Finite Element models made using python

Problem 1

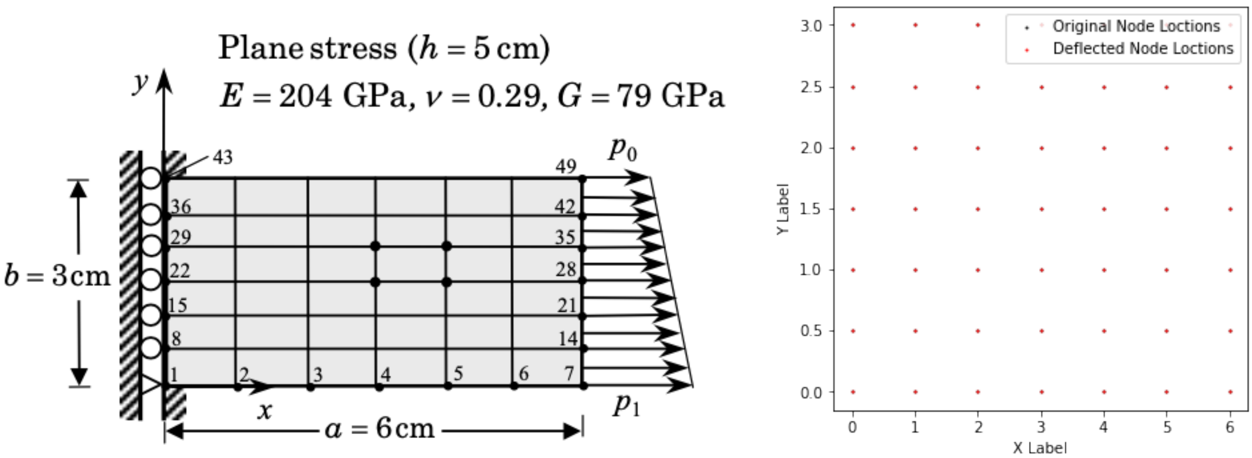


Figure 1: Question taken from JN Reddy An intro to FEM Q10.1 (Left), and solution image of displacements at all nodes (Right)

Finite Element model

The governing variational form of the equation is given as

$$\prod = U - W$$
$$\prod = \int_\Omega E_{ijkl}\epsilon_{kl}\epsilon_{ij} \quad d\Omega - \left(\int_\Gamma t_i u_i \quad d\Gamma + \int_\Omega b_i u_i \quad d\Omega \right) \tag{1}$$

The weak form of the model is written as:

$$\int_\Omega E_{ijkl}(\delta\epsilon_{kl})\epsilon_{ij} \quad d\Omega = \int_\Gamma t_i \delta u_i \quad d\Gamma + \int_\Omega b_i \delta u_i \quad d\Omega \tag{2}$$

The finite element formulation of the equation is taken as

$$\sum_p K_{qp}\tilde{u}_p = f_q \tag{3}$$

where

$$K_{qp} = \int_\Omega E_{ijkl}A_{ijp}A_{klq} \quad d\Omega = \int_\Omega E_{ijkl}(\phi_{ip,j} + \phi_{jp,i})(\phi_{ip,j} + \phi_{jp,i})^T \quad d\Omega \tag{4}$$

$$f = \int_\Gamma t_i \phi_{iq} \quad d\Gamma + \int_\Omega b_i \phi_{iq} \quad d\Omega \tag{5}$$

hence,

$$\tilde{u}_p = \frac{f_q}{\sum_p K_{qp}} \tag{6}$$

Further notes on maths solution

The D matrix, equivalent to E_{ijkl} for plane stress problems can be written in direct matrix form:

$$D = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \tag{7}$$

Obtaining interpolations:

Approximate u_i using shape functions

$$u_i \approx \sum_p \phi_{ip}\tilde{u}_p \tag{8}$$

where,

$$\phi_i = \begin{cases} \phi_i = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} x^e \\ \phi_j = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} y^e \end{cases} \tag{9}$$

Using isoparimetric quadrialteral finite element element Q4, the shape functions are

$$\begin{aligned} \phi_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\ \phi_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\ \phi_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\ \phi_4 &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{10}$$

For strains to be obtained, the interpolations must be derived one step further with the notation:

$$\phi_{i,i} = \begin{cases} \phi_{i,i} = \frac{\partial}{\partial x}\phi_i \\ \phi_{i,j} = \frac{\partial}{\partial y}\phi_i \end{cases} \tag{11}$$

$$(\phi_{ip,j} + \phi_{jp,i}) = \frac{\partial}{\partial y}\phi_i + \frac{\partial}{\partial x}\phi_j \tag{12}$$

