Exercise session 3

Topics: • Limitations and conflicts
$\Rightarrow$ RHP zeros, RHP poles, time delays
• sensitivity shaping

$\rightarrow$ 8.2, 8.3, 8.5

An intuitive explanation for limitations induced by RHP zeros:

Assume first $\quad G(s) = \frac{1}{(s+1)(s+3)}$

$\Rightarrow$ no RHP zeros, RHP poles or delays!

For

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

and when $u = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

We get from $\quad Y(s) = G(s)\,U(s)$

$$\Leftrightarrow \quad Y(s) = G(s)\,\frac{1}{s}$$

$$y(t) = \mathcal{L}^{-1}\left(G(s)\,\frac{1}{s}\right)$$

We know that

$$\frac{1}{(s+1)(s+3)s} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s}$$

$$\Leftrightarrow \quad \frac{1}{(s+1)(s+3)s} = \frac{A(s+3)s + B(s+1)s + C(s+1)(s+3)}{(s+1)(s+3)s}$$

$$\Leftrightarrow \quad \frac{1}{(s+1)(s+3)s} = \frac{As^2 + A3s + Bs^2 + Bs + Cs^2 + C4s + 3C}{(s+1)(s+3)s}$$

$\Leftrightarrow$ i) $A + B + C = 0$
ii) $A3 + B + C4 = 0$
iii) $3C = 1 \quad \Rightarrow \quad C = \frac{1}{3}$

From i) $A = -B - \frac{1}{3}$

Into ii)

$$-3B - 1 + B + \frac{4}{3} = 0$$

$$\Leftrightarrow \quad \frac{1}{3} = 2B$$

$$B = \frac{1}{6}$$

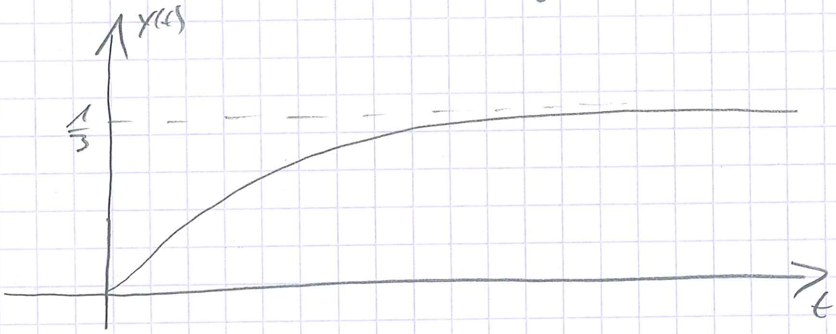
$$\Rightarrow \quad A = -\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$$

$$\Rightarrow A = -\frac{1}{2} \quad B = \frac{1}{6} \quad C = \frac{1}{3}$$

Hence

$$y(t) = \mathcal{L}^{-1}\left(-\frac{1}{2(s+1)} + \frac{1}{6(s+3)} + \frac{1}{3s}\right)$$

$$= -\frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} + \frac{1}{3}$$



Now: Insert a RHP zero!

$$G'(s) = \frac{1-s}{(s+1)(s+3)} = (1-s)\,G(s)$$

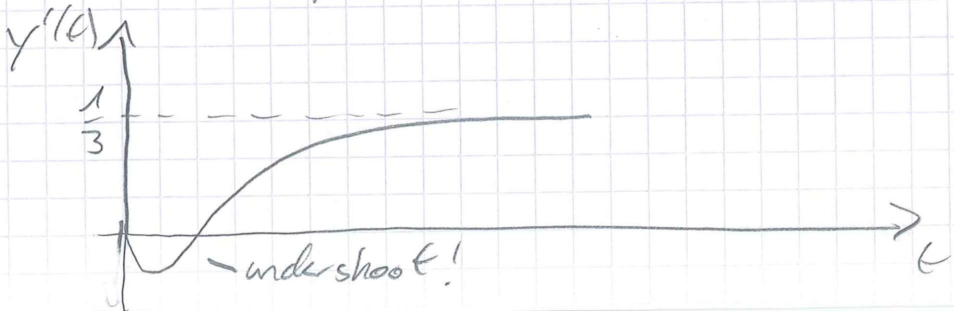
Step response is now

$$y'(t) = \mathcal{L}^{-1}\left(G'(s)\frac{1}{s}\right)$$

$$= \mathcal{L}^{-1}\left(G(s)\frac{1}{s} - s\,G(s)\frac{1}{s}\right) \qquad // \text{ Laplace rules}$$

$$= y(t) - \dot{y}(t)$$

Hence



The RHP zero limits the speed of the step response!

