A guide to the MIMO computer exercises in EL 2520 Advanced Control Theory and Practice

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1 Introduction

The purpose of this guide is to help the reader to avoid common mistakes in Computer exercise 3 and 4 and to increase the understanding on the lab. Not all parts of the labs are explained.

2 Computer exercise 3 - Decentralized Control

The problem is here to construct a decentralized controller F for the system described in Figure 1 such that the output y follows the reference r where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad r = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} r_1 - y_1 \\ r_2 - y_2 \end{bmatrix}$$

and

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

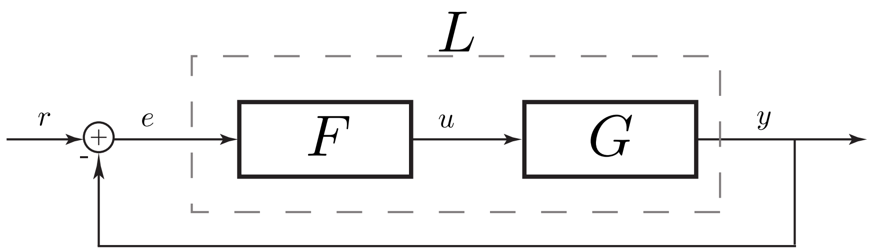

Figure 1: Block Diagram

2.1 The structure of the controller

When we talk about decentralized control we mean that the controller should have a diagonal structure where we use one input to control one output. In our case with a 2×2 system this means that

$$F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} \quad \text{or} \quad F = \begin{bmatrix} 0 & f_1 \\ f_2 & 0 \end{bmatrix}$$

which pairing we should use is decided by the RGA of G

2.2 Shaping the loop gain L

The loop gain $L = GF$ is as seen in Figure 1 the transfer function from the control error e to the output y as $y = Le$ giving

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \ell_{11}e_1 + \ell_{12}e_2 \\ \ell_{21}e_1 + \ell_{22}e_2 \end{bmatrix} \\ &= \begin{bmatrix} \ell_{11}(r_1 - y_1) + \ell_{12}(r_2 - y_2) \\ \ell_{21}(r_1 - y_1) + \ell_{22}(r_2 - y_2) \end{bmatrix} \end{aligned}$$

solving for y_1 and y_2 we get

$$\begin{aligned} y_1 &= \frac{\ell_{11}}{1+\ell_{11}} r_1 + \frac{\ell_{12}}{1+\ell_{11}} (r_2 - y_2) \\ y_2 &= \frac{\ell_{21}}{1+\ell_{22}} (r_1 - y_1) + \frac{\ell_{22}}{1+\ell_{22}} r_2 \end{aligned}$$

Since we here **always** are interested in having y_1 following r_1 and y_2 following r_2 we are interested in shaping these transfer functions

$$\begin{aligned} y_1 &= \frac{\ell_{11}}{1+\ell_{11}} r_1 \\ y_2 &= \frac{\ell_{22}}{1+\ell_{22}} r_2. \end{aligned}$$

Based on what we have learnt previously in the course this means that we should shape ℓ_{11} and ℓ_{22} in a good way, i.e., according to the specifications.

NOTE:

The reasoning on which elements in the transfer functions in L we should shape is independent of the choice of F. However, what these elements will be is dependent on the structure of F since

$$L = GF = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} = \begin{bmatrix} g_{11}f_1 & g_{12}f_2 \\ g_{21}f_1 & g_{22}f_2 \end{bmatrix}$$

or

$$L = GF = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 0 & f_1 \\ f_2 & 0 \end{bmatrix} = \begin{bmatrix} g_{12}f_2 & g_{11}f_1 \\ g_{11}f_2 & g_{21}f_1 \end{bmatrix}$$

3 Computer exercise 4 - Dynamic Decoupling

Before digging in to the problem we first try to establish an understanding of the purpose of decoupling. To do this we first study the block diagram in Figure 2. The problem we want to solve is to construct a controller F for the system G. The idea behind decoupling is to solve this problem in three steps. The first step is to find a decoupling weight W_1 which makes $\tilde{G} = GW_1$ diagonal. The second step is to construct a decentralized controller $\tilde{F}$ for the decoupled system $\tilde{G}$. In the third and final step the controller for the original system G is given by $F = W_1\tilde{F}$.

