in a and there is one index l so that $b_l > 0$ and $a_l = b_j$. Otherwise the number wouldn't be autobiographical. However, this goes against lemma 3. Consequently, we have to check only a finite number of numbers and so we can find that the only autobiographical numbers with one or two zeros are equivalent to $(1, 2, 1, 0)$, $(2, 0, 2, 0)$ or $(2, 1, 2, 0, 0)$.

We now consider the last case where the number of zeros is greater than 2. Call the number of zeros m. Let b be a describing number of a. Then an index i_m must exist such that $b_{i_m} = m$ and $a_{i_m} = 1$. Note that due to lemma 4 no index j can exist with $b_j > 2$ and $a_j > 0$. In addition, an index i_1 must exist so that $b_{i_1} = 1$. Note that $a_{i_1} \neq 1$. Otherwise, the number of ones in the row would be greater than or equal to 2, but then a_{i_1} would be equal to 1. According to lemma 3, $a_{i_1} = 2$. Hence there is also an index i_2 such that $b_{i_2} = 2$. According to lemma 3, a_{i_2} must be equal to 1. Notice that we have now determined a completely except for the order of its elements. Hence, the number a must be equivalent to $(m, 2, 1, 1, 0, 0, 0, ..., 0)$. Note that this number is an autobiographical number as shown in example 5. □

References

[1] Self-descriptive number. (2020, September 7). In *Wikipedia*. Accessed at 8 september 2020 on `https://en.wikipedia.org/wiki/Self-descriptive_number`

[2] Sloane, N. J. A. (ed.), *Autobiographical numbers (or curious numbers)*. The On-Line Encyclopedia of Integer Sequences. OEIS Foundation. Accessed at 9 september 2020 on `http://oeis.org/A046043`.