Solutions – EL2520
Exam June, 2015

Problem 1

a) The squared vector norm of $f(t)$ is given as $|f(t)|^2 = e^{-2t} + 1/(t+1)^2$ on the positive t-axis and zero elsewhere. We investigate the L_∞ norm first. The derivative of $|f(t)|^2$ is

$$\frac{d}{dt}|f(t)|^2 = -2e^{-2t} - 2/(t+1)^3$$

which is clearly negative for all $t > 0$. Hence, we realize that the maximum must occur at the boundary of the indicator function and we get

$$\|f(t)\|_\infty = \lim_{t\to 0^+} \sqrt{e^{-2t} + 1/(t+1)^2} = \sqrt{2}$$

For the L_2-norm we simply evaluate the integral

$$\|f(t)\|_2^2 = \int_0^\infty e^{-2t} + \frac{1}{(t+1)^2} dt = \left[-\frac{e^{-2t}}{2} - \frac{1}{t+1} \right]_0^\infty = \frac{3}{2}$$

where the effect of the indicator function was included by letting the lower bound equal zero. Hence, the sought norm is

$$\|f(t)\|_2 = \sqrt{\frac{3}{2}}$$

b) Since both systems are stable, the small gain theorem requires us to check that $\|\mathcal{S}_1\|\|\mathcal{S}_2\| < 1$. We get three cases:

- **Case 1** $\alpha < 0$: $\|\mathcal{S}_1\|\|\mathcal{S}_2\| = 6(\alpha^2 - \alpha)$
  We want to find the limiting α such that $6(\alpha^2 - \alpha) = 1$. Solving this quadratic expression we find a valid solution $\alpha = \frac{3-\sqrt{15}}{6}$. Hence, $\frac{3-\sqrt{15}}{6} < \alpha < 0$ yields stable solutions.
- **Case 2** $0 \le \alpha \le 1$: $\|\mathcal{S}_1\|\|\mathcal{S}_2\| = 6(\alpha - \alpha^2)$
  We now solve the quadratic equation $6(\alpha - \alpha^2) = 1$ and find that it has two solutions in the interval. $\alpha = \frac{3\pm\sqrt{3}}{6}$. Hence we find that the closed loop is stable for $0 \le \alpha < \frac{3-\sqrt{3}}{6}$ and $\frac{3+\sqrt{3}}{6} < \alpha \le 1$.

- **Case 3** $1 < \alpha$: $\|\mathcal{S}_1\|\|\mathcal{S}_2\| = 6(\alpha^2 - \alpha)$
  We want to find α such that $6(\alpha^2 - \alpha) = 1$. Solving this quadratic expression again we find a valid solution $\alpha = \frac{3+\sqrt{15}}{6}$. Hence, $1 < \alpha < \frac{3+\sqrt{15}}{6}$ yields stable solutions.

By putting it all together we find that the small gain theorem guarantees stability for α within the intervals

$$\frac{3-\sqrt{15}}{6} < \alpha < \frac{3-\sqrt{3}}{6}$$
$$\frac{3+\sqrt{3}}{6} < \alpha < \frac{3+\sqrt{15}}{6}$$

If α is outside the intervals, the small gain theorem gives no information regarding stability. The theorem is only sufficient and not necessary, i.e., a stable system could violate the conditions of the theorem.

c) From the definition of the norm we realize that we need the largest singular value of G. We get this as the square root of the largest eigenvalue of:

$$GG^* = \frac{\alpha^2}{|s+\alpha|^2} + \frac{1}{|s+1|^2}$$

But this is a scalar so it is its own eigenvalue and we have

$$\bar{\sigma}(s) = \sqrt{\frac{\alpha^2}{|s+\alpha|^2} + \frac{1}{|s+1|^2}}$$

We now get the norm by finding the supremum over all frequencies

$$\|G\| = \sup_{\omega} \sqrt{\frac{\alpha^2}{|i\omega+\alpha|^2} + \frac{1}{|i\omega+1|^2}}$$

Clearly, the terms within the square root only get smaller as ω increases and we conclude that $\omega = 0$ is the maximizer whereby we get

$$\|G\| = \sqrt{2}.$$

Problem 2

a)

- The system poles are given by the eigenvalues of A (if it is a minimal realization, see the next question), and since A is triangular, the eigenvalues are found on the diagonal, i.e., 1, 1 and γ.
- The observability and controllability matrices are