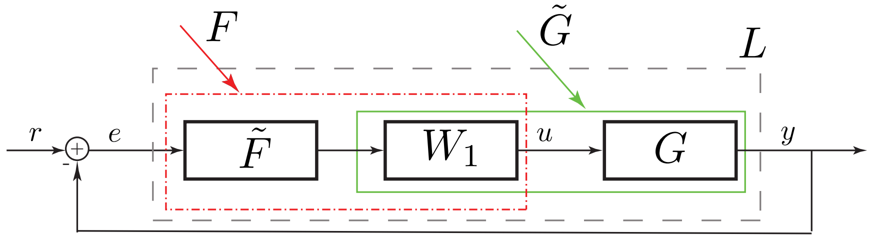

Figure 2: Block Diagram

3.1 The decoupling weight

The purpose of the decoupling weight W_1 is to make $\tilde{G} = GW_1$ diagonal so that the systems appears to be diagonal seen from the $\tilde{F}$ controller. With G and W_1 as

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad W_1 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$$

we get

$$\begin{aligned} \tilde{G} = GW_1 &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \\ &= \begin{bmatrix} g_{11}w_{11} + g_{12}w_{21} & g_{11}w_{12} + g_{12}w_{22} \\ g_{21}w_{11} + g_{22}w_{21} & g_{21}w_{12} + g_{22}w_{22} \end{bmatrix} \\ &\triangleq \begin{bmatrix} g_{11}w_{11} + g_{12}w_{21} & 0 \\ 0 & g_{21}w_{12} + g_{22}w_{22} \end{bmatrix} \end{aligned}$$

that is we have to choose the elements in W_1 such that we fulfill

$$\begin{aligned} 0 &= g_{11}w_{12} + g_{12}w_{22} \\ 0 &= g_{21}w_{11} + g_{22}w_{21} \end{aligned} \tag{1}$$

Since we have two equations and four unknowns we have several degrees of freedom on how to choose the elements in W_1. Experience tells us that it is good to set the weights corresponding to a good pairing equal to 1 and use the remaining weights to solve the equations in (1). This gives use two possible weights

$$W_1 = \begin{bmatrix} 1 & w_{21} \\ w_{12} & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{g_{21}}{g_{22}} \\ -\frac{g_{12}}{g_{11}} & 1 \end{bmatrix} \Rightarrow \tilde{G} = \begin{bmatrix} g_{11} - \frac{g12g21}{g_{22}} & 0 \\ 0 & g_{22} - \frac{g_{12}g_{21}}{g_{11}} \end{bmatrix}$$

$$W_1 = \begin{bmatrix} w_{11} & 1 \\ 1 & w_{22} \end{bmatrix} = \begin{bmatrix} -\frac{g_{22}}{g_{21}} & 1 \\ 1 & -\frac{g_{11}}{g_{12}} \end{bmatrix} \Rightarrow \tilde{G} = \begin{bmatrix} g_{12} - \frac{g11g22}{g_{21}} & 0 \\ 0 & g_{21} - \frac{g_{11}g_{22}}{g_{12}} \end{bmatrix}$$

which weight we should use is decided by the RGA of G.

