Solutions – EL2520
Exam August 19, 2014

Problem 1

(a) By the definition we get

$$\|x\|_2^2 = \int_{-\infty}^{\infty} 1_{t\geq 0} \frac{1}{(\alpha t+\beta)^2}\,\mathrm{d}t = \int_0^{\infty} \frac{\mathrm{d}t}{(\alpha t+\beta)^2}.$$

After doing the change of variable $v = \alpha t + \beta$, we get:

$$\|x\|_2^2 = \int_{\beta}^{\infty} \frac{\mathrm{d}v}{\alpha v^2} = \frac{1}{\alpha\beta}.$$

Hence, $\|x\|_2 = \frac{1}{\sqrt{\alpha\beta}}$.

$$\|x\|_\infty = \sup_t |x(t)| = \sup_{t\geq 0} \frac{1}{\alpha t+\beta} = \frac{1}{\beta}.$$

For y, we have

$$\begin{aligned}\|y\|_2^2 &= \int_0^{\infty} (e^{-t})^2 + (-5e^{-2t})^2 dt \\ &= \int_0^{\infty} e^{-2t} + 25e^{-4t} dt \\ &= \frac{1}{2} + \frac{25}{4} = \frac{27}{4}.\end{aligned}$$

Hence, $\|y\|_2 = 3\sqrt{3}/2$.

$$\|y\|_\infty = \sup_t |y(t)| = \sup_{t\geq 0} \sqrt{e^{-2t} + 25e^{-4t}} = \sqrt{26}.$$

(b) The energy gain is:

$$\|G\| = \sup_{\omega} |G(i\omega)|.$$

Now

$$|G(i\omega)|^2 = \frac{1+\alpha^2\omega^2}{1+\beta^2\omega^2}.$$

We deduce that:

$$\|G\| = \begin{cases} \alpha/\beta & \text{if } \alpha \geq \beta, \\ 1 & \text{otherwise.} \end{cases}$$

(c) Let $f(x) = x/(1+|x|)$, and observe that:

$$|f(x)| \leq |x|.$$

Thus,

$$\|y\|_2^2 = \int_{-\infty}^{\infty} |f(u(t))|^2 \,\mathrm{d}t \leq \int_{-\infty}^{\infty} |u(t)|^2 \,\mathrm{d}t = ||u||_2^2$$

Hence, we deduce that $\|\mathcal{S}\| \leq 1$.

(d) The transfer matrix of the system is:

$$G(s) = C(sI - A)^{-1}B,$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$C = [0 \quad 20], \quad D = 0.$$

We deduce that:

$$G(s) = \frac{20}{s+2}.$$

The energy gain of the system is then:

$$\sup_{\omega} |G(i\omega)| = 10.$$

Problem 2

(a) Computing the 2×2 minor of the first system yields $\frac{-2(s+1)}{(s+2)(s+3)(s+5)}$, thus we see that its poles are -2, -3 and -5, while its only zero is -1.

Compute the transfer function for the second system, $G(s) = C(sI - A)^{-1}B$. Notice that $(sI - A)^{-1} = \frac{1}{s^2-5}\begin{bmatrix} s-1 & 2 & 0 \\ 2 & s+1 & 0 \\ -2 & \frac{-s-5}{s} & \frac{s^2-5}{s} \end{bmatrix}$, thus

$$G(s) = \frac{1}{s^2-5}\begin{bmatrix} 2s-4 & -s-1 \\ -s+3 & -s-3 \end{bmatrix}.$$

Its 2×2 minor is $\frac{-3}{s^2-5}$, thus its poles are $\sqrt{5}$ and $-\sqrt{5}$, and it does not have any zeros.

(b) For the first system, we have

$$\text{RGA}(0) = \begin{bmatrix} -1.5 & 2.5 \\ 2.5 & -1.5 \end{bmatrix}$$

$$\text{RGA}(2i) = \begin{bmatrix} -1.5-i & 2.5+i \\ 2.5+i & -1.5-i \end{bmatrix}$$

Thus, for the first system we would choose the pairing $1-2$ and $2-1$ since we want to avoid negative elements at $\omega = 0$.

