EL2520− Control Theory and Practice
Laboratory experiment: The four-tank process
Process E

Jatesada Borsub
jatesada@kth.se
900112-9759

Alexandros Filotheou
alefil@kth.se
871108-5590

Daniel Finimundi
dfin@kth.se
910415-T312

Fangjin Kuang
fangjin@kth.se
940710-6450

Roberto Sanchez-Rey
rosr@kth.se
840616-9139

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Abstract

This report is a summary of the work involved in completing the project assignment of Control Theory and Practice, Advanced Course. The process to be controlled is the multivariable four-tank system, and, in particular, two of its configurations: the minimum and the non-minimum phase cases. At first we model the system in theory; then we perform experiments in order to determine the values of parameters unknown to us. Finally, we design controllers suitable for each system and evaluate their performance.

Modeling

Exercise 2.1.1

From the physical setup of the process, the following hold:

$$\begin{aligned} A_1\frac{dh_1}{dt} &= q_{L,1} + q_{out,3} - q_{out_1} \\ A_2\frac{dh_2}{dt} &= q_{L,2} + q_{out,4} - q_{out_2} \\ A_3\frac{dh_3}{dt} &= q_{U,1} - q_{out,3} \\ A_4\frac{dh_4}{dt} &= q_{U,2} - q_{out,4} \end{aligned}$$

where $q_{\{L,U\},j}$ is the flow directed from pump j to the lower or upper tank connected to valve j, and $q_{out,i}$ is the outflow of tank i. Since

$$\begin{aligned} q_{L,j} &= \gamma_j k_j u_j \\ q_{U,j} &= (1-\gamma_j)k_j u_j \\ q_{out,i} &= a_i\sqrt{2gh_i} \end{aligned}$$

the system of differential equations becomes

$$\begin{aligned} \frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1 \\ \frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2 \\ \frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2 \\ \frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1 \end{aligned}$$

Exercise 2.1.2

By definition, in an equilibrium all derivatives $\frac{dh_i}{dt}$ are zero:

$$\begin{aligned} -\frac{a_1}{A_1}\sqrt{2gh_1^0} + \frac{a_3}{A_1}\sqrt{2gh_3^0} + \frac{\gamma_1 k_1}{A_1}u_1^0 &= 0 \\ -\frac{a_2}{A_2}\sqrt{2gh_2^0} + \frac{a_4}{A_2}\sqrt{2gh_4^0} + \frac{\gamma_2 k_2}{A_2}u_2^0 &= 0 \\ -\frac{a_3}{A_3}\sqrt{2gh_3^0} + \frac{(1-\gamma_2)k_2}{A_3}u_2^0 &= 0 \\ -\frac{a_4}{A_4}\sqrt{2gh_4^0} + \frac{(1-\gamma_1)k_1}{A_4}u_1^0 &= 0 \\ y_i^0 = k_c h_i^0, i &= \{1,2,3,4\} \end{aligned}$$

where h_i^0, u_i^0 and y_i^0 denote steady-state values of their corresponding variables.

Exercise 2.1.3

Denoting the deviations from equilibrium with $\Delta u_i = u_i - u_i^0$, $\Delta h_i = h_i - h_i^0$, $\Delta y_i = y_i - y_i^0$, and introducing the vectors

$$u = \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix}, x = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{bmatrix}, y = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix}$$

we can linearize around the steady-state operating point by performing a Taylor series expansion. Neglecting the high-order terms, the system is now expressed by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where

$$A = \begin{bmatrix} \left.\frac{\partial \Delta h_1}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_1}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_1}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_1}{\partial h_4}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_2}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_2}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_2}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_2}{\partial h_4}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_3}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_3}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_3}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_3}{\partial h_4}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_4}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_4}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_4}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_4}{\partial h_4}\right|_{h_i^0,u_i^0} \end{bmatrix}$$

$$B = \begin{bmatrix} \left.\frac{\partial \Delta h_1}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_1}{\partial u_2}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_2}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_2}{\partial u_2}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_3}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_3}{\partial u_2}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta h_4}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta h_4}{\partial u_2}\right|_{h_i^0,u_i^0} \end{bmatrix}$$

