Solutions – EL2520
Exam August, 2015

Problem 1

a) By the definition we have

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{\omega^2+1}d\omega} = \sqrt{\frac{2}{\pi}\left[\tan^{-1}(\omega)\right]_{-\infty}^{\infty}} = \sqrt{2}.$$

b) We take the Laplace transform to get

$$\begin{aligned} sX(s) &= \overbrace{\begin{bmatrix} -1 & 1 \\ \alpha & -2 \end{bmatrix}}^{A} X(s) + \overbrace{\begin{bmatrix} 1 & 1 \\ 0.5 & 0 \end{bmatrix}}^{B} U(s) \\ Y &= \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0.5 \end{bmatrix}}_{C} X(s) \end{aligned}$$

We now want to eliminate $X(s)$ and solve for $Y(s)$ in terms of $U(s)$. We get

$$(sI-A)X(s) = BU(s) \Rightarrow X(s) = (sI-A)^{-1}BU(s) \Rightarrow Y(s) = \underbrace{C(sI-A)^{-1}B}_{G(s)}U(s)$$

We thus need to find the inverse

$$(sI-A)^{-1} = \begin{bmatrix} s+1 & -1 \\ -\alpha & s+2 \end{bmatrix}^{-1}$$

which by applying the inverse rule for two-by-two matrices:

$$Q^{-1} = \frac{1}{\det Q}\begin{bmatrix} Q_{22} & -Q_{12} \\ -Q_{21} & Q_{11} \end{bmatrix}$$

becomes

$$(sI-A)^{-1} = \frac{1}{(s+1)(s+2)-\alpha}\begin{bmatrix} s+2 & 1 \\ \alpha & s+1 \end{bmatrix}.$$

Pre and postmultiplication with C and B yields

$$G(s) = \frac{1}{(s+1)(s+2)-\alpha}\begin{bmatrix} 2s+3.5+2\alpha & s+2+2\alpha \\ 1.25s+2.75+0.5\alpha & s+2+0.5\alpha \end{bmatrix}.$$

For this system, we don't have to explicitly calculate the poles in order to determine stability. The poles are given be the greatest common divisor of all the minors of the system. The divisor $(s+1)(s+2)-\alpha = s^2+3s+2-\alpha$ will be present in all 1-order minors and cannot be canceled in all simultaneously meaning that its roots are going to be poles. But since we arrived at the transfer-matrix from a 2×2 state-space representation, we know that the system can't have more than two poles. Hence, the poles will be the roots of the polynomial $(s+1)(s+2)-\alpha = s^2+3s+2-\alpha$. For second order systems, stability coincides with all coefficients in the pole polynomial being positive which yields the condition $\alpha < 2$ for stability.

c) This function is known as the *sinc* function if the multiplication with the indicator function is removed. We begin by computing the L_∞ norm. The multiplication with the indicator function ensures that the signal is zero up to and including $t = 0$. The maximum must hence occur either at a critical point where the derivative is zero or at the "boundary", i.e., when $t \to 0^+$. We begin by looking at the boundary. The limit is

$$\lim_{t\to 0^+} \frac{\sin(t)}{t} = \lim_{t\to 0^+} \frac{\cos(t)}{1} = 1$$

where we used l'Hôpitals rule to evaluate the limit. We next compare this to the critical points. The derivative is

$$\frac{d}{dt}\frac{\sin t}{t} = \frac{t\cos t - \sin t}{t^2}$$

which is zero whenever $t = \tan(t)$. This equation has infinitely many roots of which the first positive root occurs approximately at $t \approx 4.5$. Since $|\sin t| \leq 1$, it follows that $|g(t)| \leq 1/4.5$ at the first critical point. The next root is approximately located at $t \approx 7.7$ which yield an even lower bound. We realize that critical points even further away from the origin will yield ever lower bounds, all less than 1, and we conclude that supremum occurs at the boundary.

$$\|g\|_\infty = \sup_t \frac{\sin t}{t} \cdot 1_{t>0} = 1$$

We next calculate the L_2-norm. From the definition we get

$$\|g\|_2^2 = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \cdot 1_{t>0}\right)^2 dt = \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

