Exercise session 4:

Topics: • SISO input-output controllability

• MIMO: poles, zeros, gains, directions

→ 8.8, 5.2, 5.3, 9.4, 9.5

Input-output controllability analysis:

"Input-output controllability is the ability to achieve acceptable control performance; that is, to keep the outputs within specified bounds (keep e small) in spite of unknown but bounded variations, such as disturbances and model-plant mismatches, using available inputs and available measurements."

⇒ not to confuse with state controllability, which is rather a theoretical concept.

8.8) Given $\hat{y} = \hat{G}\hat{u} + \hat{G}_d \hat{d}$

with $\hat{G}(s) = \dfrac{-s+1}{(5s+1)(10s+1)}$ $\quad \hat{G}_d(s) = \dfrac{5e^{-s}}{(5s+1)(10s+1)}$

and $\hat{u} \in [-10, 10]$ and $\hat{d} \in [-3, 3]$

The goal is to achieve $\hat{y} \in [-0.5, 0.5]$

a) Determine the scaled system, i.e,

$$y = Gu + G_d d \quad \text{with} \quad u \in [-1, 1], \; d \in [-1, 1]$$

and the goal to achieve $y \in [-1, 1]$.

We set

$$\hat{y} = D_y y \qquad \text{with} \qquad D_y = 0.5$$
$$\hat{u} = D_u u \qquad \text{with} \qquad D_u = 10$$
$$\hat{d} = D_d d \qquad \text{with} \qquad D_d = 3$$

so that plugging these into $\hat{y} = \hat{G}\hat{u} + \hat{G}_d \hat{d}$ gives

$$D_y y = \hat{G} D_u u + \hat{G}_d D_d d$$

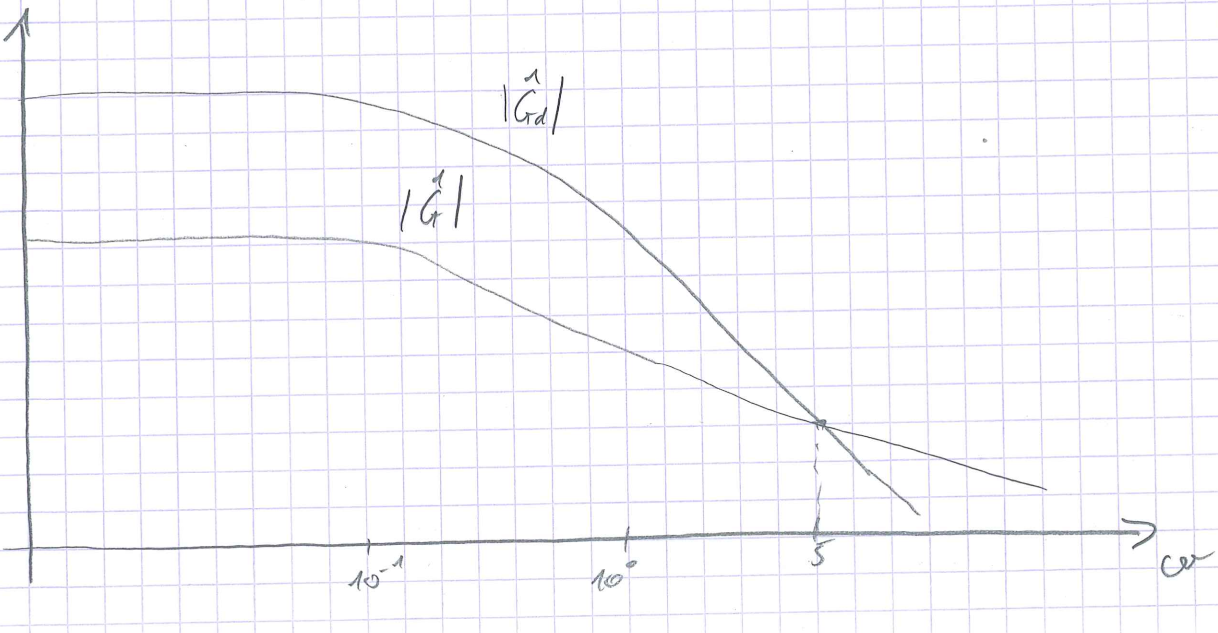
$$\Longleftrightarrow \quad y = \frac{D_u}{D_y} \hat{G} u + \frac{D_d}{D_y} \hat{G}_d d = G u + G_d d$$

