EL2420 - Control Theory and Practice - Advanced Course
Solution – 2011-05-23

1a. The minors of the system matrix G are

$$\frac{1}{s+1}\,,\,\frac{-s+1}{s+1}\,,\,\frac{2}{s+1}\,,\,\frac{1}{s+1}\,,\,\frac{2s-1}{(s+1)^2}\,.$$

The pole polynomial $(s+1)^2 = 0$ determines a pole in 1 with double multiplicity. The zero polynomial $2s-1=0$ determines a non-minimum phase zero in 0.5. The number of states in the state-space realization of the systems is such that the number of eigenvalues of the state matrix A is equal to the number of poles (with multiplicity) of the system. Therefore, the minimum size of A is $2x2$ and the state vector has at least 2 components.

According to rules of thumb, we want to avoid negative elements in $RGA(G(0))$, and we desire elements in $RGA(G(i\omega_B))$ to be close to the unity. We determine

$$RGA(G(0)) = G(0). * (G(0)^{-1})^T = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Since the diagonal elements are negative, the pairing $(u_1 \to y_2, u_2 \to y_1)$ is preferable. We find

$$RGA(G(i1)) = G(i1). * (G(i1)^{-1})^T = \begin{bmatrix} -0.2-0.4i & 1.2+0.4i \\ 1.2+0.4i & -0.2-0.4i \end{bmatrix}.$$

Elements in the off diagonal are not far from the unity, therefore the decentralized control can give acceptable performance of the closed loop. It is also acceptable to state that interactions will be weak with both pairings.

1b. By defining $x = y$, we have the following state space representation of the system

$$\begin{aligned} \dot{x} &= x + u \\ y &= x \end{aligned}$$

We can determine a LQ controller by solving the Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0\,,$$

where $A=1$, $B=1$, $M=1$, $Q_1=1$, $Q_2=\beta$. We get

$$2S + 1 - \frac{S^2}{\beta} = 0 \Rightarrow S = \beta + \sqrt{\beta^2+\beta}$$

The controller is a state feedback $u = -Lx$ given by

$$L = Q_2^{-1} B^T S = 1 + \sqrt{1 + \frac{1}{\beta}} \,.$$

Since $x = y$, we have $F_y = L$. The closed loop system has pole polynomial

$$\det(s - A + BL) = 0 \Rightarrow s + \sqrt{1 + \frac{1}{\beta}} = 0 \,,$$

which gives a pole in $s* = -\sqrt{1 + \frac{1}{\beta}}$.

If $\beta \to 0$, we have $s* \to -\infty$. A faster pole gives faster response to the system. In fact, the control effort is less penalized in the LQ problem by letting $Q_2 = \beta \ll 1$.
If $\beta \to \infty$, we have $s* \to -1$. We notice that the open loop system has a RHP pole in 1, that is mirrored in -1 in the closed loop.

2a. The H_∞-optimal controller is the one that minimizes $||W_pS||_\infty = \sup_\omega \bar{\sigma}(W_pS)$. Since for controller A we have $\sup_\omega \bar{\sigma}(W_pS) \simeq 3$, while for controller B $\sup_\omega \bar{\sigma}(W_pS) \simeq 1$, we can conclude that B is the H_∞-optimal controller.

2b. We derive the controller F_y from the expression of closed loop transfer function

$$
\begin{aligned}
&G_c = (I + GF_y)^{-1}GF_y \Leftrightarrow (I + GF_y)G_c = GF_y \\
&GF_y(I - G_c) = G_c \Leftrightarrow GF_y = G_c(I - G_c)^{-1} \\
&F_y = G^{-1}G_c(I - G_c)^{-1}
\end{aligned}
$$

By considering $G = \frac{1}{10s+1}\begin{pmatrix} 1 & s \\ 2s+1 & 1 \end{pmatrix}$ and $G_c = \frac{1}{\lambda s+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we get

$$
F_y = G^{-1}\frac{1}{\lambda s+1}\frac{\lambda s+1}{\lambda s} = \frac{(10s+1)}{\lambda s(2s^2+s-1)}\begin{pmatrix} -1 & s \\ 2s+1 & -1 \end{pmatrix}
$$

The pole polynomial of F_y is $(\lambda s)^2(2s^2+s-1)^2 = 0$, which has a root in 0, a LHP root in -1 with multiplicity 2 and a RHP root in 0.5 with multiplicity 2. F_y has RHP poles independent of λ, canceled by a RHP zero in G, which implies that we do not get internal stability. Therefore, this controller is not an appropriate choice.

