Solutions – EL2520
Exam May 28, 2014

Problem 1

(a) By the definition we get

$$\|y_1\|_2 = \sqrt{\int_{-\infty}^{\infty} |y_1|^2 dt}$$

which if we expand the euclidean vector norm becomes

$$\|y_1\|_2 = \sqrt{\int_{-\infty}^{\infty} \left[\sqrt{|e^{-t}|^2 + |3e^{-2t}|^2}\right]^2 dt} = \sqrt{\int_{-\infty}^{\infty} e^{-2t} + 9e^{-4t} dt}$$

Since the signal is zero until time zero the part of the integral before zero does not contribute to the value of the norm which leaves us with

$$\|y_1\|_2 = \sqrt{\int_{0}^{\infty} e^{-2t} + 9e^{-4t} dt}$$

which if we evaluate the integral becomes

$$\|y_1\|_2 = \sqrt{\left[\frac{-1}{2}e^{-2t} + \frac{-9}{4}e^{-4t}\right]_0^{\infty}} = \sqrt{0 - \frac{-11}{4}} = \frac{\sqrt{11}}{2}.$$

For y_2 we note that the lower limit of integration may be set to 1 since the signal is zero up to that point and we thus get

$$\|y_2\|_2 = \sqrt{\int_{1}^{\infty} \left[\sqrt{\sin^2(\omega t)/t^2 + \cos^2(\omega t)/t^2}\right]^2 dt} = \sqrt{\int_{1}^{\infty} 1/t^2 dt} = 1$$

where we used the trigonometric identity $\sin^2 + \cos^2 = 1$ to simplify.

For y_3 we get

$$\|y_3\|_2 = \sqrt{\int_{0}^{\infty} |c|^2 dt} = \begin{cases} \infty & c \neq 0 \\ 0 & c = 0 \end{cases}$$

(b) We recall that the $\mathcal{H}_2$ norm is given by

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 d\omega}$$

Hence

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\epsilon^2\omega^2+1} d\omega}$$

Using the hint, we get

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\left[\frac{1}{\epsilon}\tan^{-1}\epsilon\omega\right]_{-\infty}^{\infty}} = \frac{1}{\sqrt{2\epsilon}}$$

Clearly, as $\epsilon \to 0$ we get $\|G\|_2 \to \infty$ and as $\epsilon \to \infty$ we get $\|G\|_2 \to 0$. Graphically this may be explained as follows. The square of the norm is proportional to the area below the magnitude plot in the Bode-diagram of the squared system (plotted in absolute scale). As can be seen in the figure,

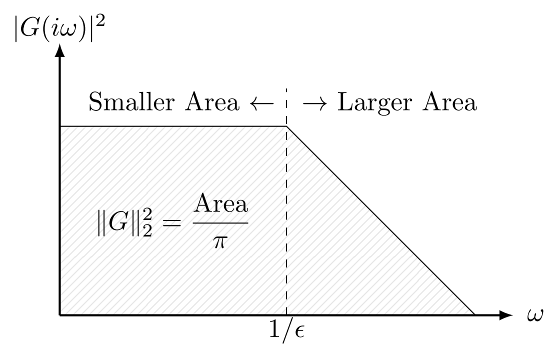

Figure 1: Area under Bode-plot of $|G(i\omega)|^2 \sim \|G\|_2^2$ (Note: The area is computed for the absolute scale and not in the log-log scale.)

the break-off frequency and hence also the area increases as ϵ is decreased. As the break-off frequency is pushed towards infinity the area goes to infinity and when the break-off frequency goes to zero the area goes to zero.

(c) We recall that the $\mathcal{H}_\infty$ norm of $G(s)$ is given by

$$\|G\|_\infty = \sup_\omega \overline{\sigma}(G(i\omega)).$$

To find the norm we need to compute the largest singular value and maximize over all frequencies. Begin by rewriting the system as

$$G(s) = \frac{1}{s+1}\begin{bmatrix} 1 & \frac{1}{\alpha} \\ \frac{-1}{\alpha} & 1 \end{bmatrix}$$

The singular values of $G(s)$ are by definition the square root of the eigenvalues of $G^*(s)G(s)$ where $G^*(s)$ is the conjugate transpose of $G(s)$. We get

$$G^*(s)G(s) = \frac{1}{|s+1|^2}\begin{bmatrix} 1 & \frac{-1}{\alpha} \\ \frac{1}{\alpha} & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{\alpha} \\ \frac{-1}{\alpha} & 1 \end{bmatrix} = \frac{1}{|s+1|^2}\begin{bmatrix} 1+\frac{1}{\alpha^2} & 0 \\ 0 & 1+\frac{1}{\alpha^2} \end{bmatrix}$$