8.2) Assume a system

$$G(s) = \frac{s-3}{s+1}$$

which has a zero at $s=3$ and a pole at $s=-1$

OBS: RHP zero.

General rule:

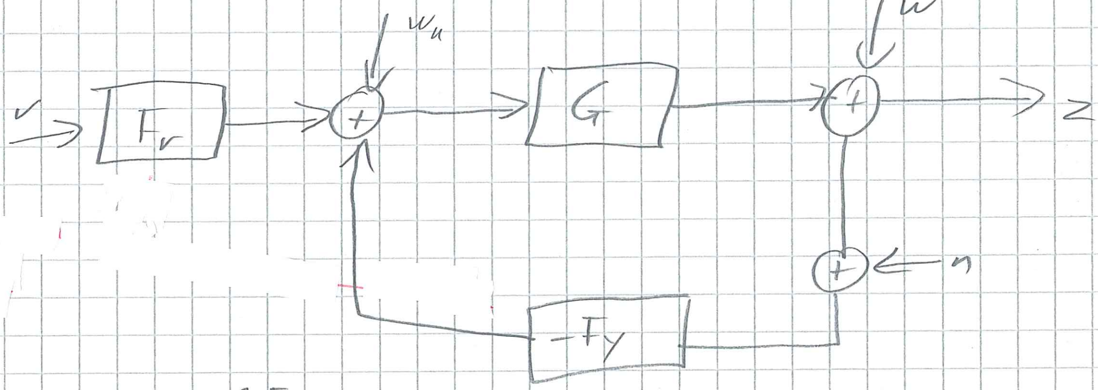
RHP zero in G carry over from open-loop into the closed-loop (into T as we noted last time)

→ otherwise internally not stable

Conclusion?!

a) We will now use: Direct Synthesis

Design $F = F_r = F_y$ such that $T = \frac{5}{s+5}$ (*)



Recall $T = \frac{GF}{1+GF}$ (**)

Hence set (*) = (**) and solve for T

$$\frac{GF}{1+GF} = T$$

$$\Leftrightarrow \quad GF = T + GFT$$

$$\Leftrightarrow \quad F = \left[(1-T)G\right]^{-1} T$$

$$\Leftrightarrow \quad F = \left[\left(1 - \frac{5}{s+5}\right)\frac{s-3}{s+1}\right]^{-1} \frac{5}{s+5}$$

$$= \left[\frac{s}{s+5} \; \frac{s-3}{s+1}\right]^{-1} \frac{5}{s+5}$$

$$= 5 \; \frac{s+1}{s(s-3)}$$

Obs: RHP pole-zero cancellation between G and F!

Now we investigate internal stability:

Recall that a SISO system is internally stable iff S, SG, SF_r and F_r are stable.

Note that $F_r = F = 5 \, \frac{s+1}{s(s-3)}$ is not stable due to a pole at $s=3$.

Note further:

$$S F_r = (1-T) F_r = \left(1 - \frac{5}{s+5}\right) 5 \frac{s+1}{s(s-3)}$$

$$= \frac{\cancel{s}}{s+5} \, 5 \, \frac{s+1}{\cancel{s}(s-3)}$$

$$= 5 \, \frac{s+1}{(s+5)(s-3)}$$

which is not stable either.

$\Rightarrow$ The controller will not work!

b) Suggest an alternative T with the same bandwidth at $5\ \mathrm{rad}/s$, but with a stable (meaning internally stable) closed-loop!

Recall: bandwidth is the frequency ω_B where $|T(i\omega_B)| = \frac{1}{\sqrt{2}}$ (-3 dB)

We need to retain the zero at $s = 3$. We choose

$$T^*(s) = 5\,\frac{(s-3)}{(s+3)(s+5)} \rightarrow \text{still bandwidth} \approx 5\,\frac{\mathrm{rad}}{s}$$

(Go back to the basic course if you forgot about how to draw Bode diagrams. Note, that the gain of $|T^*| = |T|$)