There are many ways to evaluate the integral, e.g. via Parceval's theorem or via complex analysis using the residue theorem, here we use a simple trick by introducing a helper-function:

$$
\begin{aligned}
I(a) &= \int_0^\infty \frac{\sin^2(at)}{t^2} dt \\
\frac{d}{da} I(a) &= \int_0^\infty \frac{2\cos(at)\sin(at)t}{t^2} dt = \int_0^\infty \frac{\sin(2at)}{t} dt = \frac{\pi}{2}
\end{aligned}
$$

But if the derivative of $I(a)$ with respect to a is the constant $\pi/2$, then we must have $I(a) = a\pi/2 + I_c$ where I_c is an integration constant. However, clearly $I(0) = 0 \Rightarrow I_c = 0$. So to get our integral we evaluate $I(1) = \frac{\pi}{2}$. We thus find that

$$\|g\|_2 = \sqrt{\frac{\pi}{2}}$$

Problem 2

a) The minors are $\frac{2s^2}{s+1}, \frac{s}{s+2}, \frac{2s}{s+2}, \frac{1}{s}$ and $\det(G(s)) = \frac{2s^2}{s+1}\frac{1}{s} - \frac{s}{s+2}\frac{2s}{s+2} = \frac{2s(3s+4)}{(s+1)(s+2)^2}$. Thus the pole polynomial is $s(s+1)(s+2)^2$, and the poles are 0, -1, -2, -2. The zero polynomial is given by normalizing $\det(G(s))$ with the pole polynomial, thus it is $s^2(3s+4)$, and the zeros are 0, 0, and -4/3.

b) The poles are given by the eigenvalues of A (since it is a minimal realization). Notice that A is a block triangular matrix, thus one eigenvalue is 4, and the others are given by $\operatorname{eig}\left(\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\right) = 2 \pm \sqrt{5}$.

c) (i) $RGA = G.*(G^{-1})^T = \frac{1}{199s^2+1180s+900}\begin{bmatrix} -(s+10)^2 & 200(s+1)(s+5) \\ 200(s+1)(s+5) & -(s+10)^2 \end{bmatrix}$.
$RGA(0) \approx \begin{bmatrix} -0.11 & 1.11 \\ 1.11 & -0.11 \end{bmatrix}$. $RGA(10i) \approx \begin{bmatrix} -0.005+0.008i & 1.005-0.008i \\ 1.005-0.008i & -0.005+0.008i \end{bmatrix}$.
Both RGA rules suggest using the pairing $1 \to 2$, $2 \to 1$.

(ii) One problem is that $W = G(s)^{-1}$ yields a non-proper controller.

(iii) Since the RGA method suggested using the pairing $1 \to 2$, $2 \to 1$, we could use the permutation matrix $W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compared to the RGA computed before, this will yield the permuted RGA matrix, i.e., $RGA(0) \approx \begin{bmatrix} 1.11 & -0.11 \\ -0.11 & 1.11 \end{bmatrix}$. $RGA(10i) \approx \begin{bmatrix} 1.005-0.008i & -0.005+0.008i \\ -0.005+0.008i & 1.005-0.008i \end{bmatrix}$.

(iv) Let $\tilde{G} = GW$, then $\tilde{G}_{11}(10i) = \tilde{G}_{22}(10i) = \frac{1}{10(i-1)}$. Thus, the controller becomes $f_{11} = f_{22} = 10|i-1| = 10\sqrt{2}$. The proportional controller does not change the phase, thus the phase at 10 rad/s is given by $\arg(1/(i-1)) = \arg(-0.5-0.5i) = -135°$. Thus, the phase margin is 45°, and the system is stable.