with

$$G = \underbrace{\frac{D_u}{D_y}}_{20} \hat{G} = 20 \frac{-s+1}{(5s+1)(10s+1)}$$

$$G_d = \underbrace{\frac{D_d}{D_y}}_{6} \hat{G}_d = 30 \frac{e^{-s}}{(5s+1)(10s+1)}$$

b) Investigate if an acceptable feedback control law can be derived. The plots of $\hat{G}$ and $\hat{G}_d$ look like



Fundamental limitations from RHP poles, RHP zeros, and time delays:

- We have a RHP zero at $s = 1$ so that we have bandwidth limitations of $\omega_{BS} < 1$ (if $M_s = \infty$; recall last exercise session) or even $\omega_{BS} < \frac{1}{2}$ (if $M_s = 2$, which is more reasonable)

  $\rightarrow$ Considering only this RHP zero (and since we've not given requirements on S and/or T) the control objective is in theory possible!

- The system has no time delays (time delays in G_d are here not of interest) and no RHP poles.

Disturbance and input limitations:

- Consider G_d and look for the frequency range where $|G_d(i\omega)| > 1$ and where we necessarily need control action (compare 1st lab!)

By considering the scaling factor of 6, we know

$$|G_d(i \cdot 0.75)| = 1 \quad \text{(from the figure)}$$

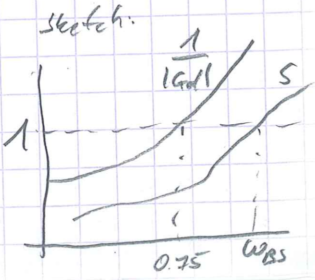
Thus we need $\omega_{BS} \geq 0.75$ (*)

$\Rightarrow$ For $M_s = \infty$, we know that this is still feasible (RHP zero limitation gave $\omega_{BS} < 1$)

(*) Note $y = SG_d\, d$ and

$$\|SG_d\|_\infty < 1 \iff |S(i\omega)| < \frac{1}{|G_d(i\omega)|} \quad \forall \omega$$

sketch:



- Now we need to check if there is enough input available!

→ Perfect control ($e=0$) is achieved by

$$u = -G^{-1} G_d d \qquad \text{(results in } y=0\text{)}$$

We hence need $|u| < 1$

$$\Leftrightarrow |G_d(i\omega)| < |G(i\omega)| \qquad \forall \omega$$

→ acceptable control ($e<1$) is achieved by

$$|G(i\omega)| > |G_d(i\omega)| - 1 \qquad \forall \omega$$

We, however, get $G(0) = 20$

and $G_d(0) = 30$

so that neither perfect nor acceptable control is possible.

→ So what can we do?

So far we dealt with SISO systems. Today we start with MIMO systems, i.e., vector-valued signals. As a consequence, we need to use signal and system norms and most importantly look at input "directions"!

Important: to note and remember

Assume $y = Gx$ with $y \in \mathbb{R}^m, x \in \mathbb{R}^r, G \in \mathbb{R}^{m \times r}$

then $\|y\|^2 = \|Gx\|^2 = x^* G^* G x$ $*$ means: – conjugate complex – transpose

and hence $\underline{\lambda}(G^*G)\|x\|^2 \leq \|y\|^2 \leq \overline{\lambda}(G^*G)\|x\|^2$