Now again $T^* = \dfrac{GF}{1+GF}$

$$\Leftrightarrow F = \left[(1 - T^*)G\right]^{-1} T^*$$

$$= \left[\left(1 - 5\,\frac{s-3}{(s+3)(s+5)}\right)\frac{(s-3)}{(s+1)}\right]^{-1} 5\,\frac{s-3}{(s+3)(s+5)}$$

$$= \left[\frac{\left[(s+3)(s+5) - 5(s-3)\right](s-3)}{(s+3)(s+5)(s+1)}\right]^{-1} 5\,\frac{s-3}{(s+3)(s+5)}$$

$$= 5\,\frac{s+1}{(s+3)(s+5) - 5(s-3)}$$

$$= 5\,\frac{s+1}{s^2+3s+30}$$

Now: Check again internal stability:

$F_r = F$ stable (pole at $s = -1.5 \pm 5.26\,i$)

$S = \dfrac{s^2+3s+30}{(s+3)(s+5)}$ stable

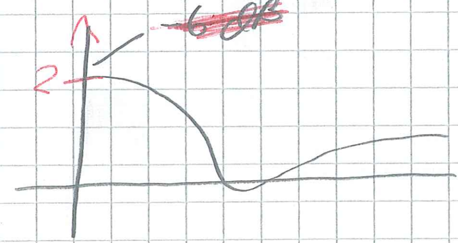
$SG = 5\,\dfrac{(s^2+3s+30)(s-3)}{(s+3)(s+5)(s+1)}$ stable

$SF_y = SF = 5\,\dfrac{s+1}{\cancel{s^2+3s+30}} \cdot \dfrac{\cancel{s^2+3s+30}}{(s+3)(s+5)} = 5\,\dfrac{s+1}{(s+3)(s+5)}$ stable

c) We have now achieved a shaping of T, but without considering S!

Matlab

Draw Bode plot! We see that $S = \frac{s^2+3s+30}{(s+3)(s+5)}$



Note: RHP zero gives the limitation:

$\omega_{BS} \lessapprox \frac{3}{2}\ \text{rad/s}$ for $M_s = 2$

while $\omega_{BS} \approx \omega_{BT} = 5\ \text{rad/s}$

$\Rightarrow$ Low-frequency noise will be amplified

$\Rightarrow$ we should consider mixed synthesis, i.e. shape T and S

d) Can we solve this issue by chosing

$F_r \neq F_y$?

with no pole at $s=3$ and s. t. the closed loop becomes $G_c = \frac{GF_r}{1+GF_y} = \frac{5}{s+5}$

Note $G_c(3) = \frac{5}{3+5} = \frac{5}{8}$

However G has a zero at $s=3$, i.e., $G(3)=0$, which cannot be cancelled by F_r (stable controller).

Basic Limitations induced by RHP poles / zeros and time delays

In the lecture, we defined

describes performance objectives

nominal performance: $|W_S(i\omega)\, S(i\omega)| \leq 1 \quad \forall \omega$

robust stability: $|W_T(i\omega)\, T(i\omega)| \leq 1 \quad \forall \omega$

describes model error

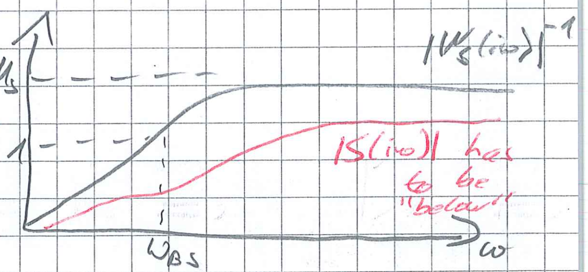
robust performance: $|W_S(i\omega) S(i\omega)| + |W_T(i\omega) \cdot T(i\omega)| \leq 1 \; \forall \omega$

$\Rightarrow$ Can we assume arbitrary $W_S(s)$ and $W_T(T)$?
The answer is: No, due to limitations such as RHP poles/zeros and time delays and since $S + T = 1$ (for SISO).

Assume the weights

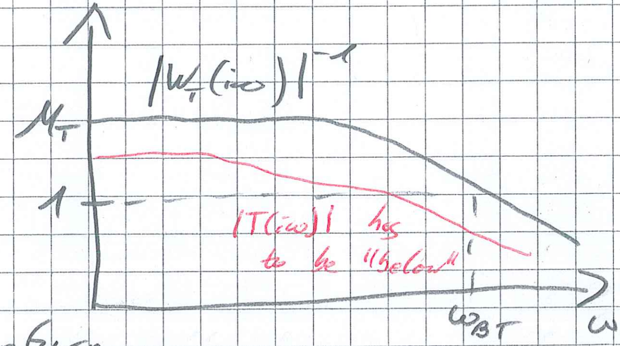
$$W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s}$$

upper bound for S $\Rightarrow$



and $W_T = \frac{1}{M_T} + \frac{s}{\omega_{BT}}$

$\Rightarrow$ upper bound for T



Limitations derived in the lecture and based on these weights (using maximum modulus principle)

