REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2015–06–08, kl 08.00–13.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures and of the blackboard notes, mathematical tables and pocket calculator.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover.

The exam consists of five problems. The distribution of points for the various problems and subproblems is indicated.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Alexandre Proutiere 08-7906351

Resultat: Will be posted no later than June 30, 2015.

Good Luck!

Problem 1

a) Compute the L_∞ and L_2 norms of the signal

$$f(t) = \begin{bmatrix} e^{-t} \\ 1/(t+1) \end{bmatrix} \cdot 1_{t>0}$$

where $1_{t>0}$ is the indicator function which is 1 whenever $t > 0$ and 0 otherwise. [3pts]

b) Given two interconnected (see Figure 1) linear and stable systems $\mathcal{S}_1$ and $\mathcal{S}_2$ which both depend on a joint parameter α such that

$$\|\mathcal{S}_1\| = 6|\alpha|$$
$$\|\mathcal{S}_2\| = |1 - \alpha|.$$

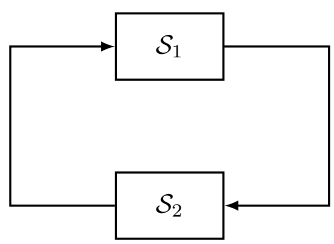

Figure 1: Two interconnected systems

For which values of α does the small gain theorem guarantee stability. What can be said about the stability for any α not fulfilling the derived condition? [4pts]

c) What is the $\mathcal{H}_\infty$-norm of the system

$$G(s) = \begin{bmatrix} \dfrac{\alpha}{s+\alpha} & \dfrac{1}{s+1} \end{bmatrix}$$

assuming $\alpha > 0$. [3pts]

Problem 2

a) Consider the following system given in state-space form:

$$\dot{x} = \begin{bmatrix} 1 & 2 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

- Determine the system poles. [1pt]
- For what values on α, β and γ is the system observable? Controllable? [2pts]

b) Consider the following system

$$G(s) = \begin{bmatrix} \frac{2}{s+2} & \frac{1}{s+1} \\ \frac{3}{s+1} & \frac{2}{s+2} \end{bmatrix}$$

- Determine the system poles and zeros. [2pts]
- We would like to control this system with a decentralized controller such that the crossover frequency is $\omega_c = 0.5$ rad/s. Determine a suitable pairing using the RGA method. [1pt]
- What would happen with the pairing if we wanted to increase the crossover frequency to 5 rad/s? [1pt]
- Propose a static decoupling matrix W such that the system GW is decoupled at the frequency 0 rad/s. [1pt]
- Design a decentralized PI $(k_i(1 + \frac{1}{s\tau_i}))$ controller F such that GWF has crossover frequency $\omega_c = 0.5$ rad/s and phase margin of 60°. [1pt]
- Design a decentralized PI controller F such that GF has crossover frequency $\omega_c = 0.5$ rad/s and phase margin of 60°. [1pt]

Problem 3

a) We wish to design a feedback controller that ensures stability of the closed-loop system for all plants of the form:

$$G_a(s) = \frac{e^{-as}}{s+1}, \quad \text{for } a \in [0,1].$$

Derive a condition (as general as possible) on the transfer function F_y (independent of a) of the feedback controller to achieve this objective. [3pts]

b) For which of the following plants can we design a closed-loop controller that rejects disturbances and exhibits low sensitivity to measurement noise at frequencies around 2 rad/s?

$$G(s) = \frac{4}{s^2+4s+3}, \quad G(s) = \frac{s^2+s-2}{s^2-12s+35}, \quad G(s) = \frac{s-5}{s^3-7s^2+16s-12}.$$

[4pts]

c) We have designed an $\mathcal{H}_\infty$ closed-loop controller so as to ensure that:

$$\left\| \begin{matrix} W_S S \\ W_T T \end{matrix} \right\|_\infty \leq 1$$

for some weight functions W_S and W_T (S denotes the sensitivity function, and T is the complementary sensitivity function).