(v) $WF = \begin{bmatrix} 0 & 10\sqrt{2} \\ 10\sqrt{2} & 0 \end{bmatrix}$ is a constant transfer function matrix. Thus we can construct the state space model for the controller

$$\dot{x} = 0x + 0u$$
$$y = 0x + \begin{bmatrix} 0 & 10\sqrt{2} \\ 10\sqrt{2} & 0 \end{bmatrix} u$$

Problem 3

a) (i) We easily find that

$$G(s) = \frac{1}{(s+1)(s-3)}.$$

There is no RHP zero, and a single RHP pole at 3. Thus the only bandwidth limitation is (by the rule of thumb): $\omega_c \geq 6$ rad/s.
(ii) We first compute the transfer function of the system.

$$(sI - A)^{-1} = \begin{pmatrix} s-1 & -3 \\ 0 & s-2 \end{pmatrix} = \begin{pmatrix} \frac{1}{s-1} & \frac{3}{(s-1)(s-2)} \\ 0 & \frac{1}{s-2} \end{pmatrix},$$

and hence after simplifications

$$G(s) = \frac{2s}{(s-1)(s-2)}.$$

The system has two RHP poles at 1, and 2. The bandwidth limitations are thus: $\omega_c \geq 4$.

b) One way of doing it consists in designing a weight function W such that $\|TW\|_\infty < 1$. To this aim we may choose W (see the course) such that, for all ω, and all $a \in [-1, 1]$,

$$W(i\omega) \geq \left| \frac{G_a(i\omega) - G(i\omega)}{G(i\omega)} \right| = \frac{|a|}{\sqrt{1+\omega^2}}.$$

Since $sup_{\omega, a\in[-1,1]} \frac{|a|}{\sqrt{1+\omega^2}} = 1$, $W(\cdot) = 1$ is a valid choice. In this case, T just satisfies $\|T\|_\infty < 1$.

c) (i) We know that the spectrum of a filtered signal $N = FE$ is given by

$$\Phi_w(\omega) = F(i\omega)\Phi_e(\omega)F(-i\omega)$$

and our task is thus to find F fulfilling this relation. Since only squares of ω turn up, i.e., no ω^4 terms etc., we may make the ansatz that the filter is of first order

$$F(s) = \frac{as+b}{cs+d}$$

Inserting the ansatz and the information given in the problem we get

$$\Phi_n(\omega) = \frac{a^2\omega^2 + b^2}{c^2\omega^2 + d^2} = \frac{16\omega^2}{49\omega^2 + 4}$$

By identifying the coefficients we get $a = 4$, $b = 0$, $c = 7$ and $d = 2$. Hence the filter is

$$F(s) = \frac{4s}{7s+2}$$

Note that we chose the positive roots to get a stable minimum-phase system.
(ii) We have requirements at all frequencies, and hence this calls for an $\mathcal{H}_\infty$ control framework where the $\| \cdot \|_\infty$ norm is minimized. We wish to reject disturbances (captured by there sensitivity function), and to minimize the sensitivity to measurement noise (captured by the complementary sensitivity function T). Thus, we can try to minimize

$$\left\| \begin{array}{c} W_S S \\ W_T T \end{array} \right\|_\infty .$$

Problem 4

a) In general, for a fixed $Q_1 > 0$, when we decrease Q_2 this means that in the objective, the cost related to the energy of the control signal u becomes less important, which in turn, allow us to get a better controller. In addition, note that multiplying the objective function by a positive number does not change the solution of the optimization problem. Hence to compare the various sets of parameters, we can re-normalised and put all Q_1 equal to 1. We get:
Set 1: $Q_1 = 1, Q_2 = 0.5$,
Set 2: $Q_1 = 1, Q_2 = 10$,
Set 3: $Q_1 = 1, Q_2 = 0.5$.
The sets 1 and 3 yield the same controller, and they provide an output with less energy than that under the set 2.

b) (i) Using the notations of the course on LQG, we have $R_1 = 1 = R_2$ and $R_{12} = \gamma$. We also have $\tilde{R}_1 = 1 - \gamma^2$. We now list the conditions under which the Kalman filter provides the optimal observer.
$\tilde{R}_1$ and R_2 are positive. This is true if and only if $|\gamma| < 1$
(A, C) is detectable which means that there exists K such that $A - KC$ is stable. Here $A - KC = a - 3K$, hence we just need to select K such that $a - 3K < 0$ which is possible.
$(A, \tilde{R}_1$ is stabilizable, which means that there exists L such that $A - L\tilde{R}_1$ is stable. Here $A - L\tilde{R}_1 = a - L(1-\gamma^2)$, so choosing $L > a/(1-\gamma^2)$ is enough.