- RHP zero of $G(s)$ at $s = z$ means $\omega_{BS} \leq (1 - M_S^{-1})\, z < z$ (if $M_S \to \infty$)
- Time delay as $G(s) = G_0 e^{-\tau s}$ means $\omega_{BS} \leq (1 - M_S^{-1}) \frac{2}{\tau} < \frac{2}{\tau}$
- RHP pole of $G(s)$ at $s = p$ means $\omega_{BT} \geq \frac{p}{1 - M_T^{-1}} > p$ (if $M_T \to \infty$)

For the particular values of

$M_S = 2$ and a RHP zero at $s = z$

we get $\omega_{BS} \leq \frac{1}{2} z$

and $M_T = 2$ and a RHP pole at $s = p$

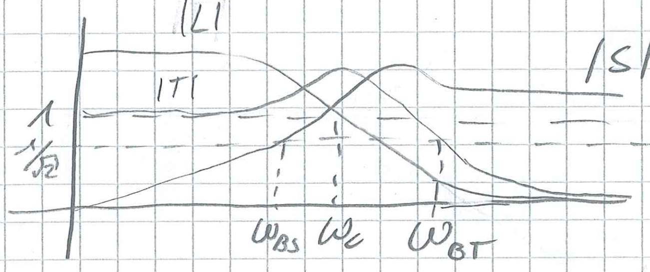
we get $\omega_{BT} \geq 2p$

8.3) Assume $G(s)$ has a zero at $s=3$ and a time delay of 1 second, i.e. $G(s) = (s-3)e^{-s}G_0(s)$.

stable, minimum phase

What is the highest realistic crossover frequency if the open loops system amplitude curve is monotonically decreasing?

Rule of thumb $\underbrace{\omega_c}_{\text{open loop}} \approx \underbrace{\omega_{BS}}_{\text{bandwidth of } S} \approx \underbrace{\omega_{BT}}_{\text{bandwidth of } T}$



From the limitations we know (see previous page)

RHP zero: $\omega_{BS} \approx \omega_c < 3$ (for a peak $M_s \to \infty$)

time delay: $\omega_{BS} \approx \omega_c < 2$

Let us analyze this in detail: Recall $L = GF$

Bode's relation establishes a connection between the amplitude and the phase curve:

If L is proper "decreasing", rational, no RHP poles, and no pole/zero on the imaginary axis, then we have an approximation

$$\arg L(i\omega) \leq \frac{\pi}{2} \frac{d}{d\log\omega} \log |L(i\omega)|$$

Now recall:

The phase margin $\phi_m = \pi + \arg(L(i\omega_c))$ is a measure of stability, i.e. $\phi_m > 0$ indicates a stable closed-loop system according to the Nyquist criterion (simplified version).

Recap these things if you're not familiar with it!

Back to the exercise:

Assume $G(s) = (s-3)\, e^{-s}\, \underbrace{G_0(s)}_{\text{minimum phase}}$

$L(s) = G(s) \cdot F(s) = (s-3)\, e^{-s}\, G_0(s)\, F(s)$

Split in minimumphase part and non-minimum phase part

$$L(s) = \underbrace{\frac{(s-3)}{(s+3)} e^{-s}}_{\text{non-minimum phase}} \underbrace{(s+3)\, G_0(s)\, F(s)}_{\text{minimum phase}} = L_{nmp}(s) \cdot L_{mp}(s)$$

Note: $\left| \frac{i\omega - 3}{i\omega + 3} e^{-i\omega} \right| = 1$ and hence $|L_{mp}(i\omega)| = |L(i\omega)| = |(i\omega+3)\, G_0(i\omega) F(i\omega)|$ which is decreasing according to assumption

Use Bode's relation

$$\arg L_{mp}(i\omega_c) \leq \frac{\pi}{2} \underbrace{\frac{d}{\partial \log \omega} \log |L_{mp}(i\omega_c)|}_{\text{decreasing}} = \frac{\pi}{2} \frac{\partial}{\partial \log \omega} \log |L(i\omega_c)| \underbrace{\leq 0}_{\text{assumption}}$$