The volume integral, Ω, is converted to interpolation coordinates

$$\int_\Omega f(x,y) \quad d\Omega = t\int_{-1}^{1}\int_{-1}^{1} \hat{f}(\xi,\eta)|J| \quad d\xi d\eta \tag{13}$$

with the expression

$$dA = detJ \quad d\xi d\eta = |J| \quad d\xi d\eta \tag{14}$$

note

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \sum_1^4 \frac{\partial \phi_i}{\partial \xi} & \sum_1^4 \frac{\partial \phi_j}{\partial \xi} \\ \sum_1^4 \frac{\partial \phi_i}{\partial \eta} & \sum_1^4 \frac{\partial \phi_j}{\partial \eta} \end{bmatrix} \tag{15}$$

to further convert the volume integral with summation notation:

$$t\int_{-1}^{1}\int_{-1}^{1}\hat{f}(\xi,\eta)|J| \quad d\xi d\eta = \sum_{k=1}^{n}\sum_{j=1}^{m} H_j H_k \hat{f}(\xi_j,\eta_k)|J| \quad d\xi d\eta \tag{16}$$

Similarly to apply the boundary conditions on surface integral region Γ, the area integral is converted to interpolation coordinates as either:

$$\int_{\Gamma} f(x,y) \quad d\Gamma = t\int_{-1}^{1}\hat{f}(\xi)|J| \quad d\xi = \sum_{j=1}^{n} H_j \hat{f}(\xi_j)|J_s| \quad d\xi \tag{17}$$

$$\int_{\Gamma} f(x,y) \quad d\Gamma = t\int_{-1}^{1}\hat{f}(\eta)|J| \quad d\eta = \sum_{m=1}^{n} H_j \hat{f}(\eta_m)|J_s| \quad d\eta \tag{18}$$

depending on what side of the element is chosen to be set as a constant, notated as J_s.

Code solution on next page...

ex_1_solution_JNB

October 12, 2020

[313]:
```
import numpy as np
import math
import sympy as sym
from sympy import *
import matplotlib.pyplot as plt
%matplotlib inline
```

Writing a script to describe the nodes:

[314]:
```
nodes = 49
number_dofs = nodes * 2
dofs = np.zeros([number_dofs,1])
force_matrix = np.zeros(number_dofs)

def function(i):
    if i <= 7:
        x = 0 + 1*(i-1)
        y = 0
        return x, y

    elif i <= 14:
        x = 0 + 1*(i-8)
        y = 0.5
        return x, y

    elif i <= 21:
        x = 0 + 1 * (i - 15)
        y = 1
        return x, y

    elif i <= 28:
        x = 0 + 1 * (i - 22)
        y = 1.5
        return x, y

    elif i <= 35:
        x = 0 + 1 * (i - 29)
        y = 2
```

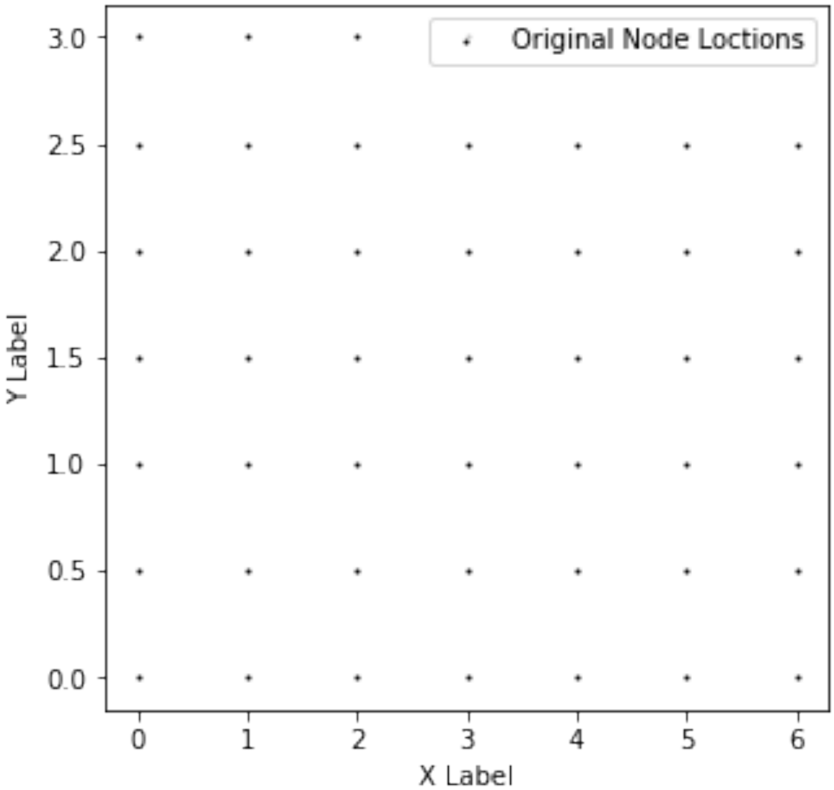
```
        return x, y

    elif i <= 42:
        x = 0 + 1 * (i - 36)
        y = 2.5
        return x, y

    else:
        x = 0 + 1 * (i - 43)
        y = 3
        return x, y
```