The Kalman filter estimates the state of the system with $\hat{x}$ such that:

$$\dot{\hat{x}} = a\hat{x} + bu + K(y - 3\hat{x}),$$

where $K = (3p + \gamma)$ and p is the positive solution of Ricatti's equation:

$$2ap + 1 - (3p + \gamma)^2 = 0.$$

Equivalently:

$$-9p^2 + (2a - 6\gamma)p + 1 - \gamma^2 = 0.$$

The positive solution of the above equation is:

$$p = \frac{1}{9} \times \left(a - 3\gamma + \sqrt{(a - 3\gamma)^2 + 1 - \gamma^2}\right).$$

We deduce that the average error $\tilde{x} = E[\hat{x} - x]$ evolves as follows:

$$\dot{\tilde{x}} = -\sqrt{(a - 3\gamma)^2 + 1 - \gamma^2}\,\tilde{x}. \tag{1}$$

(ii) First we have to verify the required assumptions. Following the notations of the course, we have $Q_1 = 1$, and $Q_2 = \rho$. Then Q_1 and Q_2 are positive, (A, B) is stabilizable, $(A, M^T Q_1 M)$ is detectable, and finally the assumptions for the applicability of the Kalman filter hold. The optimal state-feedback observer is:

$$u = -L\hat{x}, \quad L = bs/\rho,$$

where s is the positive solution of Ricatti's equation:

$$-\frac{b^2}{\rho}s^2 + 2as + 1 = 0.$$

This gives:

$$s = \frac{\rho}{b^2} \times \left(a + \sqrt{a^2 + \frac{b^2}{\rho}} \right).$$

By the decoupling principle, the solution of (P) is obtained by combining the Kalman filter and the optimal state-feedback controller.
(iii) Note that under the optimal controller, we get (removing the noise):

$$\dot{x} = -\sqrt{a^2 + \frac{b^2}{\rho}} x = -Dx.$$

The larger ρ is, the smaller D is. Hence by increasing ρ, we increase the energy of the output. Also observe that when ρ is very large, the optimal controller just consists in (1) doing nothing ($u = 0$) if the system is initial stable, i.e., if $a < 0$, and (2) in mirroring the RHP pole if the system is unstable, i.e., if $a > 0$.
The impact of the correlation γ can be seen of the optimal observer, see (1).

Problem 5

a) First compute the matrix exponential

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s+2 & -1 \\ 0 & s-1/2 \end{bmatrix}^{-1}\right\} =$$

$$\mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s-1/2)} \\ 0 & \frac{1}{s-1/2} \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+2} & \frac{2}{5}\frac{1}{s-1/2} - \frac{2}{5}\frac{1}{s+2} \\ 0 & \frac{1}{s-1/2} \end{bmatrix}\right\} =$$

$$\begin{bmatrix} e^{-2t} & \frac{2}{5}e^{\frac{1}{2}t} - \frac{2}{5}e^{-2t} \\ 0 & e^{\frac{1}{2}t} \end{bmatrix}$$

The discrete time system is given by the matrices

$$F = e^{AT} = \begin{bmatrix} e^{-2T} & \frac{2}{5}e^{\frac{1}{2}T} - \frac{2}{5}e^{-2T} \\ 0 & e^{\frac{1}{2}T} \end{bmatrix} = \begin{bmatrix} e^{-1} & \frac{2}{5}e^{\frac{1}{4}} - \frac{2}{5}e^{-1} \\ 0 & e^{\frac{1}{4}} \end{bmatrix} \approx \begin{bmatrix} 0.37 & 0.37 \\ 0 & 1.24 \end{bmatrix}$$

$$G = \int_0^T e^{At}B\,\mathrm{d}t = \int_0^T \begin{bmatrix} e^{-2t} & \frac{2}{5}e^{\frac{1}{2}t} - \frac{2}{5}e^{-2t} \\ 0 & e^{\frac{1}{2}t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\mathrm{d}t =$$

$$\begin{bmatrix} -\frac{3}{2} - \frac{1}{10}e^{-1} + \frac{8}{5}e^{\frac{1}{4}} & 1 - \frac{4}{5}e^{\frac{1}{4}} - \frac{1}{5}e^{-1} \\ -4 + 4e^{\frac{1}{4}} & 2 - 2e^{\frac{1}{4}} \end{bmatrix} \approx \begin{bmatrix} 0.52 & -0.10 \\ 1.14 & -0.57 \end{bmatrix}$$

$$H = C = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

b) Neither system is stable. The continuous system has a RHP pole at 0.5, and the discrete time system has an eigenvalue $1.24 > 1$.