For the second system, we have

$$\text{RGA}(0) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\text{RGA}(2i) \approx \begin{bmatrix} 0.74-0.15i & 0.26+0.15i \\ 0.26+0.15i & 0.74-0.15i \end{bmatrix}$$

Thus, for the second system we would choose the pairing $1-1$ and $2-2$, since we want the elements to be close to 1 at ω_c.

(c) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 & 3 & 5 & 5 \\ -1 & 2 & 1 & 4 & -5 & 10 \\ 1 & -2 & -1 & -4 & 5 & -10 \end{bmatrix}$$

which has rank 2, thus the system is not controllable.

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -2 & 0 \\ 5 & -5 & 0 \\ -5 & 0 & 0 \end{bmatrix}$$

which has rank 2, thus the system is not observable.

(d) The easiest way of choosing the stationary decoupling matrix is as

$$W = G(0)^{-1} = \left(\frac{1}{2}\begin{bmatrix} 1 & \frac{1}{3} \\ 2 & \frac{2}{5} \end{bmatrix}\right)^{-1} = \begin{bmatrix} -3 & 2.5 \\ 15 & -7.5 \end{bmatrix}.$$

We are now looking for a controller $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$, such that the diagonal elements of $G(2i) \cdot W \cdot F$ have magnitude 1.

$$G(2i) \cdot W \approx \begin{bmatrix} 0.92 - 1.62i & 0.29 + 1.06i \\ -0.41 - 1.03i & 1.21 + 0.52i \end{bmatrix}$$

thus $f_1 \approx \frac{1}{|0.92-1.62i|} \approx 0.54$ and $f_2 \approx \frac{1}{|1.21+0.52i|} \approx 0.76$.

Problem 3

(a) The transfer function from n to z is the complementary sensitivity function, thus V is equal to:

$$V(s) = \frac{G(s)F_y(s)}{1 + G(s)F_y(s)}.$$

To satisfy the imposed constraints, we need to select the weight function (ii). We have:

$$|V(i\omega)| \leq |W(i\omega)|^{-1} = \frac{1}{\sqrt{1/M^2 + \omega^2/\omega_0^2}}.$$

To have $|V(i\omega)| \leq 2$ for $\omega \leq 1$, we can select $M = 2$. To satisfy the second constraint, we can choose $\omega_0 = 5$ rad/s.

Neither of the proposed weight functions can ensure that $|V(i\omega)| < 5/\omega^2$ since they grow at most linearly with ω, and the condition is quadratic.

(b) We have:

$$G(s) = \frac{(s-1)(s-4)}{s(s-2)^2}.$$

Hence there are two RHP zeros at 1 and 4, respectively. This implies that the cross-over frequency should be roughly smaller than 4/2 = 2 rad/s. There is also a RHP pole at 2, so that the cross-over has to be larger than 4 rad/s. We deduce that it will be hard to design a controller meeting the requirements.

(c) (c-1) S is the sensitivity function, and corresponds to the input-ouput relationship from w (the disturbance) to z (the output). GF_yS corresponds to the input-output relationship from n to z. Hence by imposing a restriction on the norm of S and GF_yS, we wish to design a controller that efficiently reject disturbances, and for which there is relatively small impact of the measurement noise n on the output z.
(c-2) M_S corresponds to the maximum amplitude of the sensitivity function S. Thus to decrease the sensitivity to disturbances, we should decrease M_S.
(c-3) The transfer function from w to u is $-SF_y$, and hence if we want to limit the impact of w on u, we should impose a limitation on $-SF_y$. This can be done by designing a weight function W for SF_y that is large for frequencies up to 4 rad/s, and modify the $\mathcal{H}_\infty$ control framework so that the controller minimizes:

$$\left\| \begin{array}{c} W_S S \\ W_T T \\ W S F_y \end{array} \right\|_\infty .$$

Problem 4

(a) The open-loop system is stable iff $a > 0$.