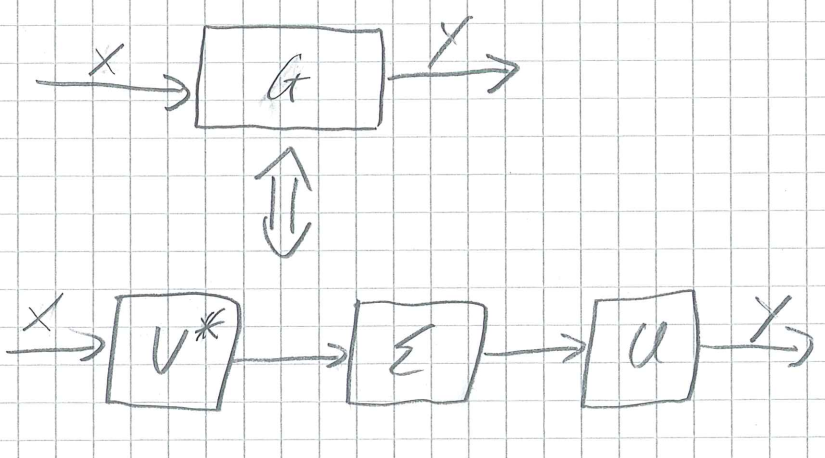
with $\underline{\lambda}$ and $\overline{\lambda}$ the minimum and maximum eigenvalue! We define $\overline{\sigma}(G) = \sqrt{\overline{\lambda}(G^*G)}$ and
$\underline{\sigma}(G) = \sqrt{\underline{\lambda}(G^*G)}$ maximum and minimum singular value

Singular value decomposition:

Then $G = U \Sigma V^*$ unitary matrices

$= [u_1 \dots u_r] \operatorname{diag}(\sigma_i) [v_1 \dots v_r]^*$ scaling (diagonal matrix)

$= \sum_{i=1}^{r} \sigma_i u_i v_i^*$



SVD interpretation (important):

Assume $G(s)x_0 = U \Sigma V^* x_0$

1) $V^* x_0$ rotates x_0 (no scaling since V is unitary)

2) $\Sigma V^* x_0$ scales all elements of $V^* x_0$

3) $U \Sigma V^* x_0$ rotates $\Sigma V^* x_0$

see animation on wikipedia!

$\Rightarrow$ 1) determines the "worst" input direction in terms of the system gain

2) determines the system gain

What is the system gain?

Assume a MIMO system $Y(i\omega) = G(i\omega)U(i\omega)$

$$\underline{\sigma}(G(i\omega)) \leq \|G(i\omega)\| = \frac{\|Y(i\omega)\|}{\|U(i\omega)\|} \leq \bar{\sigma}(G(i\omega))$$ // amplification at frequency ω!

and hence the system gain is

$$\|G\|_\infty = \underbrace{\sup_\omega}_{\text{worst frequency}} \underbrace{\bar{\sigma}(G(i\omega))}_{\text{worst input direction}}$$

5.2) Gain of a multivariable system

Consider $G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} \\ \frac{-1}{s+1} & \frac{1}{s+1} \end{bmatrix}$

Hence for $y = G(s)\,u$ we see that

$u \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ since $G(s) \in \mathbb{R}^{2\times 2}$.

<u>Determine the gain:</u> $\|G\|_\infty = \sup_\omega \bar{\sigma}(i\omega)$

First, determine the SVD: (see solutions of ex. 5.1 for detailed instructions)

① $G^*(s) \cdot G(s) = \frac{1}{(1+s)^*} \cdot \frac{1}{1+s} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{1}{|s+1|^2} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \frac{2}{|s+1|^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