This is a diagonal matrix so the eigenvalues are trivially given by the diagonal elements. Clearly, the maximum (and minimum) singular value is then equal to

$$\overline{\sigma}(G(s)) = \sqrt{\frac{1+1/\alpha^2}{|s+1|^2}}$$

Thus, to compute the norm we need to solve

$$\|G\|_\infty = \sup_\omega \sqrt{\frac{1+1/\alpha^2}{|i\omega+1|^2}}$$

This is clearly maximized when the denominator under the root sign is minimized which clearly happens for $\omega = 0$. Thus, we find that

$$\|G\|_\infty = \sqrt{1+1/\alpha^2}.$$

Problem 2

(a) For the first system, the pole polynomial is $(s+1)^2(s+3)(s+5)$, thus the poles are $-1,-1,-3,-5$. The zero polynomial is $(s+1)(s+3)-(s+4)(s+4) = -4s-13$, thus there is one zero at $-13/4$.

The second system can be written as

$$G(s) = C(sI-A)^{-1}B = \frac{1}{s}\begin{bmatrix} \frac{2}{s+1} & 1 \\ 0 & 2 \end{bmatrix},$$

where the pole polynomial is $s^2(s+1)$, thus the poles are $0,0,-1$. The zero polynomial is 4, thus there are no zeros.

(b) First system:

$$RGA(0) = \frac{1}{13}\begin{bmatrix} -3 & 16 \\ 16 & -3 \end{bmatrix}$$

with this it is enough to determine the pairing $u_1 \to y_2$ and $u_2 \to y_1$, but we can also check

$$RGA(1i) = \begin{bmatrix} -0.23-0.24i & 1.23+0.24i \\ 1.23+0.24i & -0.23-0.24i \end{bmatrix}$$

Second system:

$$RGA(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, the pairing is $u_1 \to y_1$ and $u_2 \to y_2$.

(c) Observability:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0.5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(\mathcal{O}) = 3.$$

Controllability:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(\mathcal{C}) = 3.$$

The system is both observable and controllable.

(d) At stationarity ($s = 0$) we have the system

$$G(0) = \begin{bmatrix} 1/3 & 4/3 \\ 4/5 & 3/5 \end{bmatrix}$$

We would like to design W such that $G(0)W$ is diagonal, and easiest way is to select

$$W = G(0)^{-1} = \frac{1}{13}\begin{bmatrix} -9 & 20 \\ 12 & -5 \end{bmatrix}$$

Now we design a proportional decentralized controller

$$F = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

$$G(s) * W = \frac{1}{13(1+s)} \begin{bmatrix} (3s+39)/(s+3) & 15s/(s+3) \\ 3s/(s+5) & (15s+65)/(s+5) \end{bmatrix}$$

Thus, when looking at frequency 1 rad/s ($s = i$), we get

$$k_1 = \left| \frac{13(1+i)(3+i)}{3i+39} \right| = \left| \frac{39+65i}{51} \right| \approx 1.49$$

$$k_2 = \left| \frac{13(1+i)(5+i)}{15i+65} \right| = \left| \frac{455+429i}{445} \right| \approx 1.41$$

Problem 3

(a) We have:

$$G(s) = \frac{(s-1)(s-3)}{(s-2)(s+1)^2}.$$

Hence there are two RHP zeros at 1 and 3, respectively. This implies that the cross-over frequency should be roughly smaller than 3/2 rad/s. There is also a RHP pole at 2, so that the cross-over has to be larger than 4 rad/s. We deduce that it will be hard to design a controller meeting the requirements.

(b) (b-1) When r_2 is large, we assume in the model that the level of noise is high, in which case, the control effort will be significant, reducing the control error. Hence the control error should be smaller if $r_2 = 10$.
(b-2) Similarly when r_1 is large, the model says that the disturbance is high, and hence the control effort will be significant, reducing the control error. Thus by decreasing r_1 from 2 to 1, the control error is expected to grow.

(c) (c-1) S is the sensitivity function, and corresponds to the input-ouput relationship from w (the disturbance) to z (the output). SF_y corresponds to the input-output relationship from w or n to u. Hence by imposing a restriction on the norm of S and SF_y, we wish to design a controller that efficiently reject disturbances, and for which there is relatively small impact of w or n on the input u.
(c-2) M_S corresponds to the maximum amplitude of the sensitivity function S. Thus to decrease the sensitivity to disturbances, we should decrease M_S.
(c-3) The transfer function from n to z is T, and hence if we wamt to limit the impact of measurement noise, we should impose a limitation on T. This can be done by designing a weight function W_T for T, and modify the $\mathcal{H}_\infty$ control framework so that the controller minimizes:

$$\left\| \begin{array}{c} W_S S \\ W_T T \\ W_u F_y S \end{array} \right\|_\infty .$$

Problem 4

(a) The open-loop system is stable iff $a < 0$.

