REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2014–05–28, kl 14.00–19.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures and of the blackboard notes, mathematical tables and pocket calculator. **Exercice notes on RGA and decoupling methods are allowed**, but no other exercice materials.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover.

The exam consists of five problems. The distribution of points for the various problems and subproblems is indicated.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

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Resultat: Will be posted no later than June 21, 2014.

Good Luck!

Problem 1

(a) Compute the 2-norm $\|\cdot\|_2$ of the signals

$$y_1(t) = 1_{t>0}\begin{bmatrix} e^{-t} \\ 3e^{-2t} \end{bmatrix}, \quad y_2(t) = 1_{t-1>0}\begin{bmatrix} \sin(\omega t)/t \\ \cos(\omega t)/t \end{bmatrix}, \quad y_3(t) = c.1_{t>0},$$

where $1_{t>0}$ is equal to 1 if $t > 0$ and to 0 otherwise, and where c can be any arbitrary real number. [3pts]

(b) Compute the $\mathcal{H}_2$ norm of the system

$$G(s) = \frac{1}{\epsilon s + 1}$$

where $\epsilon \in \mathbf{R}^+$. What happens as $\epsilon \to \infty$ and $\epsilon \to 0$, respectively. [3pts]

(c) Compute the $\mathcal{H}_\infty$ norm of the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{\alpha(s+1)} \\ \frac{-1}{\alpha(s+1)} & \frac{1}{s+1} \end{bmatrix}$$

for $\alpha \in \mathbf{R}$, $\alpha \neq 0$. [4pts]

Problem 2

Consider the following two systems which we would like to control using a decentralized controller, achieving a cross-over frequency at $\omega_c = 1$ rad/s.

$$G(s) = \frac{1}{s+1}\begin{bmatrix} \frac{s+1}{s+3} & \frac{s+4}{s+3} \\ \frac{s+4}{s+5} & \frac{s+3}{s+5} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} x$$

(a) Determine the poles and zeros for each system. [4pts]

(b) Use RGA analysis to determine a suitable pairing for controlling these systems using decentralized control. [2pts]

(c) Is the second system observable / controllable? [2pts]

(d) For the first system, propose a static decoupling matrix W at stationarity and a proportional decentralized controller F such that the system has a crossover frequency $\omega_c = 1$ rad/s. [2pts]

Problem 3

(a) A system is described by the rational transfer function:

$$G(s) = \frac{s^2 - 4s + 3}{s^3 - 3s - 2}.$$

We would like to design a closed-loop controller that rejects disturbances efficiently and exhibits low sensitivity to measurement noise at frequencies around 3 rad/s. Is it possible? Justify. [3pts]

(b) We have designed an LQG controller to minimize a cost function of the type:

$$J = E[\int_0^\infty (e(t)^2 + \rho u(t)^2) dt],$$

where e and u denote the control error (w.r.t. the reference signal) and control input, respectively. In this LQG problem, we have modelled the state disturbance by white noise v_1, and measurement disturbances as white noise v_2, so that the covariance of $\binom{v_1}{v_2}$ is:

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

(b-1) We try two LQG controllers, where we set r_2 equal to 1 and 10, respectively.

Which one of these controllers should lead to a lower control error? [2pts]
(b-2) We then decrease r_1 from 2 to 1. What is going to happen? [1pt]

(c) We have designed an $\mathcal{H}_\infty$ closed-loop controller so as to minimize:

$$\left\| \begin{array}{c} W_S S \\ W_u F_y S \end{array} \right\|_\infty$$

for some weight functions W_S and W_u (S denotes the sensitivity function, and the feedback control is $-F_y$).
(c-1) Which input-output relationships are S and $F_y S$ representing? Can you then infer what were the motivations behind the design of this $\mathcal{H}_\infty$ controller? [1pt]
(c-2) We haven chosen a weight function of the form $W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s}$, where M_S and ω_{BS} are two positive parameters. Should we increase or decrease M_S to further decrease the sensitivity to disturbances? [1pt]
(c-3) We discover that the obtained controller is very sensitive to measurement noise. Propose a framework to address this issue? How can we modify the $\mathcal{H}_\infty$ control framework? [2pts]