② Find singular values

$$\det(\lambda I - G^*(s) G(s))$$

$$= \det \begin{pmatrix} \lambda - \frac{2}{|s+1|^2} & \frac{2}{|s+1|^2} \\ \frac{2}{|s+1|^2} & \lambda - \frac{2}{|s+1|^2} \end{pmatrix}$$

$$= \left(\lambda - \frac{2}{|s+1|^2}\right)^2 - \left(\frac{2}{|s+1|^2}\right)^2$$

$$= \lambda^2 - \frac{4}{|s+1|^2}\lambda + \cancel{\left(\frac{2}{|s+1|^2}\right)^2} - \cancel{\left(\frac{2}{|s+1|^2}\right)^2} = 0$$

$$= \lambda\left(\lambda - \frac{4}{|s+1|^2}\right) = 0$$

$\lambda_1 = 0$

$\lambda_2 = \frac{4}{|s+1|^2}$

Hence, the singular values are

$$\sigma_1 = 0$$
$$\sigma_2 = \frac{2}{|s+1|}$$

unitary matrix

③ Calculate U: matrix of right eigenvectors u_i to GG^*

Calculate V: matrix of right eigenvectors v_i to G^*G

In our case $GG^* = G^*G$, so that $U = V$

Now solve: $G^*(s)\,G(s)\,u_i = \lambda_i\,u_i$

For $\lambda_1 = 0$: $\frac{2}{|s+1|^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_{11}\\ u_{12}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$$\Leftrightarrow \frac{2}{|s+1|^2}\begin{bmatrix}u_{11} - u_{12}\\ -u_{11} + u_{12}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow u_{11} = u_{12}$$

$$\Rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

(make sure that the norm is 1)

For $\lambda_2 = \frac{4}{|s+1|^2}$: $\frac{2}{|s+1|^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_{21}\\ u_{22}\end{bmatrix} = \frac{4}{|s+1|^2}\begin{bmatrix}u_{21}\\ u_{22}\end{bmatrix}$

$$\begin{bmatrix}u_{21} - u_{22}\\ -u_{21} + u_{22}\end{bmatrix} = 2\begin{bmatrix}u_{21}\\ u_{22}\end{bmatrix}$$

$$\Rightarrow u_{21} = -u_{22}$$

$$\Rightarrow u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Hence we have $U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = V$

which results in the SVD

$$G(s) = U\,\Sigma\,V^* \quad \text{with} \quad \Sigma = \begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \frac{2}{|s+1|}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

The gain is

$$\|G\|_\infty = \sup_\omega \bar{\sigma}(G(i\omega))$$

$$= \sup_\omega \frac{2}{|i\omega + 1|}$$

$$= 2$$

The corresponding worst case input direction is

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; !$$ — Why? Think about it!

[Note : $GV = \Sigma U$]

(recall the intuition given previously)

5.3) Note, the plots show the singular values of a rank(G)=2 matrix!

$$\|G\|_\infty = \sup_\omega \bar{\sigma}(G(i\omega))$$

$$\|G_a\|_\infty = 2$$

$$\|G_b\|_\infty = 4$$

$$\|G_c\|_\infty = 4$$

$$\|G_d\|_\infty > 20$$

Next: What are poles in a MIMO system?

Assume a state-space model:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\boxed{\text{Def.: The poles are the eigenvalues of the system matrix } A \text{ in a minimal state-space realization.}}$

both controllable and observable (minimum number of states)

$\boxed{\text{Def.: The pole polynomial is } \det(sI - A)}$

⟹ nothing new so far. What now if we are given $G(s)$ and we want to calculate the poles?

$\boxed{\text{Theorem: The pole polynomial is the least common denominator of all non-zero minors.}}$

What is a minor of a matrix G?

Determinant of G or some smaller, square submatrix of G (obtained by deleting rows/colums)

For instance:
$$\begin{bmatrix} 5 & 1 & 6 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow[\text{and form determinant}]{\text{delete 3rd row and column}} \det\left(\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix}\right)$$

this is one minor

What is the least common denominator?