$$C = \begin{bmatrix} \left.\frac{\partial \Delta y_1}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_1}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_1}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_1}{\partial h_4}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta y_2}{\partial h_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_2}{\partial h_2}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_2}{\partial h_3}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_2}{\partial h_4}\right|_{h_i^0,u_i^0} \end{bmatrix}$$

$$D = \begin{bmatrix} \left.\frac{\partial \Delta y_1}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_1}{\partial u_2}\right|_{h_i^0,u_i^0} \\ \left.\frac{\partial \Delta y_2}{\partial u_1}\right|_{h_i^0,u_i^0} & \left.\frac{\partial \Delta y_2}{\partial u_2}\right|_{h_i^0,u_i^0} \end{bmatrix}$$

After performing the necessary operations, we get

$$A = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix}, B = \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix}, C = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix}$$

and D $= 0$, with $T_i = \frac{A_i}{a_i}\sqrt{\frac{2h_i^0}{g}}$.

Exercise 2.1.4

The transfer matrix can be obtained by

$$G(s) = C(sI - A)^{-1}B =$$

$$\begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{T_1}{1+sT_1} & 0 & \frac{\frac{A_3T_1}{A_1}}{(1+sT_1)(1+sT_3)} & 0 \\ 0 & \frac{T_2}{1+sT_2} & 0 & \frac{\frac{A_4T_2}{A_2}}{(1+sT_2)(1+sT_4)} \\ 0 & 0 & \frac{T_3}{1+sT_3} & 0 \\ 0 & 0 & 0 & \frac{T_4}{1+sT_4} \end{bmatrix} \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\gamma_1 k_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)k_2c_1}{(1+sT_3)(1+sT_1)} \\ \frac{(1-\gamma_1)k_1c_2}{(1+sT_4)(1+sT_2)} & \frac{\gamma_2 k_2 c_2}{1+sT_2} \end{bmatrix}$$

Exercise 2.1.5

The zeros of $G(s)$ are given by the solution to the quadratic equation

$$T_3T_4s^2 + (T_3+T_4)s + \frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2} = 0$$

which are s_1 and s_2:

$$s_1 = -\frac{T_3+T_4}{2T_3T_4} + \frac{1}{2T_3T_4}\sqrt{(T_3+T_4)^2 - 4\frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}}$$
$$s_2 = -\frac{T_3+T_4}{2T_3T_4} - \frac{1}{2T_3T_4}\sqrt{(T_3+T_4)^2 - 4\frac{\gamma_1+\gamma_2-1}{\gamma_1\gamma_2}}$$

s_2 is strictly negative, but s_1 might not be. $G(s)$ is non-minimum phase when $s_1 \geq 0$, which happens when $\gamma_1 + \gamma_2 \leq 1$. But $0 < \gamma_i \leq 1$, hence $0 < \gamma_1 + \gamma_2 \leq 2$, which means that $G(s)$ is non-minimum phase when $0 < \gamma_1 + \gamma_2 \leq 1$ and minimum phase when $1 < \gamma_1 + \gamma_2 \leq 2$.

Exercise 2.1.6

$G(0)$ and $G(0)^{-T}$ are given by

$$G(0) = \begin{bmatrix} \gamma_1 k_1 c_1 & (1-\gamma_2)k_2 c_1 \\ (1-\gamma_1)k_1 c_2 & \gamma_2 k_2 c_2 \end{bmatrix}$$

$$G(0)^{-T} = \frac{1}{\gamma_1\gamma_2 k_1 k_2 c_1 c_2 - (1-\gamma_1)(1-\gamma_2)k_1 k_2 c_1 c_2} \begin{bmatrix} \gamma_2 k_2 c_2 & -(1-\gamma_1)k_1 c_2 \\ -(1-\gamma_2)k_2 c_1 & \gamma_1 k_1 c_1 \end{bmatrix}$$

