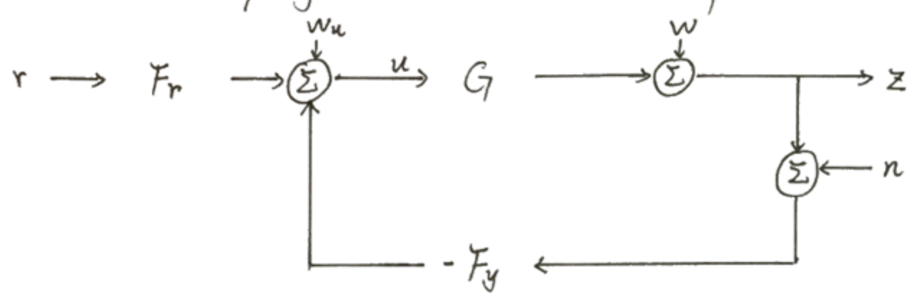
Lecture 02: The closed-loop system

Goals:

- Closed-loop is characterized by <u>7</u> transfer functions.
  - several objectives, but trade-offs exist, e.g. $S+T=1$.
  - <u>internal stability</u> requires all 7 to be stable.
- Determine, analyze, and design desired sensitivity functions

1. The closed-loop system (SISO this time)



Controller: feedback F_y feedforward F_r

Disturbances: w, w_u

Measurement noise: n.

aim: find z follow r, despite w, w_u, n.

(and model uncertainty Δ

- For Output:

$$z = w + G\left(w_u + F_r(r) - F_y(z+n) \right)$$

$$\Longrightarrow \quad z = \underbrace{\frac{1}{1+GF_y}}_{S} \cdot w + \underbrace{\frac{G}{1+GF_y}}_{SG} w_u + \underbrace{\frac{GF_r}{1+GF_y}}_{G_c} \cdot r - \underbrace{\frac{GF_y}{1+GF_y}}_{T} \cdot n$$

S: The sensitivity function.

T: The complementary sensitivity function. $\Longrightarrow S+T=1$ (trade-off)

G_c: The closed-loop transform function.

Thus, make $|S|$ small for disturbance.

make $|T|$ small for noise.

make $|T - G_c|$ small for set points.

- For Input.

$$u = W_u + F_r r - F_y(n + w + G u)$$

$$\Rightarrow u = \underbrace{\frac{1}{1+G F_y}}_{S} W_u + \underbrace{\frac{F_r}{1+G F_y}}_{S \cdot F_r} \cdot r - \underbrace{\frac{F_y}{1+G F_y}}_{S \cdot F_y} (n + w)$$

make $|S|$, $|SF_r|$, $|SF_y|$ small.

- Nominal stability = internal stability.

closed loop is internally stable if input-output stable for all 7 transfer functions.

S, SG, G_c, T, SF_r, SF_y, F_r are stable. (test poles)

Note: - if S stable, then T stable.

- if S and F_r stable, SF_r stable.
- if SG, F_r stable, G_c stable.

$\Uparrow$ check S, SG, F_r, SF_y is enough.

EX: $G = \frac{1}{s-1}$, unstable, $F_r = \frac{s-1}{s}$

$\Rightarrow S = \frac{s}{s+1}$, stable, $SG = \frac{s}{(s+1)(s-1)}$, unstable.

* Never cancell poles and zeros!

- The sensitivity S : (why to make S small).

  1) Effect of disturbance w ~~of~~ on z :

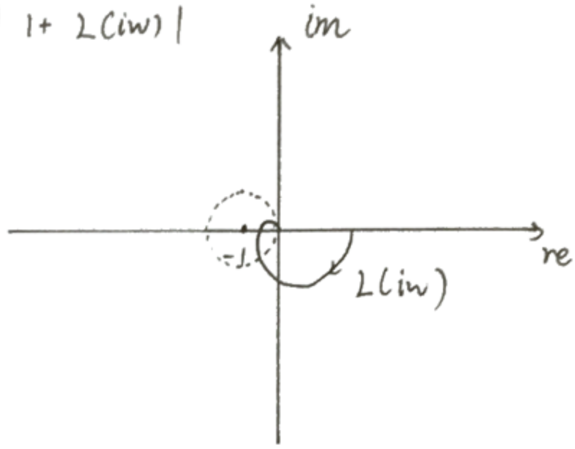
     - if without feedback $\Delta Z = 1 \cdot W$.
     - if with feedback : $\Delta Z = S \cdot W$

     $\Rightarrow$ disturbance attenuation for frequencies where $|S(iw)| < 1$,

     amplification if $|S(iw)| > 1$.

