REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2014–08–19, kl 8.00–13.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures and of the blackboard notes, mathematical tables and pocket calculator. **Exercice notes on RGA and decoupling methods are allowed**, but no other exercice materials.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover.

The exam consists of five problems. The distribution of points for the various problems and subproblems is indicated.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Alexandre Proutiere 08-7906351

Resultat: Will be posted no later than August 21, 2014.

Good Luck!

Problem 1

(a) Compute the 2-norm (also called L_2 or H_2 norm, see e.g. the course book) and the ∞-norm (also referred to as L_∞ or H_∞ norm) of the signals

$$x(t) = 1_{t\geq 0}\frac{1}{\alpha t+\beta}, \quad y(t) = 1_{t\geq 0}\begin{bmatrix} e^{-t} \\ -5e^{-2t}\end{bmatrix},$$

where $1_{t\geq 0}$ is equal to 1 if $t \geq 0$ and to 0 otherwise, and where α and β are strictly positive real numbers. [3pts]

(b) Compute the energy gain of the linear system characterized by the following transfer function:

$$G(s) = \frac{\alpha s+1}{\beta s+1}$$

where $\alpha, \beta > 0$. [2pts]

(c) Prove that the energy gain of the non-linear system $\mathcal{S} : y(t) = \frac{u(t)}{1+|u(t)|}$ is smaller than 1. *Hint: Show first that the function $x \mapsto x/(1+|x|)$ is Lipschitz.* [2pts]

(d) Determine the energy gain of the system admitting the following state-space realization:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = [0\ 20]x.$$

[3pts]

Problem 2

Consider the following two systems which we would like to control using a decentralized controller, and achieving a crossover frequency at $\omega_c = 2$ rad/s.

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s+1 & \frac{s+1}{s+3} \\ s+2 & \frac{s+2}{s+5} \end{bmatrix}$$

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & -2 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} x \end{cases}$$

(a) Determine the poles and zeros for each system. [4pts]

(b) Use an RGA analysis to determine the most suitable pairing for controlling these systems using decentralized control. Motivate. [2pts]

(c) Is the second system observable / controllable? [2pts]

(d) For the first system, propose a static decoupling matrix W at stationarity and a proportional decentralized controller F such that the system has a crossover frequency $\omega_c = 2$ rad/s. [2pts]

Hint: $\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$

Problem 3

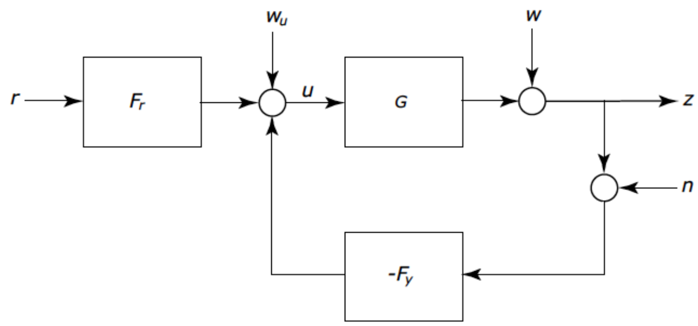

Figure 1: Closed-loop system – Block diagram.

(a) Consider a closed-loop SISO system as depicted in Figure 1. The respective transfer functions for the plant, the forward and feedback controllers are G, F_r, and $-F_y$. Provide the transfer function, here referred to as V, characterizing the input-output relationship from the measurement noise n to the ouput z.
We wish V to satisfy the following conditions: $|V(i\omega)| \leq 2$ for all $\omega \leq 1$ rad/s, and $|V(i\omega)| \leq 5/\omega$ for all $\omega \geq 5$ rad/s. To this aim, we select a weight function W, such that $\|VW\|_\infty \leq 1$. Which of the following two weight functions is appropriate, and how should the parameters M and ω_0 be chosen?

$$(i)\ W(s) = \frac{1}{M} + \frac{\omega_0}{s}, \quad (ii)\ W(s) = \frac{1}{M} + \frac{s}{\omega_0}.$$

Is any of the above weight functions appropriate if we wish to ensure that V satisfies $|V(i\omega)| \leq 5/\omega^2$ for all $\omega \geq 5$ rad/s? [4pts]
(b) A system is described by the rational transfer function:

$$G(s) = \frac{s^2 - 5s + 4}{s^3 - 4s^2 + 4s}.$$

We would like to design a closed-loop controller that rejects disturbances efficiently and exhibits low sensitivity to measurement noise at frequencies around 3 rad/s. Is it possible? Justify. [3pts]

(c) We have designed an $\mathcal{H}_\infty$ closed-loop controller so as to minimize:

$$\left\| \begin{array}{c} W_S S \\ W_T G F_y S \end{array} \right\|_\infty$$

for some weight functions W_S and W_T (S denotes the sensitivity function, and the feedback control is $-F_y$).
(c-1) Which input-output relationships are S and GF_yS representing? Can you then infer what were the motivations behind the design of this $\mathcal{H}_\infty$ controller? [1pt]

(c-2) We have chosen a weight function of the form $W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s}$, where M_S and ω_{BS} are two positive parameters. What is the impact of M_S on the sensitivity to disturbances? [1pt]

(c-3) We discover that the obtained control signal u is very sensitive to the disturbance w up to a frequency 4 rad/s. Propose a framework to address this issue? How can we modify the $\mathcal{H}_\infty$ control framework? [2pts]

Problem 4

Consider the following linear system:

$$\begin{aligned} \dot{x} &= -ax + bu + v_1 \\ y &= 2x + v_2 \\ z &= x/2 \end{aligned}$$

where v_1, v_2 are gaussian white noises such that the covariance matrix of $\binom{v_1}{v_2}$ is:

$$\begin{pmatrix} 1 & \beta \\ \beta & 2 \end{pmatrix}$$

with $\beta \geq 0$. We wish to find the controller that solves the following optimization problem:

$$\min\ J = E[\int_0^\infty (z(t)^2 + \rho u(t)^2) dt], \tag{1}$$

where $\rho > 0$.