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & \alpha \\ 1 & 4 & \alpha\gamma + \alpha + 2\beta \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & \alpha & 4 & \alpha\gamma + \alpha + 2\beta \\ 1 & 0 & 1 & \beta & 1 & \beta\gamma + \beta \\ 0 & 1 & 0 & \gamma & 0 & \gamma^2 \end{bmatrix}$$

  The controllability matrix always have full rank (=3), and the observability matrix has full rank (=3) if and only if $4\alpha \neq 2(\alpha\gamma + \alpha + 2\beta)$. Thus, the system is always controllable, and it is observable if and only if $\alpha(1-\gamma) \neq 2\beta$.

b)

- The minors of G are $\frac{2}{s+2}$, $\frac{1}{s+1}$, $\frac{3}{s+1}$ and $\det(G(s)) = \frac{4}{(s+2)^2} - \frac{3}{(s+1)^2} = \frac{s^2-4s-8}{(s+2)^2(s+1)^2}$. Thus, the pole polynomial is $(s+2)^2(s+1)^2$, and the zero polynomial is $s^2 - 4s - 8$, hence, the poles are -1, -1, -2, -2, and the zeros are $2 \pm 2\sqrt{3}$.
- The RGA matrix at stationarity is

$$RGA(0) = \begin{bmatrix} -0.5 & 1.5 \\ 1.5 & -0.5 \end{bmatrix},$$

  and the RGA matrix at 0.5 rad/s is

$$RGA(0.5i) = \begin{bmatrix} -0.45 - 0.37i & 1.45 + 0.37i \\ 1.45 + 0.37i & -0.45 - 0.37i \end{bmatrix}.$$

  Both pairing rules (avoiding negative elements of RGA(0), and using elements close to 1 of RGA($\omega_c i$)) yields the pairing $u_1 \to y_2$, $u_2 \to y_1$.
- The RGA matrix at 5 rad/s is

$$RGA(5i) = \begin{bmatrix} 1.59 - 2.18i & -0.59 + 2.18i \\ -0.59 + 2.18i & 1.59 - 2.18i \end{bmatrix}.$$

  Here, the elements on the main diagonal are closer to one, thus the pairing rules at stationarity and at crossover frequency does not agree, hence we do not find a suitable pairing and should not use a decentralized controller without decoupling.
- A possible static decoupling matrix at 0 rad/s is

$$W = G(0)^{-1} = \begin{bmatrix} -0.5 & 0.5 \\ 1.5 & -0.5 \end{bmatrix}.$$

- With the decoupling matrix W above, the decentralized controller should be on the form $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$. Let $\tilde{G} = GW$, and consider the element $\tilde{G}_{11}(0.5i) = \tilde{G}_{22}(0.5i) = 0.72 - 0.48i$, giving a controller $f_{11} = f_{22} = 0.07(1 + \frac{1}{0.12s})$.
- Remember the RGA pairing $u_1 \to y_2$, $u_2 \to y_1$, thus the decentralized controller should be on the form $F = \begin{bmatrix} 0 & f_1 \\ f_2 & 0 \end{bmatrix}$. Consider the elements $G_{12}(0.5i) = 0.8 - 0.4i$ and $G_{21}(0.5i) = 2.4 - 1.2i$, and notice that the phase margin is 153°, thus using a pure I-controller (with 90° phase reduction) would yield 63° in phase margin. Thus, we can use the controllers $f_1 = 0.19/s$ and $f_2 = 0.56/s$.

Problem 3

a) We apply the result from robust stability. The system will be stable for all plants G_a, $a \in [0,1]$ if the complementary sensitivity function T satisfies:

$$\|TW\|_\infty \leq 1,$$

where for all ω,

$$|W(i\omega)| \geq \left| \frac{G_a(i\omega) - G_{1/2}(i\omega)}{G_{1/2}(i\omega)} \right|.$$

Note that we chose $G_{1/2}$ as the transfer function of the nominal plant. Now: we have

$$\sup_\omega \left| \frac{G_a(i\omega) - G_{1/2}(i\omega)}{G_{1/2}(i\omega)} \right| = \sup_\omega \sqrt{2(1 - \cos((a-1/2)\omega))} = \sqrt{2}.$$