For $\frac{1}{(s+1)}$ and $\frac{1}{(s+3)}$ the least common denominator is $\frac{1}{(s+1)(s+3)}$

9.4)

Assume the system

$$G(s) = \frac{1}{s+1}\begin{bmatrix} 1-s & \frac{1}{3}-s \\ 2-s & 1-s \end{bmatrix} = \begin{bmatrix} \frac{1-s}{s+1} & \frac{\frac{1}{3}-s}{s+1} \\ \frac{2-s}{s+1} & \frac{1-s}{s+1} \end{bmatrix}$$

The minors are $\frac{1-s}{s+1}, \frac{\frac{1}{3}-s}{s+1}, \frac{2-s}{s+1}, \frac{1-s}{s+1}$ ← 1st order

and $\left(\frac{1-s}{s+1}\right)^2 - \left(\frac{\frac{1}{3}-s}{s+1}\,\frac{2-s}{s+1}\right)$ ← 2nd order

$$= \frac{(1-s)^2 - (\frac{1}{3}-s)(2-s)}{(s+1)^2}$$

$$= \frac{1 \cancel{-2s} \cancel{+s^2} - \left(\frac{2}{3} - \frac{1}{3}s \cancel{-2s} \cancel{+s^2}\right)}{(s+1)^2}$$

$$= \frac{1}{3}\,\frac{s+1}{(s+1)^2} = \frac{1}{3(s+1)}$$

Hence, one pole at $s=-1$ and a minimal state space realization has order 1.

MIMO zeros:

For the next exercise we need the concept of zeros. There exist different definitions of zeros in the literature. However, we use the following concept:
(These zeros are sometimes referred to as submission zeros)

Def: "Zeros are the values of "s" where G(s) looses rank"

Theorem: The zero polynomial of G(s) is the greatest common divisor of the maximal minors of G(s), normed so that they have the pole polynomial of G(s) as denominator. The zeros are the roots of its zero polynomial.

What is the greatest common divisor?

For instance assume $(s+1)(s+2)(s+3)$ and $(s+1)(s+10)(s+4)$ has the greatest common diviso $(s+1)$

9.5) Consider

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{-s+3}{s+1} & \frac{2}{s+2} \end{bmatrix}$$

1st order minors: $\frac{2}{s+1}$, $\frac{1}{s+2}$, $\frac{-s+3}{s+1}$, $\frac{2}{s+2}$

2nd order minor: $\frac{4}{(s+1)(s+2)} - \frac{3-s}{(s+1)(s+2)} = \frac{\cancel{1+s}}{\cancel{(s+1)}(s+2)} = \frac{1}{s+2}$

Poles: The least common denominator of all minors is $(s+1)(s+2)$ and we have hence two poles at -1 and -2

Zeros: The maximal minor is $\frac{1}{s+2}$ and normed with the pole polynomial makes it $\frac{(s+1)}{(s+1)(s+2)}$

We hence have a zero at -1.

Note that we do not have a cancellation of the pole and zero, which are both located at -1.

Why? The poles and zeros have different directions!

Recall: For <u>scalar</u> $G(s)$

we have $G(\underset{\text{zero}}{z}) = 0$ and $G(\underset{\text{pole}}{p}) = \infty$

For MIMO $G(s)$, however, we have

$$G(z)v_z = 0\, u_z \quad \text{and} \quad G(p)v_p = \infty\, u_p$$

$p = z = -1$ which gives $G(-1) = \begin{bmatrix} \frac{2}{0} & 1 \\ \frac{4}{0} & 2 \end{bmatrix}$

so that $v_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $u_z = \begin{bmatrix} -0.894 \\ 0.447 \end{bmatrix}$

$v_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $u_p = \begin{bmatrix} -0.447 \\ -0.894 \end{bmatrix}$

$U = \begin{bmatrix} -0.447 & -0.894 \\ -0.894 & 0.447 \end{bmatrix}$ $\Sigma = \begin{bmatrix} \infty & 0 \\ 0 & 0 \end{bmatrix}$

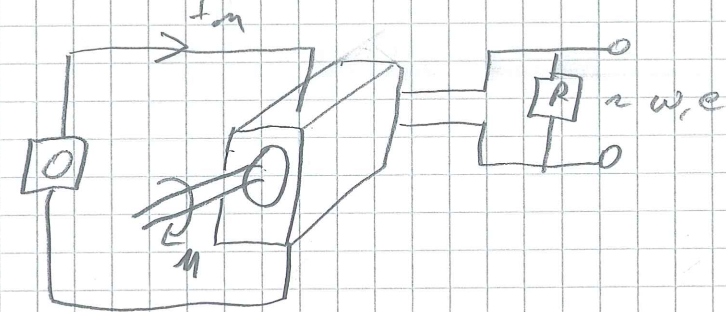
$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ calculate the SVD to get these directions.