(b) In the following, we use the same notations as those used in the course. We have here a first-order system, $n = 1$. We have: $A = -a$, $B = b$, $N = 1$, $C = 2$, $M = 1/2$, $R_1 = 1$, $R_2 = 2$, and $R_{12} = \beta$. To use Kalman filter to derive the optimal observer, we need to verify the required assumptions:

- R_2 is symmetric and positive definite. Here $R_2 = 2$, and this assumption holds.
- $\tilde{R}_1 = R_1 - R_{12}R_2^{-1}R_{12}^T$ is positive definite. Here $\tilde{R}_1 = 1 - \beta^2/2$. Hence the assumption holds iff $\beta \leq \sqrt{2}$.
- (A, C) is detectable, which means that there exists K such that $A - KC$ is stable. Here $A - KC = -a - 2K$. The assumption holds for the choice $K = 0$ for example.
- $(A - R_{12}R_2^{-1}C, \tilde{R}_1)$ is stabilizable, which means that there exists L such that $A - R_{12}R_2^{-1}C - \tilde{R}_1 L$ is stable. Here we have:

  $$A - R_{12}R_2^{-1}C - \tilde{R}_1 L = -a - \beta - (1 - \beta^2/2)L.$$

  Again the assumption holds for $L = 0$.

Since, for $\beta \leq \sqrt{2}$, we can use Kalman filter to determine the optimal observer, the latter is given by:

$$\dot{\hat{x}} = -a\hat{x} + bu + K(y - 2\hat{x}),$$

where $K = p + \beta/2$ and p is the positive solution of Riccati's equation:

$$-2ap - (2p + \beta)^2/2 + 1 = 0.$$

We obtain:

$$p = \frac{1}{2}\left[-a - \beta + \sqrt{(a + \beta)^2 + 2 - \beta^2}\right].$$

The estimation error $\tilde{x} = x - \hat{x}$ satisfies:

$$\begin{aligned}\dot{\tilde{x}} &= -\sqrt{(a + \beta)^2 + 2 - \beta^2}\,\tilde{x} + v_1 - Kv_2 \\ &= -\sqrt{a^2 + 2a\beta + 2}\,\tilde{x} + v_1 - Kv_2.\end{aligned}$$

Hence, when β increases, the speed at which the estimation error tends to 0 also increases.

(c) The optimal state feedback controller is $u = -Lx$. Here we have $L = bS/\rho$ where $S \geq 0$ solves the Riccati's equation:

$$-2aS + \frac{1}{4} - S^2b^2/\rho = 0.$$

We obtain:

$$L = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} + \frac{1}{4\rho}}.$$

From the separation principle, we know that the solution of the optimization problem combines the optimal observer and the optimal state feedback controller, as computed above.

(d) ρ has no impact on the state estimation error. When ρ increases, the cost of manipulating the control signal u is increased, which is at the expense of the controller accuracy – the latter decreases.

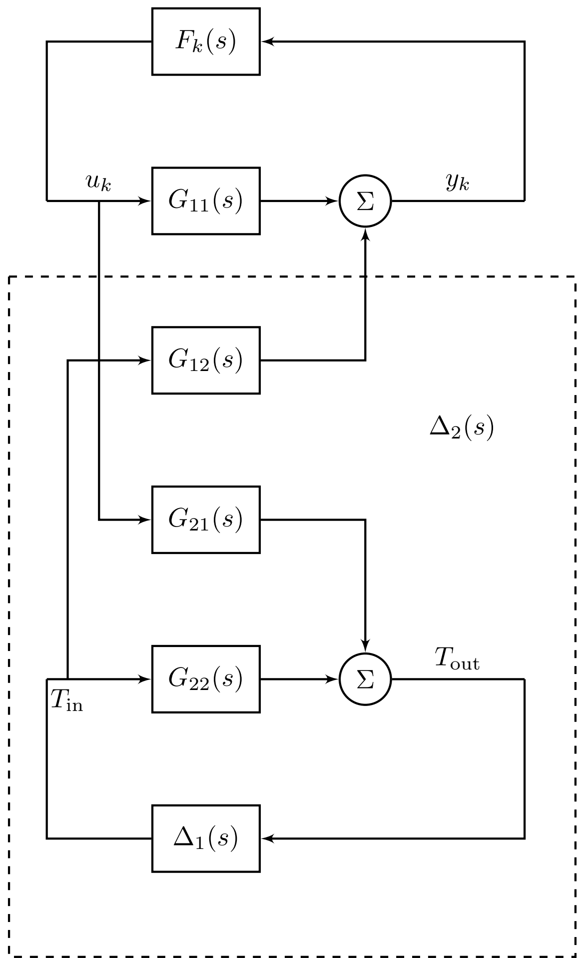

Figure 1: SISO block version

Problem 5

(a) One way to find the wanted expression is to redraw the block diagram using only SISO blocks as in Figure 1. It is then easy to identify $\Delta_2(s)$, and finding an expression reduces to some block-diagram calculations. The part of the block diagram inside the dashed box then correspond to $\Delta_2(s)$. The transfer function $\Delta_2(s)$ from u_k to the sum near y_k is easily seen to be