Writing a script to denote the node coordinates for each element.

[315]:
```
elements = 36
element_connection_matrix = []

for j in range(0, 6):
    element_connections = [(1+j), (2+j), (9+j), (8+j)]
    element_connection_matrix.append(element_connections)

for j in range(0, 6):
    element_connections2 = [(8+j), (9+j), (16+j), (15+j)]
    element_connection_matrix.append(element_connections2)

for j in range(0, 6):
    element_connections3 = [(15+j), (16+j), (23+j), (22+j)]
    element_connection_matrix.append(element_connections3)

for j in range(0, 6):
    element_connections4 = [(22+j), (23+j), (30+j), (29+j)]
    element_connection_matrix.append(element_connections4)

for j in range(0, 6):
    element_connections5 = [(29+j), (30+j), (37+j), (36+j)]
    element_connection_matrix.append(element_connections5)

for j in range(0, 6):
    element_connections6 = [(36+j), (37+j), (44+j), (43+j)]
    element_connection_matrix.append(element_connections6)
```

[316]:
```
node_locations = [function(i) for i in range(1, int(nodes+1))]

def coords_x(i):
    x = node_locations[i][0]
    return x
```

```
def coords_y(i):
    y = node_locations[i][1]
    return y

x = [coords_x(i) for i in range(nodes)]
y = [coords_y(i) for i in range(nodes)]


def image_view(x, y):
    fig, ax = plt.subplots(figsize=(5,5))

    ax.scatter(x, y, color='black', s=1)

    ax.legend(['Original Node Loctions'], loc='upper right')

    ax.set_xlabel('X Label')
    ax.set_ylabel('Y Label')

    plt.show(block=True)
    plt.show()

image_view(x, y)
```



Applying the known parameters to E and v to obtain the D matrix. Note - this is also E_{ijkl}

[317]:
```
E = 204000 # N/mm^2
v = 0.29

def D_matrix(E, v):
    value_d_matrix = E/(1-v**2) * np.array([[1, v, 0], [v, 1, 0], [0, 0, (1-v)/
↪2]])
    return value_d_matrix

E_ijkl = D_matrix(E, v)
```

Derivative functions made using sympy

[318]:
```
def differentiation_wrtx(val):
    x = sym.Symbol('x')
    derivative = sym.diff(val)
    return derivative

def differentiation_wrty(val):
    y = sym.Symbol('y')
    derivative = sym.diff(val)
    return derivative

def differentiation_wrtxi(val):
    xi = sym.Symbol('xi')
    derivative = sym.diff(val)
    return derivative

def differentiation_wrteta(val):
    eta = sym.Symbol('eta')
    derivative = sym.diff(val)
    return derivative
```

[319]:
```
def individual_phi_matrix(xi, eta):
    phi_1 = 1/4*(1-xi)*(1-eta)
    phi_2 = 1/4*(1+xi)*(1-eta)
    phi_3 = 1/4*(1+xi)*(1+eta)
    phi_4 = 1/4*(1-xi)*(1+eta)
    phi_matrix = [phi_1, phi_2, phi_3, phi_4]
    return phi_matrix
```

The next block of code allows for derivations of the interpolations with respect to ξ.