We can modify the desired G_c, such that the RHP zero of G in 0.5 is not canceled, i.e.,

$$
G'_c = \frac{(2s-1)}{(\lambda s+1)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

and in order to keep the same singular values as before, we add a LHP pole with same amplitude of the zero

$$
G''_c = \frac{(2s-1)}{(\lambda s+1)(2s+1)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

We can derive the new controller F_y

$$
\begin{aligned}
F_y &= G^{-1}\frac{(2s-1)}{(\lambda s+1)(2s+1)}\frac{(\lambda s+1)(2s+1)}{2\lambda s^2+\lambda s+2} \\
&= \frac{(2s-1)(10s+1)}{(s+1)(2\lambda s^2+\lambda s+2)}\begin{pmatrix} -1 & s \\ 2s+1 & -1 \end{pmatrix}
\end{aligned}
$$

F_y is now stable $\forall\, \lambda > 0$ and the system is internally stable.

3a. To verify if we can obtain acceptable control, we check

- Input constraint: y is in the interval $[-1\,,\,1]$ for the the worst-case frequency for the disturbance $d = \sin(\omega t)$, with $|u| < 1$. The smallest u keeping $y \in [-1, 1]$ is the u^* for which the sinusoid Gu^* is fully out of phase with the sinusoid $G_d d$, giving

$$1 = |y| = |G_d d| - |Gu|$$

Hence,

$$|u^*| = |G^{-1}|(|G_d| - 1) < 1 \Leftrightarrow |G| > |G_d| - 1$$

Both in $i)$ and $ii)$, the maximum effect is in $s = 0$, where $|G| - |G_d| = -0.5$. Therefore input constraints are fulfilled.

- Bandwidth limitations: $|S| < 1$ where $|G_d| > 1$, i.e., where

$$\frac{3}{|10j\omega_{BD} + 1|} > 1 \Leftrightarrow \omega_{BD} < 0.28\,\text{rad/s}$$

System $ii)$ has a RHP zero $z = 0.2$ which gives a limitation to the achievable bandwidth of $|S|$, i.e., $\omega_{BS} < 0.1$. Therefore, acceptable control cannot be achieved with the system $ii)$.

The maximum measurement delay T that can be tolerated by $i)$ can be derived from the relation $\omega_{BS} < T^{-1}$, and the bandwidth limitation $\omega_{BS} > \omega_{BD}$.

$$T < (0.28)^{-1} \simeq 3.5s$$

3b. $i)$ The transfer function between disturbance d and output y is the sensitivity $S(i\omega)$. The transfer function between measurement noise n and output y is the complementary sensitivity $T(i\omega)$. The requirements can be formulated as

$$\begin{aligned} &|S(i\omega)| < 0.1 && \omega < 0.5\,\text{rad/s} \\ &|S(0)| < 0.01 && \\ &|T(i\omega)| < 0.1 && \omega > 1\,\text{rad/s} \end{aligned}$$

$ii)$ In terms of loop gain $L = GF_y$, the sensitivity and complementary sensitivity can be approximated as

$$\begin{aligned} |S(i\omega)| \approx |L^{-1}(i\omega)| &\quad \Rightarrow |L(i\omega)| > 10 &\quad \omega < 0.5\,\text{rad/s} \\ |T(i\omega)| \approx |L(i\omega)| &\quad \Rightarrow |L(i\omega)| < 0.1 &\quad \omega > 1\,\text{rad/s} \end{aligned}$$

iii) A sketch of the loop gain L is not included in this solution. However L has a lower bound of 10 for $\omega < 0.5$ and upper bound of 0.1 for $\omega > 1$, therefore L is required to have a negative slope of at least -2 for frequencies $0.5 < \omega < 1$. This limits the achievable phase margin of the system.

iv) The requirement can be translated in $|T(i\omega)| < |W_s^{-1}(i\omega)|\,\forall\omega$. We consider a weight

$$W_T(s) = K_T \frac{s}{1+\tau s}$$

and determine K_T and τ s.t. $|W_T^{-1}(i\omega)| < 0.1$ for $\omega > 1$ rad/s. One possibility is to choose $\tau = 1$ (i.e., breakpoint in $\omega = 1$), and K_T s.t.