(b) (b-1) In the following, we use the same notations as those used in the course. We have here a first-order system, $n = 1$. We have: $A = a$, $B = b$, $N = 1 = C = M$, $R_1 = 1 = R_2$, and $R_{12} = \alpha$. To apply Kalman filter, we need to verify the assumptions required for the Kalman filter to lead to the optimal observer. These assumptions are:

- R_2 is symmetric and positive definite. Here $R_2 = 1$, and this assumption holds.
- $\tilde{R}_1 = R_1 - R_{12}R_2^{-1}R_{12}^T$ is positive definite. Here $\tilde{R}_1 = 1 - \alpha^2$. Hence the assumption holds since $\alpha \in [0, 1)$.
- (A, C) is detectable, which means that there exists K such that $A-KC$ is stable. Here $A - KC = a - K$. The assumption holds for the choice $K > a$.
- $(A - R_{12}R_2^{-1}C, \tilde{R}_1)$ is stabilizable, which means that there exists L such that $A - R_{12}R_2^{-1}C - \tilde{R}_1 L$ is stable. Here we have:

$$A - R_{12}R_2^{-1}C - \tilde{R}_1 L = a - \alpha - (1 - \alpha^2)L.$$

Again the assumption holds for a choice of L large enough.

(b-2) Since we can use Kalman filter to determine the optimal observer, the latter is given by:

$$\dot{\hat{x}} = a\hat{x} + bu + K(y - \hat{x}),$$

where $K = p + \alpha$ and p is the positive solution of Riccati's equation:

$$2ap - (p + \alpha)^2 + 1 = 0.$$

We obtain:

$$p = (a - \alpha) + \sqrt{(a - \alpha)^2 + (1 - \alpha)^2},$$

and

$$K = a + \sqrt{(a - \alpha)^2 + (1 - \alpha)^2}.$$

(c) The error $\tilde{x} = x - \hat{x}$ satisfies:

$$\dot{\tilde{x}} = -\sqrt{(a - \alpha)^2 + (1 - \alpha)^2}\tilde{x} + v_1 - (a + \sqrt{(a - \alpha)^2 + (1 - \alpha)^2})v_2.$$

When $\alpha > 0$, $E[\tilde{x}]$ decreases to zero slower than if α would be equal to 0. The noise correlations increase the estimation errors.

(d) The optimal state feedback controller is $u = -Lx$. Here we have $L = bS$ where $S \geq 0$ solves Riccati's equation:

$$2aS + \rho - S^2 b^2 = 0.$$

We obtain:

$$S = \frac{1}{b^2}(a + \sqrt{a^2 + b^2\rho}).$$

Finally:

$$u = -\left(\frac{a}{b} + \sqrt{(\frac{a}{b})^2 + \rho}\right) x.$$

(e) From the separation principle, we know that the solution of the optimization problem combines the optimal observer and the optimal state feedback controller:

$$\dot{\hat{x}} = a\hat{x} + bu + K(y - \hat{x}),$$

$$u = -\left(\frac{a}{b} + \sqrt{(\frac{a}{b})^2 + \rho}\right) \hat{x},$$

where $K = a + \sqrt{(a-\alpha)^2 + (1-\alpha)^2}$.

(f) The system is decoupled. Hence the solution of the optimization problem is obtained by combining the LQG controllers for x_1 and x_2:
For x_1:

$$\dot{\hat{x}}_1 = a_1\hat{x}_1 + b_1u_1 + K_1(y_1 - \hat{x}_1),$$

$$u_1 = -\left(\frac{a_1}{b_1} + \sqrt{(\frac{a_1}{b_1})^2 + \rho}\right) \hat{x}_1,$$

where $K_1 = a_1 + \sqrt{(a_1-\alpha)^2 + (1-\alpha)^2}$.
For x_2:

$$\dot{\hat{x}}_2 = a_2\hat{x}_2 + b_2u_2 + K_2(y_2 - \hat{x}_2),$$

$$u_2 = -\left(\frac{a_2}{b_2} + \sqrt{(\frac{a_2}{b_2})^2 + \rho}\right) \hat{x}_2,$$

where $K_2 = a_2 + \sqrt{(a_2-\beta)^2 + (1-\beta)^2}$.