Hence, the RGA of $G(0)$ is given by

$$RGA(G(0)) = G(0).*G(0)^{-T} =$$
$$\frac{1}{\gamma_1\gamma_2 k_1 k_2 c_1 c_2 - (1-\gamma_1)(1-\gamma_2)k_1 k_2 c_1 c_2} \begin{bmatrix} \gamma_1\gamma_2 k_1 k_2 c_1 c_2 & -(1-\gamma_1)(1-\gamma_2)k_1 k_2 c_1 c_2 \\ -(1-\gamma_1)(1-\gamma_2)k_1 k_2 c_1 c_2 & \gamma_1\gamma_2 k_1 k_2 c_1 c_2 \end{bmatrix}$$

and

$$\lambda = \frac{\gamma_1\gamma_2 k_1 k_2 c_1 c_2}{\gamma_1\gamma_2 k_1 k_2 c_1 c_2 - (1 - \gamma_1 - \gamma_2 + \gamma_1\gamma_2)k_1 k_2 c_1 c_2)} = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - 1}$$

In the minimum phase case, where $\gamma_1 = \gamma_2 = 0.625$, $\lambda_{mp} = 1.5625$, and

$$RGA(G_{mp}(0) = \begin{bmatrix} 1.5625 & -0.5625 \\ -0.5625 & 1.5625 \end{bmatrix}$$

In the non-minimum phase case, where $\gamma_1 = \gamma_2 = 0.375$, $\lambda_{nmp} = 0.5625$, and

$$RGA(G_{nmp}(0) = \begin{bmatrix} -0.5625 & 1.5625 \\ 1.5625 & -0.5625 \end{bmatrix}$$

Exercise 2.1.7

In order to accurately determine the values of k_1, k_2, we turn to tanks 1 and 2, since only there can the height of the water be recorded. The process we followed for resolving k_1 is completely analogous for k_2. Shutting the outflow of tank 1 while having only pump 1 being active, with both of its tubes feeding water to tank 1, with constant voltage applied results in a situation where no extraneous flow comes into tank 1 (other than that from pump 1), and no outflow happens. Hence, we can see that the rate at which a tank is filled in this setting is linear in time:

$$\frac{dh_1}{dt} = \cancelto{0}{-\frac{a_1}{A_1}\sqrt{2gh_1}} + \cancelto{0}{\frac{a_3}{A_1}\sqrt{2gh_3}} + \frac{k_1}{A_1}u_1$$
$$\frac{dh_2}{dt} = \cancelto{0}{-\frac{a_2}{A_2}\sqrt{2gh_2}} + \cancelto{0}{\frac{a_4}{A_2}\sqrt{2gh_4}} + \frac{k_2}{A_2}u_2$$

According to multiple measurements[1] with different input voltages, the values of k_1 and k_2 were calculated to be:

$$k_1 = 3.8454\ cm^3/sV$$
$$k_2 = 3.9925\ cm^3/sV$$

Exercise 2.1.8

In order to calculate the effective outlet areas a_i for the minimum phase case we focus our attention to the equations describing the equilibrium state and, at first, to the last two equations of exercise 2.1.2, where we can see that if we drive tanks 3 and 4 to equilibrium, we can directly calculate a_3 and a_4. In this case we are not interested in what happens to the lower tanks.

$$a_3 = a_4 = \frac{(1-\gamma_2)k_2}{A_3} u_2^0 \frac{A_3}{\sqrt{2gh_3^0}} = 0.0827\ cm^2$$

As for the effective outlet areas of tanks 1 and 2, since their values are constant irrespective of the conditions in which the system is running, they can be calculated in the same manner by driving the tanks to equilibrium without the intervention of flows from the upper tanks. In this case:

$$a_1 = a_2 = \frac{\gamma_1 k_1}{A_1} u_1^0 \frac{A_1}{\sqrt{2gh_1^0}} = 0.1282\ cm^2$$

It should be noted at this point that since the process is subject to changes due to other groups interchanging the outlet nozzles, and since their areas are approximately equal, as they are different versions of the same thing, we can regard them in our calculations as being equal, since measurements can be noisy.

