Sum of powers of consecutive natural numbers

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Summary

In this short paper a sequence of integer numbers is introduced. The sequence is about the sum of powers of consecutive natural numbers. Some properties about the sequence are proven, but using the theorems the n-th term of the sequence cannot be completely determined.

1 Introduction

We first introduce a function $a : \mathbb{N}_0 \to \mathbb{N}_0$.

Definition 1. *Let $k, n \in \mathbb{N}$ and let $a : \mathbb{N}_0 \to \mathbb{N}_0$. Then $a(k) = n$ if and only if n is the smallest non-zero natural number for which the following statement applies:*

$$n^k \leqslant \sum_{m=1}^{n-1} m^k$$

Before going into the properties of the given function, let's first take a look at the first few terms of the sequence determined by $a(k)$ (starting at $k = 1$):

$$3, 5, 6, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, 22, ...$$

The sequence is sequence A332101 on OEIS ([1]).

It seems that the function grows steadily and that all terms exist. The first few theorems prove these statements. The last theorem predicts the n-th term of the sequence to an accuracy of 2 possible integers.

2 Properties of the sequence

Theorem 1. *$a(k)$ exists.*

Proof. We prove this theorem by showing that there is a natural number N such that for every natural number $n > N$, the following applies:

$$(n+1)^k - \sum_{m=1}^{n} m^k < n^k - \sum_{m=1}^{n-1} m^k$$

This way, we prove that the difference between n^k and $\sum_{m=1}^{n-1} m^k$ gets smaller and smaller as n gets bigger. Because the difference between integers cannot get arbitrarily small, there needs to be a number n such that $n^k \leqslant \sum_{m=1}^{n-1} m^k$.

Let $N > \frac{1}{2^{\frac{1}{k}}-1}$. Then for every $n > N$,

$$n > \frac{1}{2^{\frac{1}{k}} - 1}$$

Thus:

$$\frac{1}{n} < \sqrt[k]{2} - 1$$

Rewriting gives us:

$$\left(\frac{n+1}{n}\right)^k < 2$$

Or:

$$(n+1)^k - n^k < n^k$$

Adding a few terms gives us:

$$(n+1)^k - n^k - \sum_{m=1}^{n-1} m^k < n^k - \sum_{m=1}^{n-1} m^k$$

Or writing n^k at the left-hand side within the sum:

$$(n+1)^k - \sum_{m=1}^{n} m^k < n^k - \sum_{m=1}^{n-1} m^k$$

□

The second theorem is not really necessary as the final theorem is a much stronger version and the proof of it doesn't need the following theorem. However, the proof is short and simple so we'll still mention it.

Theorem 2. *Let $k \in \mathbb{N}_0$. Then $a(k+1) \geqslant a(k)$.*

Proof. Let $n = a(k+1)$. Then:

$$n^{k+1} \leqslant \sum_{m=1}^{n-1} m^{k+1}$$

Or:

$$n^k \leqslant \sum_{m=1}^{n-1} \frac{m^{k+1}}{n}$$

$\frac{m}{n} < 1$ for every m smaller than n. Thus we get:

$$n^k < \sum_{m=1}^{n-1} m^k$$

This proves the theorem. □

The last theorem gives two possibilities for the value of $a(k)$.

Theorem 3. *Let $k \in \mathbb{N}_0$. Then one of the following is correct:*

$$a(k) = \left\lceil \frac{1}{2^{\frac{1}{k+1}} - 1} \right\rceil$$
$$a(k) = \left\lceil \frac{1}{2^{\frac{1}{k+1}} - 1} + 1 \right\rceil$$

Proof. First we define a function g:

$$g : \mathbb{R}^+ \times \mathbb{N}_0 \to \mathbb{R} : (x,k) \longmapsto x^k - \sum_{n=1}^{\lfloor x \rfloor} (x-n)^k$$

We prove the following statement for a random $(x,k) \in \mathbb{R}^+ \times \mathbb{N}_0$:

$$\frac{\partial g(x,k+1)}{\partial x} = (k+1)g(x,k) \tag{1}$$

We distinguish between two cases. First, let's assume that x isn't an integer. Then we know that $\frac{d\lfloor x\rfloor}{dx} = 0$. Thus g is a combination of functions that are derivable with respect to x. Using the chain rule, one can obtain the required result easily. Now assume x is an integer. We obtain the result by using the definition of a derivative. We calculate the right and the left limit seperately. First the right limit:

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{g(x+h,k+1) - g(x,k+1)}{h} \\
=&\lim_{h\downarrow 0} \frac{(x+h)^{k+1} - \sum\limits_{n=0}^{x-1}(h+n)^{k+1} - x^{k+1} + \sum\limits_{n=0}^{x-1} n^{k+1}}{h}
\end{aligned}$$