(a) For which values of a is the open-loop system stable? [1pt]

(b) For which values of β is the optimal observer determined by the Kalman Filter? Compute the optimal observer for these values. Under the Kalman Filter, how is the state estimation error evolving over time? How is the speed at which this estimate tends to 0 impacted by β? [5pts]

(c) Determine the optimal linear state feedback controller, and describe a solution of optimization problem (1). Justify. [3pts]

(d) What is the impact of ρ on the state estimation error? What is the impact of ρ on the accuracy of the controller? [2pts]

Problem 5

We first provide the motivation of the problem and then formulate the questions to be addressed. These questions are independent of the motivation part.

Motivation: In many process industries, it is common for multiple processes to share certain utilities such as cooling water and pressurized air. This may however introduce couplings and complex dynamics in unexpected ways. It is therefore of crucial importance to make controllers robust in such situations. Consider the situation in Figure 2 where a utility T is shared by N different processes and where the utility input to the first process, T_{in}, is regulated by a feedback controller $F_T(s)$ acting on measurements of the utility coming out of the last process. From the perspective of a process engineer

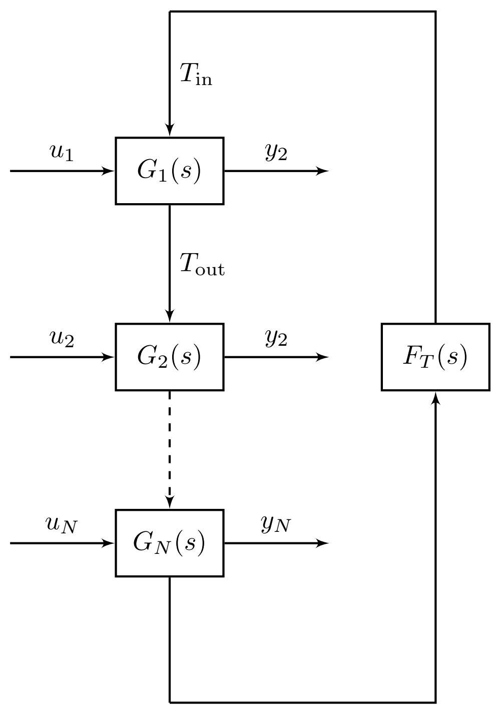

Figure 2: A shared utility controlled by feedback

designing a controller for the k:th process, the plant may be represented as in the block diagram in Figure 3 where the effect on the utility T from all other processes including the feedback $F_T(s)$ is lumped into an uncertainty term $\Delta_1(s)$.

Questions:

(a) Show that the block diagram in Figure 3 may be represented by the block diagram in Figure 4 and derive an expression for Δ_2 given that the plant is on the form

$$G_k(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

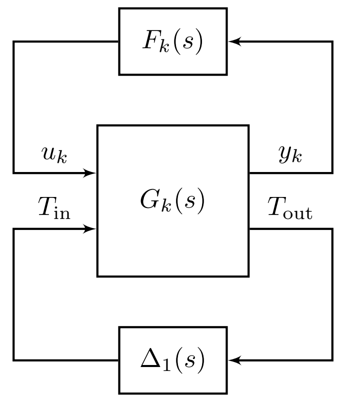

Figure 3: MIMO formulation of uncertain feedback.

where $G_{ij}(s) \in \mathbb{C}$. *Hint: you may use a SISO representation of the system.* [3pts]

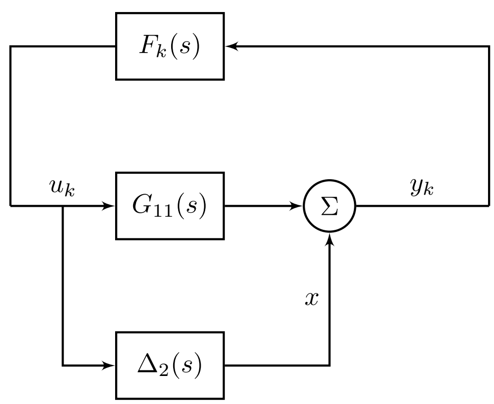

Figure 4: SISO formulation in terms of additive uncertainty

(b) Use the small gain theorem to derive a criterion in terms of $F_k(s)$, $G_{11}(s)$ and $\Delta_2(s)$ which if fulfilled guarantees robust stability. [3pts]

(c) Assume now that the plant is

$$G_k(s) = \begin{bmatrix} \frac{1}{s+1} & 0.2 \\ 0.99 & 1 \end{bmatrix},$$

that the uncertainty is stable and fulfills $|\Delta_1(i\omega)| < 0.5$ for all frequencies ω and that we want to design a proportional controller $F_k(s) = K$. Use the criterion derived above to determine for which K we can guarantee stability. [4pts]