[1] The maximum height of each tank as perceived by the sensors was not taken to be 25cm, but was calculated for each tank. Furthermore the pressure sensors at tanks 1 and 2 were found to have an offset of 9% and 15% respectively the last time they were measured. These offsets can be seen in figures 2 and 4 as at time $t = 0$ the levels of the tanks are not zero.

Manual control

Exercise 2.2.1

Table 1 presents the experimental and calculated steady-state values for the height in each tank for both the minimum and non-minimum phase cases. It is evident that not all real values agree with their theoretically derived ones. This is reasonable: non-zero steady-state in this setting means that the amount of flow coming into a tank is equal to that coming out of it, during a given amount of time. This means that there are infinite stationary points, depending on the amount of water present in the tanks at time $t = 0$, or at later points of time.

Height of tank i	minimum phase [cm] / calculated	non-minimum phase [cm] / calculated
1	17.5 / 26.545	28 / 27.0478
2	22 / 27.0478	14 / 26.5450
3	6.5 / 9.3885	12 / 10.8637
4	7.5 / 8.7095	9.5 / 10.0779

Table 1: Steady-state water height in each tank, experimentally derived and calculated from the system's model. Initially the tanks were empty. The inputs were set to $0.5u_{max} = 7.5V$ and the system was permitted to reach steady-state from these initial conditions.

Exercise 2.2.2

Figure 1 shows the step responses from when one input is ON for the minimum and non-minimum phase cases.

The system is indeed coupled. In accordance to their respective RGA matrices, we can see that, in the minimum phase case, input i drives tank i but is coupled to output j, while, in the non-minimum phase case, input i drives tank j but is coupled to output i.

Exercise 2.2.3

We set the reference levels to 50% of the maximum height for tanks 1 and 2. In the minimum phase case we found it easier to control the system, and the transient, in our experiment, was roughly 500 seconds. Contrary to this, the non-minimum phase case is much harder to control manually, if it is actually feasible to do so. In the end we could not reach our goal, however, in any case it is reasonable to assume that its transient would be more than in the minimum phase case.

Exercise 2.2.4

Among the most important differences between the minimum and non-minimum phase cases are the actual physical setup of the plant for each case, the fact that the latter is more difficult to control by hand, and more difficult to obtain an intuitive understanding due to the increased complexity of the intrinsics of the physical process itself.