Hence we can choose $W = \sqrt{2}$, and the condition for robust stability is $|T(i\omega)| \leq 1/\sqrt{2}$ for all ω.

b) The cross-over frequency ω_c must be around 2 rad/s. We apply the rules of thumbs stating that we need $\omega_c \leq z/2$ and $\omega_c \geq 2p$ for all RHP zeros z and poles p of the plant.
First plant. RHP zeros: none, RHP poles: none; ok.Note however that we should also investigate the limitation $S + T = 1$, which could well makes the design of an appropriate controller challenging.
Second plant. RHP zeros: 1, RHP poles: 5,7; not ok.
Third plant. RHP zeros: 5, RHP poles: 2, 3; not ok.

c)
- The condition for robust stability is:

  $$\forall \omega, \forall G \in \Pi, \quad |W_T(i\omega)| \geq \left| \frac{G(i\omega) - G_p(i\omega)}{G(i\omega)} \right|.$$

  $G_p \in \Pi$ represents the nominal plant, and can be chosen arbitrarily.
- To decrease the sensitivity to noise, we should decrease the amplitude of T, and so we should decrease $1/|W_T|$, and hence M_T should be decreased.
- The transfer function from the noise n to the input signal u is SF_y. To limit the impact of n on u, we can design a weight function W_u (according to the requirements of $n \to u$), and use the $\mathcal{H}_\infty$ control to ensure that:

  $$\left\| \begin{matrix} W_S S \\ W_T T \\ W_u S F_y \end{matrix} \right\|_\infty \leq 1.$$

Problem 4

a) The open-loop system has a RHP pole at 1, thus it is unstable.

b) (b-1) In the following, we use the same notations as those used in the course. We have here a first-order system, $n = 1$. We have: $A = 1$, $B = b$, $N = 1 = C$, $M = 2$, $R_1 = 1$, $R_2 = \alpha$ and $R_{12} = R_{21} = \beta$. To apply Kalman filter, we need to verify the assumptions required for the Kalman filter to lead to the optimal observer. These assumptions are:

- R_2 is symmetric and positive definite if $\alpha > 0$.
- $\tilde{R}_1 = R_1 - R_{12}R_2^{-1}R_{12}^T$ is positive definite. Here $\tilde{R}_1 = 1 - \beta^2\alpha^{-1}$. Hence the assumption holds if $\beta^2\alpha^{-1} \leq 1$.
- (A, C) is detectable, which means that there exists K such that $A - KC$ is stable. Here $A - KC = 1 - K$. The assumption holds for the choice $K > 1$.
- $(A - R_{12}R_2^{-1}C, \tilde{R}_1)$ is stabilizable, which means that there exists L such that $A - R_{12}R_2^{-1}C - \tilde{R}_1 L$ is stable. Here we have:

$$A - R_{12}R_2^{-1}C - \tilde{R}_1 L = 1 - \beta\alpha^{-1} - (1 - \beta^2\alpha^{-1})L.$$

(b-2) Since we can use Kalman filter to determine the optimal observer, the latter is given by:

$$\dot{\hat{x}} = a\hat{x} + bu + K(y - \hat{x}),$$

where $K = (p + \beta)\alpha^{-1}$ and p is the positive solution of Riccati's equation:

$$2p - (p + \beta)^2\alpha^{-1} + 1 = 0.$$

We obtain:

$$p = \alpha - \beta + \sqrt{\alpha^2 + \alpha - 2\alpha\beta},$$

and

$$K = 1 + (\sqrt{\alpha^2 + \alpha - 2\alpha\beta})\alpha^{-1}.$$

(b-3) The error $\tilde{x} = x - \hat{x}$ satisfies:

$$\dot{\tilde{x}} = -(\sqrt{\alpha^2 + \alpha - 2\alpha\beta})\alpha^{-1}\tilde{x} + v_1 - (1 + (\sqrt{\alpha^2 + \alpha - 2\alpha\beta})\alpha^{-1})v_2.$$

When $\beta > 0$, $E[\tilde{x}]$ decreases to zero slower than if β would be equal to 0. The noise correlations increase the estimation errors.

c) The optimal state feedback controller is $u = -Lx = -Q_2^{-1}B^T Sx$. Here we have $L = bS/\rho$ where $S \geq 0$ solves Riccati's equation:

$$2S + 4 - S^2 b^2 \rho^{-1} = 0.$$

We obtain:

$$S = \frac{\rho}{b^2}\left(1 + \sqrt{1 + \frac{4b^2}{\rho}}\right).$$

Finally:

$$u = -\frac{1}{b}\left(1 + \sqrt{1 + \frac{4b^2}{\rho}}\right)x.$$

d) The optimal state feedback satisfies:

$$\dot{x} = x + bu = -\left(\sqrt{1 + \frac{4}{\rho}}\right)x$$

By choosing $\rho = 4/3$ one can put the pole at -2 and as ρ is non-negative, it is impossible to put the pole at -1/2 by the optimal state feedback.