$$|W_T^{-1}(i1)| = \frac{\sqrt{1+1}}{K_T \cdot 1} = 0.1 \Rightarrow K_T = 14$$

4a. With the prediction horizon $N_p = 1$, the objective function becomes

$$V = Q_y y_{k+1}^2 + Q_y y_k^2 + u_k^2$$

With the state-space model inserted

$$\begin{aligned} V &= Q_y(-2y_k + 2u_k)^2 + Q_y y_k^2 + u_k^2 = \\ &= 4Q_y y_k^2 + Q_y y_k^2 + (4Q_y + 1)u_k^2 - 8Q_y y_k u_k \end{aligned}$$

As the value of y_k is assumed given, the first two terms of the objective function will not affect the optimal input sequence and can be removed. Thus, $H = (4Q_y + 1)$, and $h = -8Q_y y_k$.
The constraint gives

$$Lu_k \leq b$$

where $L = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4b. The gain of the saturation is given by

$$||\text{sat}(\cdot)||_\infty = \sup_{\text{u}\neq 0} \frac{||\text{sat(u)}||_2}{||\text{u}||_2} = 1\,.$$

A sufficient condition for stability of the closed loop can be derived with the small gain theorem $||\text{sat}(\cdot)||_\infty ||\text{M}||_\infty \leq 1$, where M is the transfer function from u_p to u

$$\begin{aligned} u &= W_s(u_p - u) - F_y y = W_s(u_p - u) - F_y G u_p \\ u &= \frac{W_s - F_y G}{1 + W_s} u_p = M u_p \end{aligned}$$

After substitutions, we get

$$M(s) = \frac{(1 - 10K)s + 1}{(s + 1)(10s + 1)}$$

We need then to guarantee $||M||_\infty = \sup_\omega |M(i\omega)| \leq 1$. Since $|M(0)| = 1$, the condition is verified if the frequency breakpoint of the zero is higher than the frequency breakpoint of the first pole, i.e.,

$$\frac{1}{|1 - 10K|} \geq \frac{1}{10}$$

which gives $-0.9 \leq K \leq 1.1$.

5a. Introduce $G_p(s) = (I + \Delta_G(s))G(s)$ which gives

$$\Delta_G(s) = G_p(s)G^{-1}(s) - I = \begin{pmatrix} 0 & \frac{\epsilon}{3} \\ \epsilon & 0 \end{pmatrix}$$

5b. Two poles in -1 for a decoupled system implies a pole in -1 for each of the two channels. That is, we seek $T_{ii} = 1/(s+1)$ which yields $K_1 = 100, T_1 = 100$ and $K_2 = 33.33, T_2 = 100$.

5c. The robustness criterion for relative output uncertainty as employed above is $\|T\Delta_G\|_\infty \leq 1$, which corresponds to the maximum singular value $\bar{\sigma}(T\Delta_G) \leq 1 \ \forall\omega$. Here

$$T\Delta_G = \frac{1}{s+1}\begin{pmatrix} 0 & \frac{\epsilon}{3} \\ \epsilon & 0 \end{pmatrix}$$

and the singular values are trivially given by the magnitude of the elements in this case, i.e,

$$\bar{\sigma}(T\Delta_G) = \frac{|\epsilon|}{\sqrt{\omega^2 + 1}}$$

which peaks at $|\epsilon|$. Hence the robustness criterion guarantees that the true closed-loop system with G_p and the controller from (b) is stable if $|\epsilon| < 1$.

5d We assume that ϵ is real and constant. The loop-gain becomes

$$GF = \frac{1}{s}\begin{pmatrix} 1 & \frac{\epsilon}{3} \\ \epsilon & 1 \end{pmatrix}$$

and the closed-loop

$$S = (I + GF)^{-1} = \frac{s}{s^2 + 2s + 1 - \frac{\epsilon^2}{3}}\begin{pmatrix} s+1 & -\frac{\epsilon}{3} \\ -\epsilon & s+1 \end{pmatrix}$$

and thus the closed-loop is stable for $|\epsilon| < \sqrt{3}$. We can therefore conclude that the robustness criterion was relatively conservative in this case.