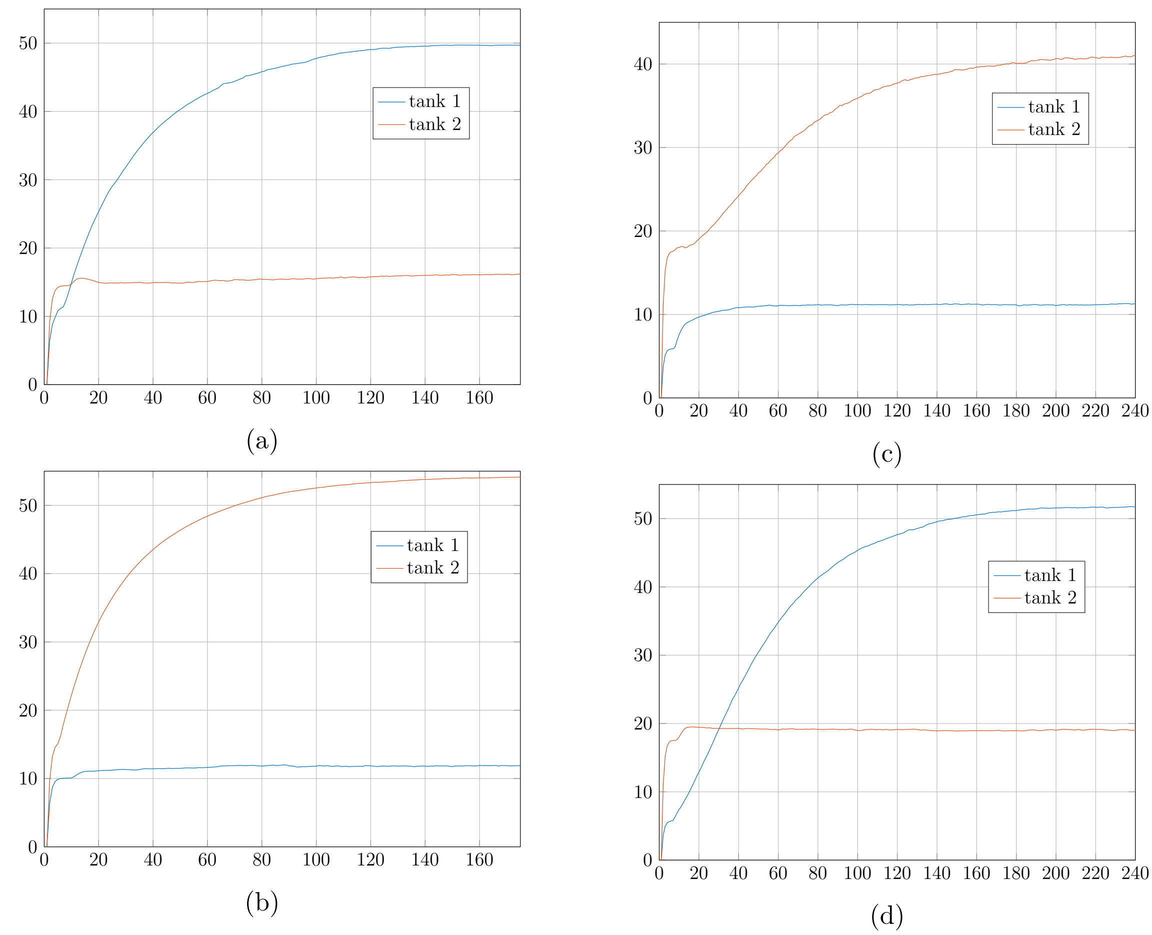

Figure 1: Step responses for the minimum phase case (left) and non-minumum phase case (right). The input step was set to $0.5u_{max} = 7.5V$. The vertical axes show the height of water in each tank in percentage of its maximum height value. Figures (a) and (c) correspond to input 1 being ON, while figures (b) and (d) correspond to input 2 being ON.

Decentralized control

This and the next section are organised as follows: the system is driven to equilibrium by means of the controller, or, if it is not feasible (i.e. in the non-minimum phase case) close to the reference point. Then, the system is subjected to (a) a step change of 5% in one of the inputs (marked with orange colour), (b) a disturbance w_1 which, physically, is the pouring of a cup of water directly in either tank 1 or 2 (marked with purple colour), and (c) a disturbance w_2 which, physically, is the opening of the extra outlet connected to tanks 3 or 4 (marked with green colour).

Exercise 4.1.1

Figure 2 illustrates the response of the minimum phase system to step input changes and introduction of disturbances. Table 2 summarizes some important performance metrics under this experiment.