$$\Delta_2(s) = G_{12}(s)\Delta_1(s)(1 - G_{22}(s)\Delta_1(s))^{-1}G_{21}(s)$$

(b) We begin by deriving the transfer function from x to u_k. This is

$$U_k(s) = F_k(s)(1 - G_{11}(s)F_k(s))^{-1}X(s)$$

Hence, the small gain theorem states that the system is robustly stable if

$$\|\Delta_2(s)\| \|F_k(s)(1 - G_{11}(s)F_k(s))^{-1}\| < 1$$

assuming that the nominal closed loop is stable and that $\Delta_2(s)$ is stable.

(c) By the definition of the norm and the submultiplicative property of the norm we get

$$\|\Delta_2\| \leq 0.2 \cdot \sup_\omega |\Delta_1(i\omega)| \cdot \sup_\omega |(1 - \Delta_1(i\omega))^{-1}| \cdot 0.99.$$

We also have that $\sup_\omega |\Delta_1(i\omega)| < 0.5$ so in the worst case we get $\sup_\omega |(1 - \Delta_1(i\omega))^{-1}| = (1 - 0.5)^{-1} = 2$. Thus we find

$$\|\Delta_2\| \leq 0.2 \cdot 0.99$$

For the proportional controller we get

$$\|F_k(s)(1 - G_{11}(s)F_k(s))^{-1}\| = \left\| \frac{K}{1 - \frac{K}{s+1}} \right\| = |K| \left\| \frac{s+1}{s+1-K} \right\|$$

We calculate the norm

$$\left\| \frac{s+1}{s+1-K} \right\| = \sup_\omega \left| \frac{i\omega + 1}{i\omega + 1 - K} \right| = \sup_\omega \sqrt{\left| \frac{\omega^2 + 1}{\omega^2 + (1-K)^2} \right|}$$

since the ω minimizing the norm will also minimize the square of the norm we may remove the square-root, differentiate with respect to ω and equate to zero in order to find the critical ω

$$\frac{2\omega(K^2 - 2K)}{(\omega^2 + (1-K)^2)^2} = 0$$

From this we conclude that we have a critical point at $\omega = 0$ for all K and for $K = 0$ or $K = 2$ all ω are critical. However, if we look carefully we see that we have several cases. When $0 \leq K \leq 2$, then $\omega = 0$ is a maximum such that

$$\left\| \frac{s+1}{s+1-K} \right\| = \sup_\omega \sqrt{\left| \frac{0+1}{0 + (1-K)^2} \right|} = \frac{1}{|1-K|}$$

but if $K < 0$ or $K > 2$, then $\omega = 0$ is a minimizer and to find the supremum we must look at the limit when $\omega \to \infty$. Thus we find that in these cases we have

$$\left\| \frac{s+1}{s+1-K} \right\| = \lim_{\omega \to \infty} \sqrt{\left| \frac{\omega^2 + 1}{\omega^2 + (1-K)^2} \right|} = 1$$

By inserting this into our stability criterion we get for $0 \leq K \leq 2$

$$0.2 \cdot 0.99 \frac{|K|}{|1-K|} < 1$$

which is fulfilled for $0 \leq K < \frac{1}{1+0.2 \cdot 0.99}$ and for $\frac{1}{1-0.2 \cdot 0.99} < K \leq 2$.

In the case of $K < 0$ our criterion becomes

$$0.2 \cdot 0.99|K| < 1$$

which is true as long as $\frac{-1}{0.2 \cdot 0.99} < K < 0$. For $K > 2$ we get

$$0.2 \cdot 0.99|K| < 1$$

which is fulfilled whenever $2 < K < \frac{1}{0.2 \cdot 0.99}$

We conclude that the system is robustly stable for $\frac{-1}{0.2 \cdot 0.99} < K < \frac{1}{1+0.2 \cdot 0.99}$ and $\frac{1}{1-0.2 \cdot 0.99} < K < \frac{1}{0.2 \cdot 0.99}$.