Problem 4

Consider the following linear system:

$$\begin{aligned} \dot{x} &= ax + bu + v_1 \\ y &= x + v_2 \\ z &= x \end{aligned}$$

where v_1, v_2 are gaussian white noises such that the covariance matrix of $\binom{v_1}{v_2}$ is:

$$\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$$

with $\alpha \in [0, 1)$. We wish to find the controller that solves the following optimization problem:

$$\min \ J = E[\int_0^\infty (\rho z(t)^2 + u(t)^2) dt], \tag{1}$$

where $\rho > 0$.

(a) For which values of a is the open-loop system stable? [1pt]

(b-1) Can we use Kalman filter to determine the optimal observer? (By optimal, we mean minimizing the estimation error). Check the assumptions needed to apply Kalman filter. [1pt]

(b-2) Compute the optimal observer. [2pts]

(b-3) What is the impact of the positive correlation α on the state estimation error? [1pt]

(c) Determine the optimal linear state feedback controller. [2pts]

(d) Provide a complete solution of optimization problem (1). Justify. [3pts]

(e) We now replace in the problem (1) z by the error $e = z - r$ where r is a reference signal. Assume that the step responses for two choices of ρ correspond to the two curves presented in Figure 1. Which curve corresponds to the largest ρ? Justify. [2pts]

(f) Now consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu + v_1 \\ y &= x + v_2 \\ z &= x \end{aligned}$$

where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

We have $v_1 = \binom{v_{11}}{v_{12}}$ and $v_2 = \binom{v_{21}}{v_{22}}$, and we assume that the covairance matrix of

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{12} \\ v_{22} \end{bmatrix} \quad \text{is} \quad \begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & \beta \\ 0 & 0 & \beta & 1 \end{pmatrix}$$

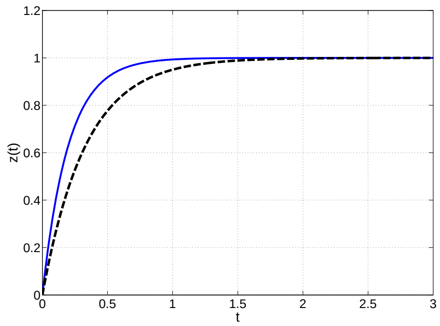

Figure 1: Step response for two values of parameter ρ.

with $\alpha, \beta \in [0, 1]$. Desing a controller solving:

$$\min\ J_2 = E[\int_0^\infty (\rho|z(s)|^2 + |u(s)|^2)ds], \tag{2}$$

where $\rho > 0$, and $|\cdot|$ denotes the euclidian norm. [2pts]

Problem 5

(a) Consider the continuous time system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Construct the discrete time system

$$\begin{aligned} x_{k+1} &= Fx_k + Gu_k \\ y_k &= Hx_k \end{aligned}$$

by sampling the original system with interval $T = 1$s. *Hint: remember that for any square matrix A,* $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$. [1pt]

(b) Prove that in general, a sampled discrete system is stable if and only if the original continuous time system is stable, for any sampling interval $T > 0$. To this aim, you can restrict your attention to scenarios where A is a diagonal matrix. [2pts]

(c) We consider the following discrete time system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.5 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \\ y_k &= x_k. \end{aligned}$$

Two different MPC problems are considered, which handle the constraint on the control signal u differently. The first controller solves the following optimization problem:

$$\min_u J_1(u, y)$$

where

$$J_1(u, y) = y_{k+1}^T y_{k+1} + \frac{1}{2} y_{k+2}^T y_{k+2} + 2u_k^T u_k + 2u_{k+1}^T u_{k+1}.$$

The second controller solves:

$$\min_u \; J_2(u, y)$$
$$\text{subject to } |u| \leq 1$$

where

$$J_2(u, y) = y_{k+1}^T y_{k+1} + \frac{1}{2} y_{k+2}^T y_{k+2}.$$

Determine these two controllers. [5pts]
Hint: Only one step control is applied.

(d) Will these controllers guarantee the stability of the discrete time system? [2pts]