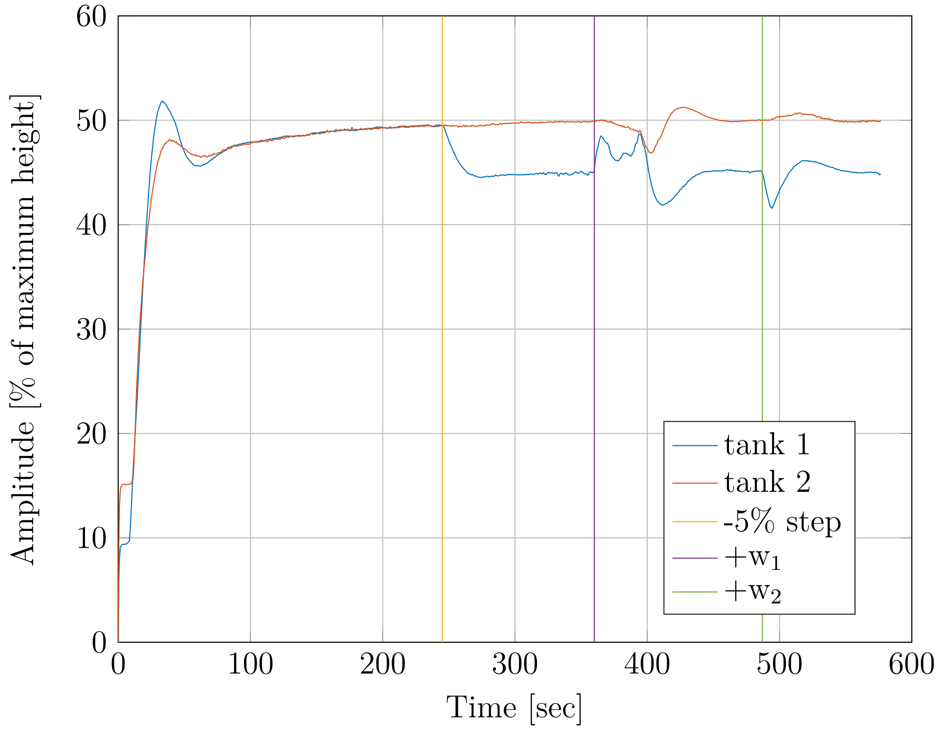

Figure 2: Response of the minimum phase system while under control from a decentralized controller. At time $t = 245$ a step of -5% is introduced in u_1. At time $t = 360$ a cup of water is poured into tank 1. At time $t = 487$ the extra outlet of tank 3 is opened.

Step response rise time	16
Overshoot	0
Recovery time from w_1	110
Recovery time from w_2	80

Table 2: Rise time and overshoot of the step response, and recovery times from the introduction of disturbances w_1 and w_2.

Figure 3 illustrates the response of the minimum phase system to step input changes and introduction of disturbances. Table 3 summarizes some important performance metrics under this experiment.

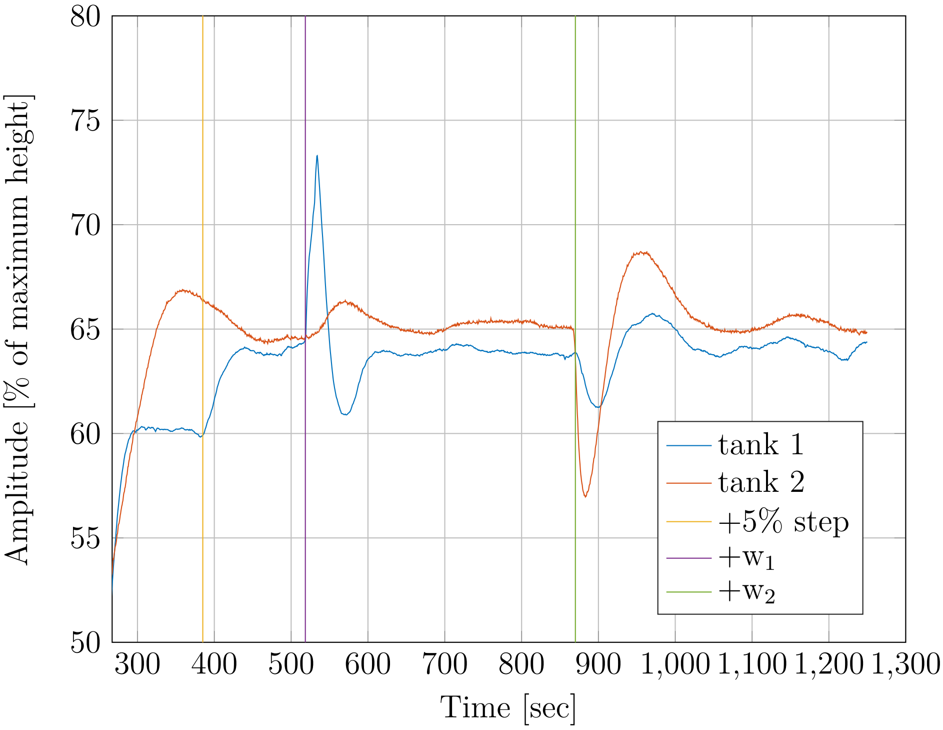

Figure 3: Response of the non-minimum phase system while under control from a decentralized controller. At time $t = 385$ a step of +5% is introduced in u_1. At time $t = 519$ a cup of water is poured into tank 1. At time $t = 870$ the extra outlet of tank 4 is opened.