- We want to ensure the stability of the closed-loop system for all plants belonging to a set Π. What condition should the weight function W_T satisfy to achieve this objective? [1pt]
- We have chosen a weight function of the form $W_T = \frac{1}{M_T} + \frac{s}{\omega_{BS}}$, where M_T and ω_{BS} are two positive parameters. Should we increase or decrease M_T to further decrease the sensitivity to measurement noise? [1pt]
- We discover that the impact of the measurement noise on the input signal u is too significant. Propose a framework to address this issue. How can we modify the $\mathcal{H}_\infty$ control framework? [1pt]

Problem 4

Consider the following linear system:

$$\begin{aligned} \dot{x} &= x + bu + v_1 \\ y &= x + v_2 \\ z &= 2x \end{aligned}$$

where v_1, v_2 are gaussian white noises such that the covariance matrix of $\binom{v_1}{v_2}$ is:

$$\begin{pmatrix} 1 & \beta \\ \beta & \alpha \end{pmatrix}.$$

We wish to find the controller that solves the following optimization problem:

$$\min_u E\left[\int_0^\infty (z(t)^2 + \rho u(t)^2)\,\mathrm{d}t\right], \tag{1}$$

where $\rho > 0$.

a) Is the open-loop system stable? [1pt]

b)
- Can we use a Kalman filter to determine the optimal observer? [2pt]
- Compute the optimal observer. [2pts]
- What is the impact of the positive correlation β on the state estimation error? [1pt]

c) Determine the optimal linear state feedback controller, and provide a full solution of the problem (1). [2pts]

d) Consider now the same system without noise, i.e., $v_1 = v_2 = 0$. Assume that $b = 1$. By playing with the parameter ρ, can we move the pole of the controlled system to -2 or -1/2? How would we need to choose ρ? [2pts]

Problem 5

a) Briefly explain the principle of Model Predictive Control. [1pt]

b) Consider an MPC problem where we have constraints on the output. After solving this MPC accounting for these constraints, we realize that in practice, implementing the solution does not always meet the constraints (this can be due for example to neglected disturbances or measurement noise). How can we deal with this infeasability of the solution, i.e., how can we modified the MPC formulation? [2pts]

c) Consider the following continuous-time first-order system:

$$\begin{aligned}\dot{x} &= a_c x + b_c u\\ y &= x,\end{aligned}$$

where a_c and b_c are scalars.

Construct the corresponding discrete-time system (i.e., $x_{k+1} = ax_k + bu_k$) by sampling the original system periodically every 1s. [1pt]

d) The discrete time system is controlled using MPC, while solving the constrained optimization problem:

$$\begin{aligned}\min_u \quad & Q_N x_{t_0+N}^2 + \sum_{t=t_0}^{t_0+N-1} Q_0 x_t^2 + \sum_{t=t_0}^{t_0+N-1} W u_t^2\\ \text{subject to } & |u_t| \leq 1, \quad \forall t = t_0, ..., t_0+N-1\\ & x_{t+1} = ax_t + bu_t, \quad \forall t.\end{aligned}$$

Considering the prediction horizon $N = 1$, compute the control action u_{t_0} as a function of the initial condition x_{t_0}.

With the values $a = 1/2$, $b = 1$, and $Q_0 = Q_N = 1$, and assuming $W \geq 0$, how does W affect the performance of the controlled system? [3pts]

e) Consider the same optimization problem as in the previous part, where the constraint $|u_t| \leq 1$ is substituted by $|x_t| \leq 1$, and the prediction and control horizon is $N = 2$. Translate the MPC formulation into a Quadratic Programming (QP) problem

$$\begin{aligned}\min_u \quad & u^\top H u + h^\top u\\ \text{subject to } & Lu \leq f.\end{aligned}$$

and determine H, h, L and f. *Note that $Lu \leq f$ is an element-wise inequality.* [3pts]