Problem 5

(a) First, let us compute

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \begin{bmatrix} 1 & -2k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & 2t\, e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Then, we have

$$F = e^{AT} = \frac{1}{e} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$G = \int_0^T e^{At} B\, dt = \begin{bmatrix} 1 - e^{-1} & 2 - 4e^{-1} \\ 0 & 1 - e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 - 9e^{-1} & -1 + e^{-1} \\ 2 - 2e^{-1} & 0 \end{bmatrix}$$

$$H = C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

(b) The continuous system is stable if and only if all eigenvalues of A have negative real part. Let $\{\lambda_i\}$ denote the eigenvalues of A, then the eigenvalues of $F = e^{AT}$ are $\{e^{\lambda_i T}\}$. The sampled system is stable if and only if the eigenvalues of F are less than 1 in magnitude, which corresponds to the real part of $\lambda_i T < 0$, but since $T > 0$ this is equivalent to A being stable.

(c) Let us start with the second controller.

$$\begin{aligned} J_2 = &\left(\begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \right)^T \left(\begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \right) + \\ &\frac{1}{2} \left(\begin{bmatrix} 0.25 & 2.5 \\ 0 & 4 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k+1} \right)^T \left(\begin{bmatrix} 0.25 & 2.5 \\ 0 & 4 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{k+1} \right) \\ = &x_k^T \frac{1}{32} \begin{bmatrix} 9 & 26 \\ 26 & 516 \end{bmatrix} x_k + 3.5u_k^2 + 0.5u_{k+1}^2 + x_k^T \frac{1}{4} \begin{bmatrix} 1 \\ 58 \end{bmatrix} u_k + x_k^T \begin{bmatrix} 0 \\ 4 \end{bmatrix} u_{k+1} + 2u_k u_{k+1} \end{aligned}$$

Introduce the variable $\tilde{u} = \begin{bmatrix} u_k \\ u_{k+1} \end{bmatrix}$, and then write J_2 as the quadratic form

$$J_2 = \tilde{u}^T \begin{bmatrix} 3.5 & 1 \\ 1 & 0.5 \end{bmatrix} \tilde{u} + x_k^T \begin{bmatrix} 1/4 & 0 \\ 58/4 & 4 \end{bmatrix} \tilde{u} + x_k^T \begin{bmatrix} 9/32 & 26/32 \\ 26/32 & 516/32 \end{bmatrix} x_k$$

We are now solving the convex problem

$$\begin{aligned} &\min_{\tilde{u}} J_2 \\ &\text{subject to } \|\tilde{u}\|_\infty \leq 1 \end{aligned}$$

Which has the extreme point given by

$$\tilde{u} = -\frac{1}{2} \begin{bmatrix} 3.5 & 1 \\ 1 & 0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1/4 & 58/4 \\ 0 & 4 \end{bmatrix} x_k = \begin{bmatrix} -1/12 & -13/6 \\ 1/6 & 1/3 \end{bmatrix} x_k$$

Only the first control signal is applied by the MPC, thus we get the controller

$$u_k = \begin{cases} \begin{bmatrix} -1/12 & -13/6 \end{bmatrix} x_k & \text{if } \left| \begin{bmatrix} -1/12 & -13/6 \end{bmatrix} x_k \right| < 1 \\ \operatorname{sgn}\left(\begin{bmatrix} -1/12 & -13/6 \end{bmatrix} x_k \right) & \text{otherwise} \end{cases}$$

The first controller is similar, notice that

$$J_1 = J_2 + 2u_k^T u_k + 2u_{k+1}^T u_{k+1} = \tilde{u}^T \begin{bmatrix} 5.5 & 1 \\ 1 & 2.5 \end{bmatrix} \tilde{u} + x_k^T \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 58 & 16 \end{bmatrix} \tilde{u} + x_k^T \frac{1}{32} \begin{bmatrix} 9 & 26 \\ 26 & 516 \end{bmatrix} x_k$$

Thus

$$\tilde{u} = -\frac{1}{2} \begin{bmatrix} 5.5 & 1 \\ 1 & 2.5 \end{bmatrix}^{-1} \begin{bmatrix} 1/4 & 58/4 \\ 0 & 4 \end{bmatrix} x_k = \begin{bmatrix} -5/204 & -43/34 \\ 1/102 & -5/17 \end{bmatrix} x_k$$

and

$$u_k = \begin{bmatrix} -5/204 & -43/34 \end{bmatrix} x_k$$

(d) Using the first controller, $u_k = Kx_k$, we can compute the closed loop poles from the eigenvalues of

$$A + BK = \begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -5/204 & -43/34 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \\ -5/204 & 25/34 \end{bmatrix}$$

whose eigenvalues have the magnitude 0.63, thus the system is stabilized with this controller.

The second controller does not stabilize the system, which can be seen for example when the initial condition is $x_0 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$. Then, in particular, the second component will evolve as $x_{k,2} = 1 + 9 \cdot 2^k$.