Step response rise time	30
Overshoot	0
Recovery time from w_1	100
Recovery time from w_2	170

Table 3: Rise time and overshoot of the step response, and recovery times from the introduction of disturbances w_1 and w_2.

Exercise 4.1.2

The non-minimum phase system reacts slower both in its step response and the attenuation of disturbances, although it takes both systems roughly the same time to recover from the introduction of w_1. Furthermore, the non-minimum phase case system is more sensitive to w_2, the disturbance introduced from the upper tanks, than the minimum phase system: not only it takes twice the time to recover from w_2, but the amplitude of the oscillation in the counterpart tank is significantly larger.

Robust control

Exercise 4.2.1

Figure 4 illustrates the response of the minimum phase system to step input changes and introduction of disturbances. Table 4 summarizes some important performance metrics under this experiment.

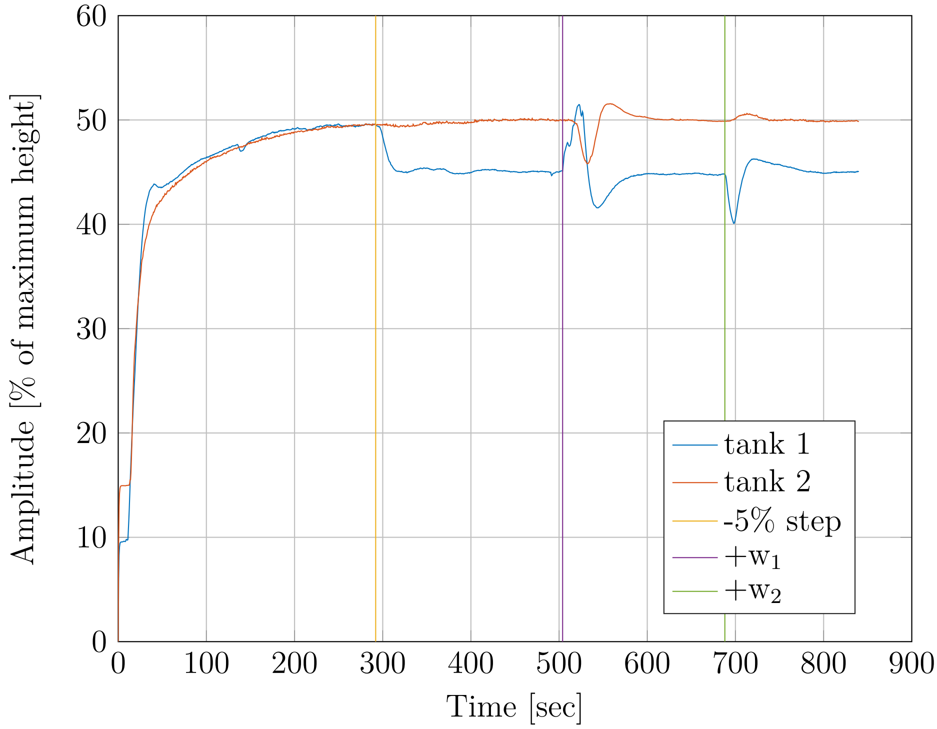

Figure 4: Response of the minimum phase system while under control from a controller derived with the Glover-McFarlae method. At time $t = 292$ a step of -5% is introduced in u_1. At time $t = 504$ a cup of water is poured into tank 1. At time $t = 688$ the extra outlet of tank 3 is opened.

Step response rise time	14
Overshoot	0
Recovery time from w_1	100
Recovery time from w_2	120

Table 4: Rise time and overshoot of the step response, and recovery times from the introduction of disturbances w_1 and w_2.