Use phase margin ϕ_m

$$\phi_m = \pi + \arg(L(i\omega_c)) = \pi + \arg\left(\frac{i\omega_c - 3}{i\omega_c + 3} e^{-i\omega_c}\right) + \arg(L_{mp}(i\omega_c))$$

$$\underbrace{\leq}_{\text{Use Bode's relation}} \pi + \arg\left(\frac{i\omega_c - 3}{i\omega_c + 3} e^{-i\omega_c}\right) \quad (*)$$

Consider (*) at the stability threshold

$$\pi + \operatorname{atan}\left(\frac{\omega_c}{-3}\right) - \operatorname{atan}\left(\frac{\omega_c}{3}\right) - \omega_c = 0$$

Solve with Matlab and get

$\omega_c = 1.9765$

which agrees with our initial investigation!

Hence $\omega_c \leq 1.9765 \ \mathrm{rad/s}$ is achievable

$\Rightarrow$ no lead lag with $\omega_c \geq 2 \ \mathrm{rad/s}$ possible!

8.5) RHP zero and disturbance attenuation

Given $\bar{y} = \frac{s-1}{(s+1)^2}\bar{u} + 0.5\frac{s-10}{(s+1)^2}\bar{d} = \bar{G}(s)\bar{u} + \bar{G}_d(s)\bar{d}$

Aim: $|\bar{y}| < 0.5$ f.a. $|\bar{d}| < 1$

$\Rightarrow$ no limitations on the input signal $\bar{u}$
(we will consider this later)

First: scale the system

Set $y = 2 \cdot \bar{y}$
and $d = \bar{d}$ $\quad u = \bar{u}$

so that we aim for $|y| < 1$ f.a. $|d| < 1$

and we obtain hence

$$y = 2\bar{G}(s)u + 2\bar{G}_d(s)d$$
$$= G(s)u + G_d(s)$$
$$= 2\frac{s-1}{(s+1)^2}u + \frac{s-10}{(s+1)^2}d$$

Note first of all that we have a limitation induced by the RHP zero at $s=1$. The controller is also not allowed to cancel this zero due to internal stability reasons. (see last exercise)

We also note that

$G(z) = 0 \Rightarrow L(z) = 0 \Rightarrow S(z) = 1$ so

① Note that

$y = S G_d d$ and we want $\|S G_d d\|_\infty < 1$ to achieve the specifications.

From maximum modulus principle, it follows

$$\|S G_d\|_\infty \geq |S(2) G_d(z)| = |G_d(z)| = \frac{9}{4}$$

Therefore, it follows that $\|S G_d\|_\infty \geq \frac{9}{4} > 1$

so that the specification can not be achieved.

② An alternative view: Consider a weight

$$W_s(s) = \frac{1}{M_s} + \frac{\omega_{BS}}{s}$$

and require

$$\|W_s S\|_\infty \leq 1$$

Can we design $W_s(s)$ "appropriately"?

We have from the maximum modulus principle.

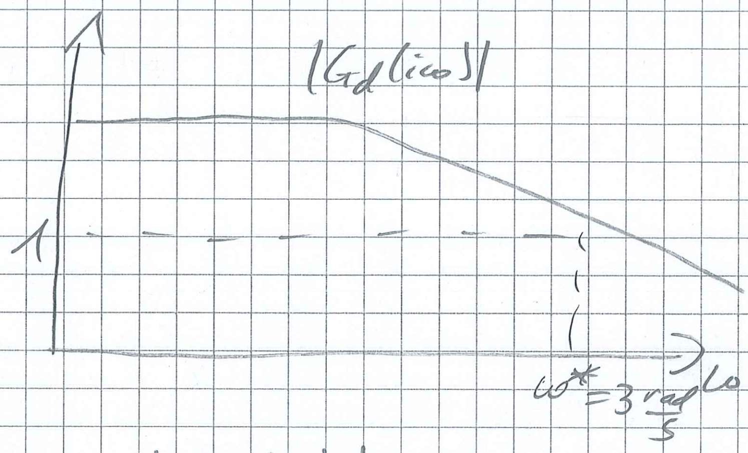
$$1 \geq \|W_s S\|_\infty \geq |W_s(z) S(z)| = |W_s(z)|$$

Which means $\omega_{BS} \leq (1 - M_s^{-1}) z < z$

If we assume $M_s = 2$, then this means

$$\omega_{BS} \leq \frac{1}{2}$$

We however note that



with $|G_d(i3)| = 1$

so that we will require $\omega_{BS} > 3$.

$\Rightarrow$ Control specifications are not feasible.