[320]:
```python
def phi_matrix_comma_i(eta):
    xi = sym.Symbol('xi')
    phi_1_i = diff(1/4*(1-eta)- 1/4*(1-eta)*xi)
    phi_2_i = diff(1/4*(1-eta)+ 1/4*(1-eta)*xi)
    phi_3_i = diff(1/4*(1+eta)+ 1/4*(1+eta)*xi)
    phi_4_i = diff(1/4*(1+eta)- 1/4*(1+eta)*xi)
    phi_matrix_comma_i = [phi_1_i, phi_2_i, phi_3_i, phi_4_i]
    return phi_matrix_comma_i
```

The next block of code allows for derivations of the interpolations with respect to η.

[321]:
```python
def phi_matrix_comma_j(xi):
    eta = sym.Symbol('eta')
    phi_1_j = diff(1/4*(1-xi)- 1/4*(1-xi)*eta)
    phi_2_j = diff(1/4*(1+xi)- 1/4*(1+xi)*eta)
    phi_3_j = diff(1/4*(1+xi)+ 1/4*(1+xi)*eta)
    phi_4_j = diff(1/4*(1-xi)+ 1/4*(1-xi)*eta)
    phi_matrix_comma_j = [phi_1_j, phi_2_j, phi_3_j, phi_4_j]
    return phi_matrix_comma_j
```

Using 1x1 Gaussian integration of the elements, ξ and η have a value of 0. However this must be applied manually as a global allocation of values to ξ and η will disrupt the differential equations.

The following block of code allowcates the rest of the strain energy equation to the relevant matrix spaces. IT is based on one element with the arbitary values allocated earlier.

[322]:
```python
class strain_energy:

    individual_k = []

    for i in range(0, len(element_connection_matrix)):

        element = element_connection_matrix[i]

        x_1 = element[0]
        x_2 = element[1]
        x_3 = element[2]
        x_4 = element[3]

        y_1 = element[0]
        y_2 = element[1]
        y_3 = element[2]
        y_4 = element[3]

        element_x_coords = [x[x_1-1], x[x_2-1], x[x_3-1], x[x_4-1]]
        element_y_coords = [y[y_1-1], y[y_2-1], y[y_3-1], y[y_4-1]]
```

```
        J_11 = phi_matrix_comma_i(1)[0] * element_x_coords[0]  + phi_matrix_comma_i(1)[1] * element_x_coords[1] + phi_matrix_comma_i(1)[2] * element_x_coords[2] + phi_matrix_comma_i(1)[3] * element_x_coords[3]
        J_12 = phi_matrix_comma_i(1)[0] * element_y_coords[0]  + phi_matrix_comma_i(1)[1] * element_y_coords[1] + phi_matrix_comma_i(1)[2] * element_y_coords[2] + phi_matrix_comma_i(1)[3] * element_y_coords[3]
        J_21 = phi_matrix_comma_j(1)[0] * element_x_coords[0]  + phi_matrix_comma_j(1)[1] * element_x_coords[1] + phi_matrix_comma_j(1)[2] * element_x_coords[2] + phi_matrix_comma_j(1)[3] * element_x_coords[3]
        J_22 = phi_matrix_comma_j(1)[0] * element_y_coords[0]  + phi_matrix_comma_j(1)[1] * element_y_coords[1] + phi_matrix_comma_j(1)[2] * element_y_coords[2] + phi_matrix_comma_j(1)[3] * element_y_coords[3]

        J = np.array([[J_11, J_12], [J_21, J_22]])
        det_J = J[0][0]*J[1][1] - J[0][1]*J[1][0]
        J_minus_1 = np.array([[J_22, J_21], [J_12, J_11]]) * 1/det_J

        d_N_1_d = np.matmul(J_minus_1, [phi_matrix_comma_i(0)[0], phi_matrix_comma_j(0)[0]])
        d_N_2_d = np.matmul(J_minus_1, [phi_matrix_comma_i(0)[1], phi_matrix_comma_j(0)[1]])
        d_N_3_d = np.matmul(J_minus_1, [phi_matrix_comma_i(0)[2], phi_matrix_comma_j(0)[2]])
        d_N_4_d = np.matmul(J_minus_1, [phi_matrix_comma_i(0)[3], phi_matrix_comma_j(0)[3]])


        B_matrix = np.array([[d_N_1_d[0], 0, d_N_2_d[0], 0, d_N_3_d[0], 0, d_N_4_d[0], 0], [0, d_N_1_d[1], 0, d_N_2_d[1], 0, d_N_3_d[1], 0, d_N_4_d[1]], [d_N_1_d[1], d_N_1_d[0], d_N_2_d[1], d_N_2_d[0], d_N_3_d[1], d_N_3_d[0], d_N_4_d[1], d_N_4_d[0]]])

        B_T = np.transpose(B_matrix)

        # Carrying out the integral equations with the summation varation:
        H_i = 2
        H_j = 2
        I = H_i * H_j * det_J

        k = I * np.matmul(B_T, np.matmul(E_ijkl,B_matrix))

        individual_k.append(k)
```

There is a total of 36 elements so also 36 values of local matrix, k. Each of these must be split into the appropriate global position.