Figure 5 illustrates the response of the minimum phase system to step input changes and introduction of disturbances. Table 5 summarizes some important performance metrics under this experiment.

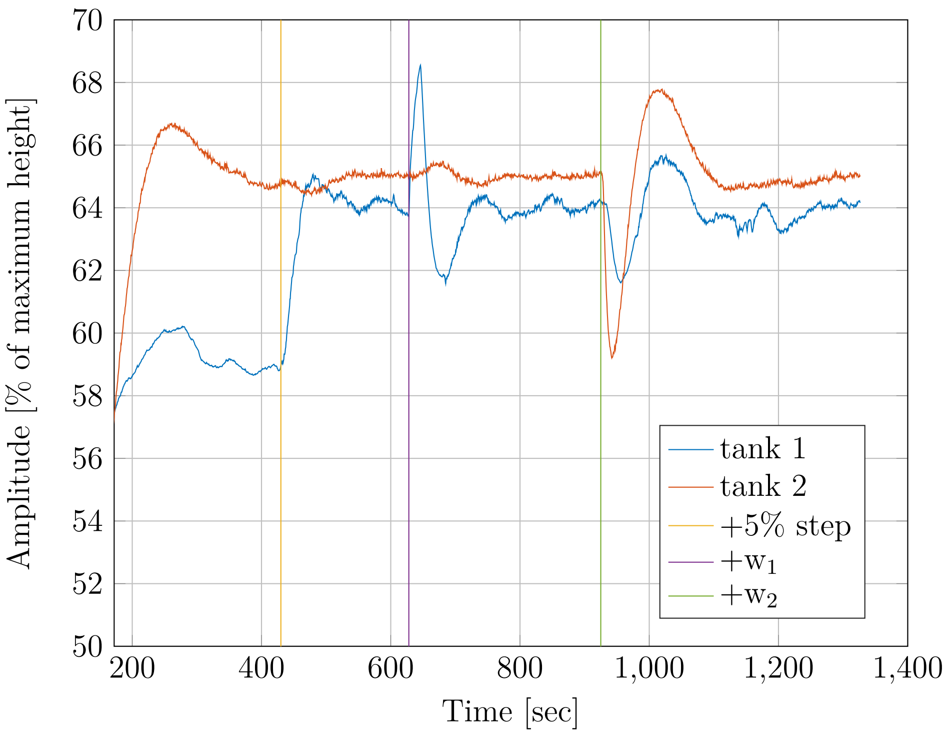

Figure 5: Response of the non-minimum phase system while under control from a controller derived with the Glover-McFarlae method. At time $t = 430$ a step of +5% is introduced in u_1. At time $t = 628$ a cup of water is poured into tank 1. At time $t = 925$ the extra outlet of tank 4 is opened.

Step response rise time	22
Overshoot	2%
Recovery time from w_1	180
Recovery time from w_2	200

Table 5: Rise time and overshoot of the step response, and recovery times from the introduction of disturbances w_1 and w_2.

Exercise 4.2.2

The Glover-McFarlane controllers make each system respond quicker to steps introduced in the input, although the rejection of disturbances is somewhat slower, especially in the non-minimum phase case. However, on the whole, the amplitude of the oscillations caused by the introduction of disturbances are significantly decreased.

Exercise 4.2.3

Overall, we notice that the response of the non-minimum phase system is slower than that of the minimum-phase system, while the introduction of disturbances that involve tanks 3 and 4 are more influential on the former system. Overall, the non-minimum phase system is proved to be harder to control due to constraints relative to its nature.

Conclusions

This report was a summary of our experience in modelling, controlling, and experimenting in real conditions with the physical process of the four-tank coupled system. After determining crucial parameters of the plant, controllers were designed in order to obtain a practical understanding of the difficulty of control pertaining to minimum and non-minimum phase case plants. Both the decentralized and robust controllers were proved to be successful in their operation, albeit with different performances. Lastly, we note on the increased arduousness of control of non-minimum phase case systems.