     Note: must have $|S(iw)| > 1$ for some w, if ~~G(s) has~~ $L(s) = G(s) F_y(s)$ pole excess of at least 2.

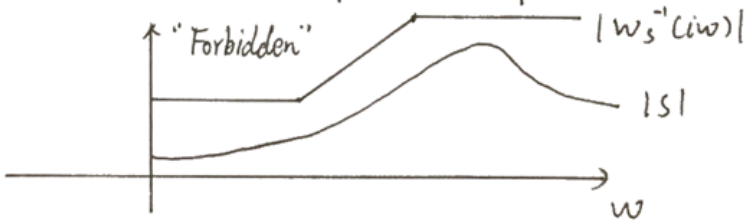
     psudo-proof: $|1 + L(iw)|$



$$|1 + L(iw)| \leq 1 \Rightarrow \cancel{|S(iw)|} = \frac{1}{|1 + L(iw)|} > 1$$

     - shaping the sensitivity S.
       * can not make $|S(iw)|$ small at all w.
       * resonable design specification.

$$|S(iw)| \leq |W_s^{-1}(iw)| \quad \forall w. \iff |W_s \cdot S| < 1 \ \forall w$$



$$\iff \|W_s \cdot S\|_\infty < 1$$

  2) Effect of uncertainty on closed-loop G_c.

$$G_c = \frac{G F_r}{1 + G F_y} \Rightarrow \frac{dG_c}{dG} = \cancel{\frac{F_r}{1+F_y}} \ \frac{F_r(1 + G F_y) - F_y G F_r}{(1 + G F_y)^2}$$

$$= \frac{F_r}{(1+GF_y)^2} = S \cdot \frac{G_c}{G} \implies \frac{dG_c/G_c}{dG/G} = S$$

Thus, relative unc in model G is attenuation in G_c by a factor S.

- The complementary sensitivity: T

  1) Noise: $\Delta z = T \cdot n \implies$ make $|T|$ small where n large.

  2) Robust stability: assume true system is

$$\tilde{G} = G(1+\Delta G)$$

how large can ΔG be

employ small gain theorem (SGT)

identify M: $u_\Delta = -GF_y(u_\Delta + y_\Delta)$

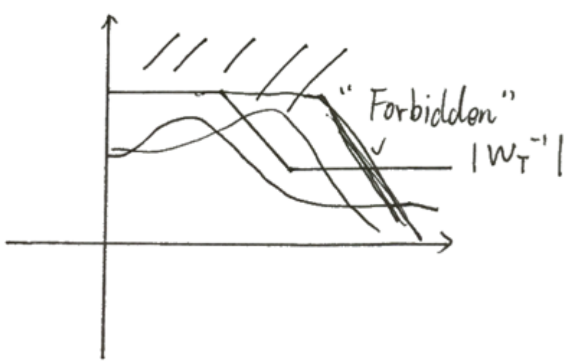
$$\implies M = \frac{GF_y}{1+GF_y} = T$$

stable if M is stable, ΔG stable, and

$$\|M \cdot \Delta G\|_\infty < 1 \implies \|T \cdot \Delta G\|_\infty < 1$$

$$\iff \underset{\forall w}{|T(iw) \cdot \Delta G(iw)| < 1} \iff |T| < \frac{1}{|\Delta G|} \quad \forall w.$$

Thus, make $|T|$ small where n and ΔG large.



recall: $S+T=1$, can not make $|W_S|$ and $|W_T|$ large at same time.