Note that we replaced $(x+h-n)$ with $(h+n)$ using a substitution. Analogously we replaced $(x-n)$ with n using a substitution. Using L'Hôpital and evaluating in $h = 0$:

$$= (k+1)x^k - (k+1)\sum_{n=1}^{x-1} n^k = (k+1)g(x,k)$$

Now, we calculate the left limit:

$$\begin{aligned}
&\lim_{h\uparrow 0} \frac{g(x+h,k+1) - g(x,k+1)}{h} \\
=&\lim_{h\uparrow 0} \frac{(x+h)^{k+1} - \sum\limits_{n=1}^{x-1}(h+n)^{k+1} - x^{k+1} + \sum\limits_{n=0}^{x-1} n^{k+1}}{h}
\end{aligned}$$

Note the very small change in the second equation were the second term in the numerator starts from 1 and not from 0. However, using L'Hôpital again and evaluating in $h = 0$, we get the exact same result:

$$= (k+1)x^k - (k+1)\sum_{n=1}^{x-1} n^k = (k+1)g(x,k)$$

So the right derivative and left derivative both exist and are equal to each other. Thus the derivative exists and (1) is correct for all values of $(x,k) \in \mathbb{R}^+ \times \mathbb{N}_0$. Let $f_k : \mathbb{R}^+ \to \mathbb{R}$ be defined as $f_k(x) = g(x,k)$. We know now that $f_{k+1}(x)$ can only reach a local maximum or minimum if and only if $f_k(x) = 0$. Note that $f_k(x)$ has exactly one root. This can easily be shown using induction and our recursive derivative (1). Moreover, one can also show using induction that $f_k(x)$ is bigger than 0 when x is smaller than this root and smaller than 0 when x is bigger than this root. This means there exists only one maximum of $f_{k+1}(x)$. If we find this maximum, we can find the root of $f_k(x)$ and this would solve the problem: the smallest integer bigger than this root, would be the k-th term of our sequence (because this would be the smallest integer for which f_k is smaller than 0). We won't find the exact spot of the maximum of $f_{k+1}(x)$. However, we will be able to narrow down the possibilities.

Let $n \in \mathbb{N}$ and:

$$\frac{1}{2^{\frac{1}{k+1}} - 1} < n \leqslant \frac{1}{2^{\frac{1}{k+1}} - 1} + 1$$

. We prove the following inequalities:

$$f_{k+1}(n) > f_{k+1}(n+1) \tag{2}$$
$$f_{k+1}(n) \geqslant f_{k+1}(n-1) \tag{3}$$

First, let's prove the first of these inequalities:

$$n > \frac{1}{2^{\frac{1}{k+1}} - 1}$$

Thus:

$$\frac{1}{n} < 2^{\frac{1}{k+1}} - 1$$

Or:

$$\frac{n+1}{n} < 2^{\frac{1}{k+1}}$$

Giving:

$$\left(\frac{n+1}{n}\right)^{k+1} < 2$$

Thus:

$$(n+1)^{k+1} - n^{k+1} < n^{k+1}$$

Rewriting:

$$(n+1)^{k+1} - \sum_{i=1}^{n} i^{k+1} < n^{k+1} - \sum_{i=1}^{n-1} i^{k+1}$$

Which gives us the first equality. We now prove the second equality:

$$n \leqslant \frac{1}{2^{\frac{1}{k+1}} - 1} + 1$$

Thus:

$$\frac{1}{n-1} \geqslant 2^{\frac{1}{k+1}} - 1$$

Or:

$$\frac{n}{n-1} \geqslant 2^{\frac{1}{k+1}}$$

Giving:

$$\left(\frac{n}{n-1}\right)^{k+1} \geqslant 2$$

Thus:

$$n^{k+1} - (n-1)^{k+1} \geqslant (n-1)^{k+1}$$

Rewriting:

$$n^{k+1} - \sum_{i=1}^{n-1} i^{k+1} \geqslant (n-1)^{k+1} - \sum_{i=1}^{n-2} i^{k+1}$$

This gives us the second inequality. The inequalities ensure that $f_{k+1}(x)$ reaches a maximum between $n-1$ and $n+1$. Because of our previous remark, this means that $a(k)$ can only be n or $n+1$, proving our theorem.

□

3 Suspicion

In [1] a claim is mentioned about the exact value of $a(k)$. The claim is:

$$a(k) = \text{round}\left(\frac{k}{\log 2} + 2\right)$$

Using our result, we can disprove this statement. Even though small values of k (smaller than 10^7) seem to confirm the claim, larger values of k do not. For example, for $k = 10^8$, the claim states that $a(k) = 144269506$. However, the result presented here, says that $144269504 \leqslant a(k) \leqslant 144269505$.

4 Conclusion

We looked into a sequence concerning consecutive powers of natural numbers. We proved that the sequence exists, that it increases and we were able to show a very strict lower and upper bound of the sequence. At the end we disproved a claim written in [1].

References

[1] Sloane, N. J. A. (ed.), *Sequence A332101: Least m such that m ˆn<= Sum_{k<m} k ˆn*. The On-Line Encyclopedia of Integer Sequences. OEIS Foundation. Accessed at 9 september 2020 on `https://oeis.org/A332101`.