[323]:
```
Global_matrix = np.zeros([number_dofs, number_dofs])

for i in range(0, len(strain_energy.individual_k)):
    k_e = strain_energy.individual_k[i]
    element_nodes = element_connection_matrix[i]

    element_node_1 = element_nodes[0]
    positioning_node_1_global = [int((element_node_1*2)-1),␣
↪int(element_node_1*2)]

    element_node_2 = element_nodes[1]
    positioning_node_2_global = [int((element_node_2*2)-1),␣
↪int(element_node_2*2)]

    element_node_3 = element_nodes[2]
    positioning_node_3_global = [int((element_node_3*2)-1),␣
↪int(element_node_3*2)]

    element_node_4 = element_nodes[3]
    positioning_node_4_global = [int((element_node_4*2)-1),␣
↪int(element_node_4*2)]

    relevant_dofs = [positioning_node_1_global, positioning_node_2_global,␣
↪positioning_node_3_global, positioning_node_4_global]
    dofs_vec = np.concatenate(relevant_dofs).ravel()

    for m in range(0, len(k_e[0])):
        col_val = dofs_vec[m]

        for j in range(0, len(k_e[0])):
            k_e_m_j = k_e[m][j]
            row_val = dofs_vec[j]
            Global_matrix[col_val-1][row_val-1] += k_e_m_j
```

[324]:
```
class load_conditions:

    forces_applied_to = [7, 28, 42, 56, 70, 82, 98]
    forces = [7, 6.5, 6, 5.5, 5, 4.5, 4.0]

    for i in range(0, len(forces)):
        force_matrix[forces_applied_to[i]-1] = forces[i]

    b_i = force_matrix
```

[325]:
```
inverse_k = np.linalg.pinv(Global_matrix)
```

The following block of code applies the forces to the known stiffness matrix in order to determine displacement values at the known loading locations.

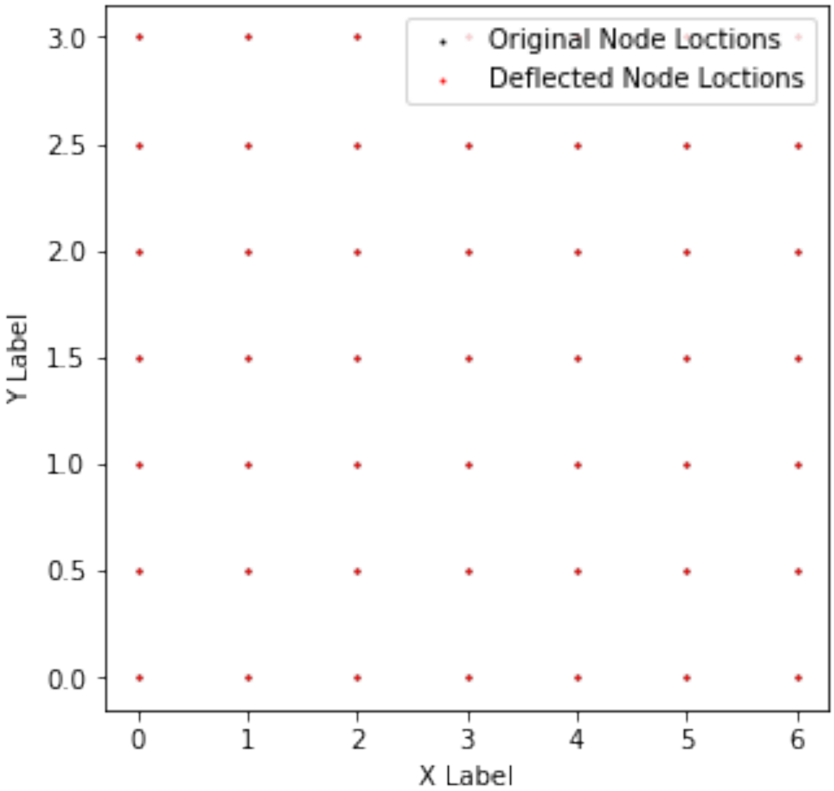
[326]:
```
for i in range(0, len(load_conditions.forces_applied_to)):
    val = load_conditions.forces_applied_to[i]
    dofs[val-1] += np.matmul(inverse_k[val-1], load_conditions.b_i)
```