8.1) MIMO Modeling

We look at a current generator:

input
I_m : magnetizing current
M : mechanical momentum
e : voltage peak value
disturbance
R : generator load
ω : angular velocity
output
I_f : peak value of current
M_e : forcing moment
$\mathfrak{J}, k_e, c_e$: constants



We are given:

I $\quad e = R \cdot I_f$

II $\quad \mathfrak{J}\,\dot{\omega} = M - M_e$

III $\quad M_e = k_e\, \omega\, I_f$

IV $\quad e = c_e I_m \cdot \omega$

e, ω : outputs $\quad$ with $y = [e, \omega]$

M, I_m : inputs $\quad$ with $u = [M, I_m, R]$

R : disturbance

1. Find a model for this generator:

$$\text{as} \quad \dot{x} = f(x, u, R) \qquad \text{(nonlinear description)}$$
$$y = h(x, u, R)$$

Note that I and IV give

$$I_f = \frac{e}{R} = \frac{C_e I_m \omega}{R}$$

Hence III gives

$$M_e = K_e \omega^2 \frac{C_e I_m}{R}$$

and hence we get

$$\dot{\omega} = \frac{1}{J}\left(M - K_e \omega^2 \frac{C_e I_m}{R}\right) = f(\omega, u)$$

note: ω is the state (one-dimensional) x

For $h(\cdot)$ we get

$$y = \begin{bmatrix} \omega \\ e \end{bmatrix} = \begin{bmatrix} \omega \\ C_e I_m \omega \end{bmatrix} = h(\omega, u)$$

Plugging in $K_e = C_e = J = 1$

$$\boxed{\begin{aligned} \dot{\omega} &= M - \omega^2 \frac{I_m}{R} \\ \begin{bmatrix} \omega \\ e \end{bmatrix} &= \begin{bmatrix} \omega \\ I_m \omega \end{bmatrix} \end{aligned}}$$

2. linearize this system around $\omega_0 = R_0 = I_{m0} = M_0 = 1$

Define
$$\begin{aligned} \Delta \omega &= \omega - \omega_0 \\ \Delta R &= R - R_0 \\ \Delta I_m &= I_m - I_{m0} \\ \Delta M &= M - M_0 \end{aligned}$$

Now we apply (classically) the Taylor expansion resulting in:

(a) $$\Delta\dot{\omega} \approx \frac{\partial}{\partial\omega} f(\omega,u)\Big|_{\substack{u=u_0\\ \omega=\omega_0}} \Delta\omega + \frac{\partial}{\partial u} f(\omega,u)\Big|_{\substack{u=u_0\\ \omega=\omega_0}} \Delta u$$

(b) $$\Delta y \approx \frac{\partial}{\partial\omega} h(\omega,u)\Big|_{\substack{u=u_0\\ \omega=\omega_0}} \Delta\omega + \frac{\partial}{\partial u} h(\omega,u)\Big|_{\substack{u=u_0\\ \omega=\omega_0}} \Delta u$$

which gives for (a)

$$\Delta\dot{\omega} = -2\Delta\omega + \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\Delta u = A\Delta\omega + B\Delta u$$

$$\Delta y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\Delta\omega + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\Delta u = C\Delta\omega + D\Delta u$$

3. Find the transfer function

We know $G(s) = C(sI-A)^{-1}B + D$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}(s+2)^{-1}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{(s+2)}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{1}{(s+2)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & s+2 & 0 \end{bmatrix}$$

$$= \frac{1}{s+2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & s+1 & 1 \end{bmatrix}$$