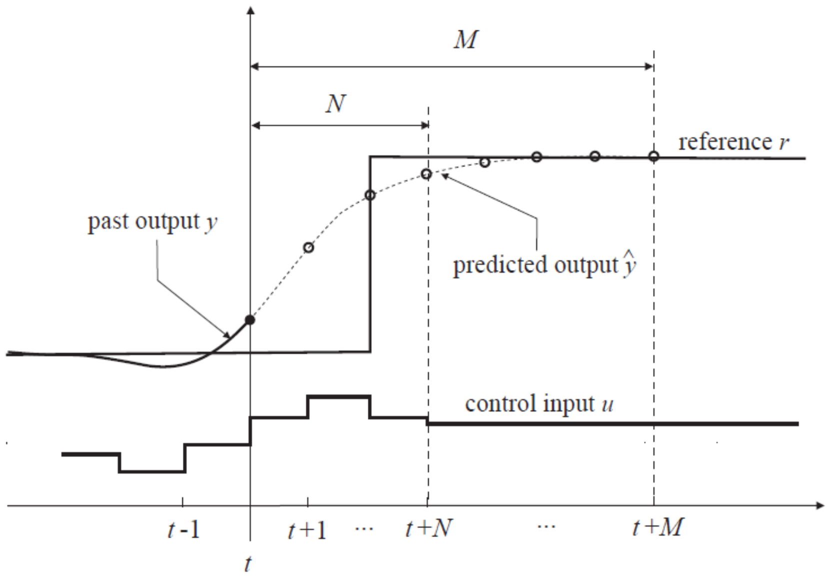

Figure 1: Illustration of MPC principle.

Problem 5

a) The receding horizon principle is illustrated in the Fig. 1. Given the current state measurement (or estimate), a sequence of future control inputs are determined by minimizing a cost function penalizing predicted inputs and outputs of the system. The first input is implemented, and the optimization is repeated at the next step.

b) Problems may arise if an output constraint is violated, or the current state of the system does not allow a feasible solution to be found. This may be caused by, e.g., disturbances or estimation errors when state measurements are not available. One way of limiting these issues is to use 'soft' constraints on outputs and states. For instance, a constraint on the form $x_{min} \leq x(k) \leq x_{max}$ can be replaced by $x_{min} - \epsilon(V_k^x)_{min} \leq x(k) \leq x_{max} + \epsilon(V_k^x)_{max}$ where ϵ is a slack variable and $(V_k^x)_{min}$ and $(V_k^x)_{max}$ are relaxed vectors. An extra term may then be added to the cost function that penalizes ϵ^2. This allows the constraints to be violated but at a high cost.

c) The discrete-time parameters are given by (with $T = 1$)

$$\begin{aligned} a &= e^{a_c T} = e^{a_c}. \\ b &= \int_0^T e^{a_c T} b_c \, \mathrm{d}t = \frac{b_c}{a_c}\left(e^{a_c} - 1\right). \end{aligned}$$

d) Considering $N = 1$:

$$\frac{\partial}{\partial u_{t_0}}\left(Q_1(ax_{t_0}+bu_{t_0})^2+Q_0x_{t_0}^2+Wu_{t_0}^2\right)=0.$$

Thus, we have

$$u_{t_0}=-\frac{abQ_1}{b^2Q_1+W}x_{t_0},$$

and by having $a=1/2$, $b=1$ and $Q_1=1:$ the initial control signal would be $u_{t_0}=-\frac{1/2}{1+W}x_{t_0}$
$\Rightarrow x_{t_0+1}=\frac{1}{2}x_{t_0}-\frac{1/2}{1+W}x_{t_0}=\frac{1}{2}\left(1-\frac{1}{1+W}\right)x_{t_0},$