3.2 Shaping the loop gain L

The loop gain $L = GF = \tilde{G}\tilde{F}$ as seen in Figure 2. Ideally we would like the loop gain to be diagonal, since we then have $y = Le$ as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 \\ 0 & \ell_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \ell_{11}e_1 \\ \ell_{22}e_2 \end{bmatrix} = \begin{bmatrix} \ell_{11}(r_1 - y_1) \\ \ell_{22}(r_2 - y_2) \end{bmatrix}$$

which gives

$$y_1 = \frac{\ell_{11}}{1 + \ell_{11}} r_1$$
$$y_2 = \frac{\ell_{22}}{1 + \ell_{22}} r_2$$

meaning that y_1 is dependent **only** on r_1 and that y_2 is dependent **only** on r_2. For comparison with the decentralized control case compare with (2.2). To get the desired behavior we should shape ℓ_{11} and ℓ_{22} according to the specifications.

3.2.1 The controller $\tilde{F}$

Since the decoupling W_1 makes $\tilde{G}$ diagonal in order for us to get L to be diagonal we have to choose $\tilde{F}$ diagonal as

$$\tilde{F} = \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix}.$$

How $\tilde{f}_1$ and $\tilde{f}_2$ should be chosen is dependent of the specifications of the loop gain.

NOTE:

The reasoning on which elements in the transfer functions in L we should shape is independent of the choice of W_1. However, what these elements will be is dependent on the structure of W_1 since

$$
\begin{aligned}
L = GF = GW_1\tilde{F} &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & w_{21} \\ w_{12} & 1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & -\frac{g_{21}}{g_{22}} \\ -\frac{g_{12}}{g_{11}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} g_{11} - \frac{g12g21}{g_{22}} & 0 \\ 0 & g_{22} - \frac{g_{12}g_{21}}{g_{11}} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{f}_1\left(g_{11} - \frac{g12g21}{g_{22}}\right) & 0 \\ 0 & \tilde{f}_2\left(g_{22} - \frac{g_{12}g_{21}}{g_{11}}\right) \end{bmatrix} \\
&= \begin{bmatrix} \ell_{11} & 0 \\ 0 & \ell_{22} \end{bmatrix}
\end{aligned}
$$

or

$$
\begin{aligned}
L = GF = GW_1\tilde{F} &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} w_{11} & 1 \\ 1 & w_{22} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} -\frac{g_{22}}{g_{21}} & 1 \\ 1 & -\frac{g_{11}}{g_{12}} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} g_{12} - \frac{g11g22}{g_{21}} & 0 \\ 0 & g_{21} - \frac{g_{11}g_{22}}{g_{12}} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} \\
&= \begin{bmatrix} \tilde{f}_1\left(g_{12} - \frac{g11g22}{g_{21}}\right) & 0 \\ 0 & \tilde{f}_2\left(g_{21} - \frac{g_{11}g_{22}}{g_{12}}\right) \end{bmatrix} \\
&= \begin{bmatrix} \ell_{11} & 0 \\ 0 & \ell_{22} \end{bmatrix}
\end{aligned}
$$

3.3 The controller F

The controller F is now given by

$$F = W_1 \tilde{F} = \begin{bmatrix} 1 & w_{21} \\ w_{12} & 1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{g_{21}}{g_{22}} \\ -\frac{g_{12}}{g_{11}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{f}_1 & -\frac{g_{21}}{g_{22}}\tilde{f}_2 \\ -\frac{g_{12}}{g_{11}}\tilde{f}_1 & \tilde{f}_2 \end{bmatrix}$$

or

$$F = W_1 \tilde{F} = \begin{bmatrix} w_{11} & 1 \\ 1 & w_{22} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix} = \begin{bmatrix} -\frac{g_{22}}{g_{21}} & 1 \\ 1 & -\frac{g_{11}}{g_{12}} \end{bmatrix} \begin{bmatrix} \tilde{f}_1 & 0 \\ 0 & \tilde{f}_2 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{g_{22}}{g_{21}}\tilde{f}_1 & \tilde{f}_2 \\ \tilde{f}_1 & -\frac{g_{11}}{g_{12}}\tilde{f}_2 \end{bmatrix}$$

Compare this with the controllers suggested in Section 2.1