c) Substituting

$$\begin{aligned}
x_0 &= x_0 \\
x_1 &= Ax_0 + Bu_0 = Ax_0 + \begin{bmatrix} B & 0 & 0 \end{bmatrix}\mathbf{u} \\
x_2 &= A^2x_0 + ABu_0 + Bu_1 = A^2x_0 + \begin{bmatrix} AB & B & 0 \end{bmatrix}\mathbf{u} \\
x_3 &= A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2 = A^3x_0 + \begin{bmatrix} A^2B & AB & B \end{bmatrix}\mathbf{u}
\end{aligned}$$

into

$$\sum_{k=0}^{2}(x_k^TQx_k + u_k^TRu_k) + x_3^TPx_3 =$$

$$x_0^TQx_0 + u_0^TRu_0 + x_1^TQx_1 + u_1^TRu_1 + x_2^TQx_2 + u_2^TRu_2 + x_3^TRx_3 =$$

yields

$$\mathbf{u}^T S \mathbf{u} + h^T \mathbf{u} + c$$

with

$$S = \begin{bmatrix} R & & \\ & R & \\ & & R \end{bmatrix} + \begin{bmatrix} B^T \\ 0 \\ 0 \end{bmatrix} Q \begin{bmatrix} B & 0 & 0 \end{bmatrix} + \begin{bmatrix} B^T A^T \\ B^T \\ 0 \end{bmatrix} Q \begin{bmatrix} AB & B & 0 \end{bmatrix} +$$

$$\begin{bmatrix} B^T {A^T}^2 \\ B^T A^T \\ B^T \end{bmatrix} P \begin{bmatrix} A^2 B & AB & B \end{bmatrix} =$$

$$\begin{bmatrix} R + B^T Q B + B^T A^T Q A B + B^T {A^T}^2 P A^2 B & B^T A^T Q B + B^T {A^T}^2 P A B & B^T {A^T}^2 P B \\ B^T Q A B + B^T A^T P A^2 B & R + B^T Q B + B^T A^T P A B & B^T A^T P B \\ B^T P A^2 B & B^T P A B & R + B^T P B \end{bmatrix}$$

$$h = 2x_0^T A^T Q \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} + 2x_0^T {A^T}^2 Q \begin{bmatrix} AB \\ B \\ 0 \end{bmatrix} + 2x_0^T {A^T}^3 P \begin{bmatrix} A^2 B \\ AB \\ B \end{bmatrix} =$$

$$2x_0^T A^T \begin{bmatrix} Q + A^T Q A + {A^T}^2 P A^2 \\ A^T Q + {A^T}^2 P A \\ {A^T}^2 P \end{bmatrix} B$$

$$c = x_0^T Q x_0 + x_0^T A^T Q A x_0 + x_0^T {A^T}^2 Q A^2 x_0 + x_0^T {A^T}^3 P A^3 x_0$$

d) Notice first that the solution does not depend on c, thus we consider the optimal control problem

$$\min_{\mathbf{u}} \mathbf{u}^T S \mathbf{u} + h^T \mathbf{u}$$

with

$$S = \begin{bmatrix} 37 & 24 & 16 \\ 24 & 29 & 16 \\ 16 & 16 & 21 \end{bmatrix}$$

$$h = x_0 \begin{bmatrix} 32 \\ 24 \\ 16 \end{bmatrix}$$

Since S is positive definite, the solution is given by the linear equation system

$$2S\mathbf{u} + h = 0$$

thus

$$\mathbf{u} = -\frac{1}{2} S^{-1} h \approx - \begin{bmatrix} 0.35 \\ 0.11 \\ 0.03 \end{bmatrix} x_0.$$

e) Using only the first control input, $u_k = -0.35x_k$ yields the closed loop

$$x_{k+1} = Ax_k + Bu_k = x_k + 2 * (-0.35x_k) = 0.3x_k,$$

thus the system will be stable.