To satisfy the constraint $|u_t|\leq 1$, the inequality $\left|-\frac{1/2}{1+W}x_{t_0}\right|\leq 1$ should hold. Thus, we conclude

$$x_{t_0+1}=\begin{cases}\frac{1}{2}\left(1-\frac{1}{1+W}\right)x_{t_0} & \text{if } |x_{t_0}|\leq 2(1+W)\\ \frac{1}{2}x_{t_0}+1 & \text{if } x_{t_0}\leq -2(1+W)\\ \frac{1}{2}x_{t_0}-1 & \text{if } x_{t_0}\geq 2(1+W),\end{cases} \tag{1}$$

and as it is obvious, by decreasing the W, the convergence rate increases for the first case.

e)

$$\begin{cases}x_{t_0+1} &= ax_{t_0}+bu_{t_0}\\ x_{t_0+2} &= ax_{t_0+1}+bu_{t_0+1}=a^2x_{t_0}+abu_{t_0}+bu_{t_0+1}.\end{cases} \tag{2}$$

Thus, based on (2) and considering the control horizon $N=2$, the cost function in MPC problem can be written as

$$Q_2x_{t_0+2}^2+Q_0\left(x_{t_0}^2+x_{t_0+1}^2\right)+W\left(u_{t_0}^2+u_{t_0+1}^2\right)=\left(Q_2a^4x_{t_0}^2+Q_0x_{t_0}^2(a^2+1)\right)+$$

$$\begin{bmatrix}u_{t_0} & u_{t_0+1}\end{bmatrix}\begin{bmatrix}Q_2a^2b^2+Q_0b^2+W & Q_2ab\\ Q_2ab & Q_2b^2+W\end{bmatrix}\begin{bmatrix}u_{t_0}\\ u_{t_0+1}\end{bmatrix}+\begin{bmatrix}2Q_2a^3bx_{t_0}+2Q_0abx_{t_0}\\ 2Q_2a^2bx_{t_0}\end{bmatrix}^T\begin{bmatrix}u_{t_0}\\ u_{t_0+1}\end{bmatrix}.$$

which turns to a Quadratic Programming (QP) problem.

The term $\left(Q_2a^4x_{t_0}^2+Q_0x_{t_0}^2(a^2+1)\right)$ is not effective in the optimization problem as it is not a function of control signal.

Thus,

$$H=\begin{bmatrix}Q_2a^2b^2+Q_0b^2+W & Q_2ab\\ Q_2ab & Q_2b^2+W\end{bmatrix}$$

and

$$h = \begin{bmatrix} 2Q_2a^3bx_{t_0} + 2Q_0abx_{t_0} & 2Q_2a^2bx_{t_0} \end{bmatrix}.$$

Based on the (2) and the constraint $|x_t| \leq 1$, we have:

$$\begin{cases} |\ ax_{t_0} + bu_{t_0}\ | & \leq 1 \\ |\ a^2x_{t_0} + abu_{t_0} + bu_{t_0+1}\ | & \leq 1, \end{cases} \tag{3}$$

which results in

$$\begin{cases} bu_{t_0} & \leq 1 - ax_{t_0} \\ -bu_{t_0} & \leq 1 + ax_{t_0} \end{cases} \tag{4}$$

and

$$\begin{cases} abu_{t_0} + bu_{t_0+1} & \leq 1 - a^2x_{t_0} \\ -abu_{t_0} - bu_{t_0+1} & \leq 1 + a^2x_{t_0}. \end{cases} \tag{5}$$

Therefore, based on inequalities (4) and (5), we can write:

$$\begin{bmatrix} b & 0 \\ -b & 0 \\ ab & b \\ -ab & -b \end{bmatrix} \begin{bmatrix} u_{t_0} \\ u_{t_0+1} \end{bmatrix} \leq \begin{bmatrix} 1 - axt_0 \\ 1 + axt_0 \\ 1 - a^2xt_0 \\ 1 + a^2xt_0 \end{bmatrix}.$$

Thus,

$$L = \begin{bmatrix} b & 0 \\ -b & 0 \\ ab & b \\ -ab & -b \end{bmatrix}$$

and

$$f = \begin{bmatrix} 1 - axt_0 \\ 1 + axt_0 \\ 1 - a^2xt_0 \\ 1 + a^2xt_0 \end{bmatrix}.$$