[327]:
```
final_forces = np.matmul(Global_matrix, dofs)
final_dofs = np.matmul(inverse_k, final_forces)
```

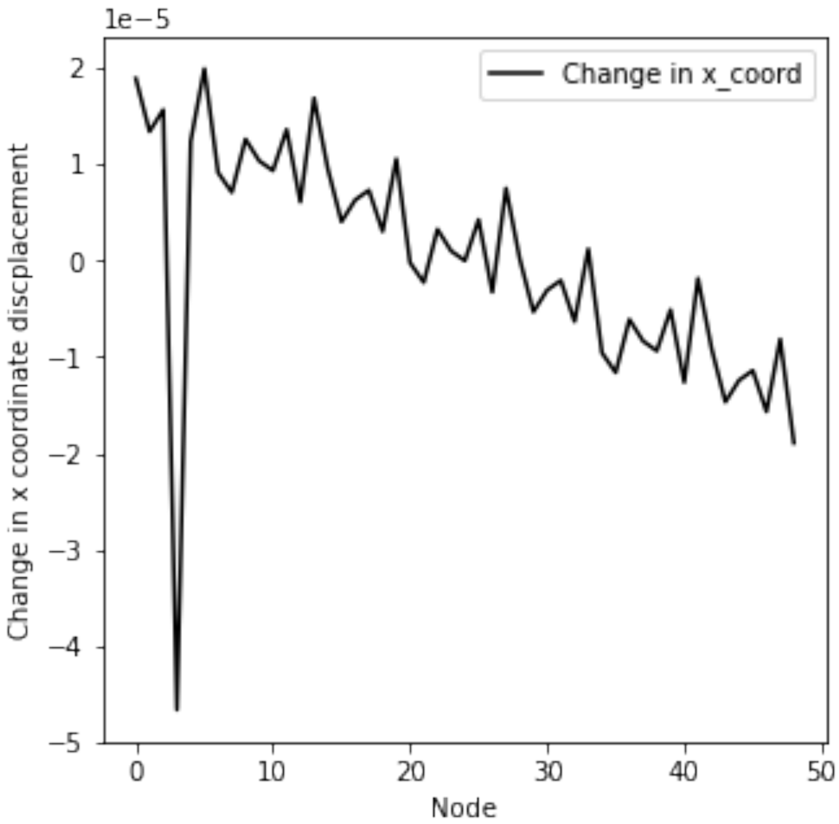
[328]:
```
x_deflec = []
y_deflec = []

for i in range(0, len(dofs), 2):
    x_dofs = final_dofs[i]
    y_dofs = final_dofs[i+1]
    x_deflec.append(x_dofs)
    y_deflec.append(y_dofs)
```

[329]:
```
final_x_array = [x[i] + x_deflec[i] for i in range(len(x))]
final_x = np.concatenate(final_x_array).ravel()
final_y_array = [y[i] + y_deflec[i] for i in range(len(y))]
final_y = np.concatenate(final_y_array).ravel()
```

[330]:
```
def image_view(x, y, final_x, final_y):
    fig, ax = plt.subplots(figsize=(5,5))

    ax.scatter(x, y, color='black', s=1)
    ax.scatter(final_x, final_y, color='red', s=1)

    ax.legend(['Original Node Loctions', 'Deflected Node Loctions'], loc='upper␣
↪right')

    ax.set_xlabel('X Label')
    ax.set_ylabel('Y Label')

    plt.show(block=True)
    plt.show()

image_view(x, y, final_x, final_y)
```



[331]:
```
x_difference = x - final_x
y_axis = x_difference
x_axis = [[i] for i in range(0,len(x_difference))]

def image_view(x_axis, y_axis):
    fig, ax = plt.subplots(figsize=(5,5))

    ax.plot(x_axis, y_axis, color='black')

    ax.legend(['Change in x_coord'], loc='upper right')

    ax.set_xlabel('Node')
    ax.set_ylabel('Change in x coordinate discplacement')

    plt.show(block=True)
    plt.show()

image_view(x_axis, y_axis)
```


1e−5
Change in x_coord
Change in x coordinate discplacement
Node
2
1
0
−1
−2
−3
−4
−5
0
10
20
30
40
50