REGLERTEKNIK

School of Electrical Engineering, KTH

EL2520 Control Theory and Practice – Advanced Course

Exam (tentamen) 2015–08–18, kl 08.00–13.00

Aids: The course book for EL2520 (advanced course) and EL1000/EL1100 (basic course), copies of slides from this year's lectures and of the blackboard notes, mathematical tables and pocket calculator.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be motivated.
Unjustified answers will results in point deductions.
Write a clear answer to each question.
Write name and personal number on each page.
Please use only one side of each sheet.
Mark the total number of pages on the cover.

The exam consists of five problems. The distribution of points for the various problems and subproblems is indicated.

Grading: Grade A: ≥ 43, Grade B: ≥ 38,
Grade C: ≥ 33, Grade D: ≥ 28,
Grade E: ≥ 23, Grade Fx: ≥ 21.

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Results: Will be posted no later than September 4, 2015.

Good Luck!

Problem 1

a) What is the $\mathcal{H}_2$-norm of the system characterized by the transfer function

$$G(s) = \frac{2}{s+1}?$$

[3 pts]

b) Let a system be defined through its state-space representation

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 \\ \alpha & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0.5 & 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 2 \\ 1 & 0.5 \end{bmatrix} x. \end{aligned}$$

Derive its transfer matrix. For which values of α is this system stable? [4 pts]

c) Compute the L_∞ and the L_2 norms of the signal

$$g(t) = \frac{\sin t}{t} \cdot 1_{t>0},$$

where $1_{t>0}$ is the indicator function which is 1 whenever $t > 0$ and 0 otherwise. *Hint:* $\int_0^\infty \frac{sin(bx)}{x} dx = \frac{\pi}{2}$ *for any* $b > 0$ *and differentiate the integral* $\int_0^\infty \frac{sin^2(bx)}{x^2} dx$ *with respect to* b. [3 pts]

Problem 2

a) Determine the poles and zeros of the following system:

$$G(s) = \begin{bmatrix} \frac{2s^2}{s+1} & \frac{s}{s+2} \\ \frac{2s}{s+2} & \frac{1}{s} \end{bmatrix}.$$

[2 pts]

b) Determine the poles of the following system:

$$\dot{x} = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 14 \\ 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x.$$

[1 pt]

c) Now consider the system:

$$G(s) = \frac{1}{s} \begin{bmatrix} \frac{1}{2s+2} & \frac{10}{s+10} \\ \frac{10}{s+10} & \frac{1}{s+5} \end{bmatrix}.$$

We would like to control this system with a decentralized controller such that the crossover frequency is $\omega_c = 10$ rad/s.

(i) Determine a suitable pairing using the RGA method. [2 pts]

(ii) Using the decoupling matrix $W = G(s)^{-1}$ could be problematic, why? [1 pt]

(iii) Propose another decoupling matrix W for frequencies up to the desired crossover frequency, such that the system can be controlled by a decentralized controller $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$. Validate the decoupling by checking the RGA conditions for GW.

[1 pt]

(iv) Design a proportional decentralized controller $F = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$ for the decoupled system GW, such that the crossover frequency is $\omega_c = 10$ rad/s. What is the corresponding phase margin? Is the system stable? [2 pts]

(v) Derive a minimal state space representation for the controller, given by the transfer matrix WF. [1 pt]

Problem 3

a) We wish to design a closed-loop controller of the two following linear systems.
(i) The first system is characterized by the transfer function:

$$G(s) = \frac{s-2}{s^3 - 4s^2 + s + 6},$$

(ii) whereas for the second system, we have the following minimal state-space representation:

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

Determine whether these systems have bandwidth limitations. [5 pts]

b) We aim at devising a feedback controller that stabilizes all plants whose transfer functions can be written as:

$$G_a(s) = G(s)\left(1 + \frac{a}{s+1}\right),$$

for some $a \in [-1, 1]$. G is stable and minimum phase. Propose a condition on the complementary sensitivity function T that the controller could satisfy to meet this goal. [2 pts]

c) Consider a plant G to be controlled using a feedback controller. The measurement of the output is corrupted by noise.
(i) The spectrum of this noise n is:

$$\phi_n(\omega) = \frac{16\omega^2}{49\omega^2 + 4}.$$

Model the disturbance of this noise as filtered white noise, i.e., $N(s) = F(s)E(s)$ where E is the transfer function of the white noise e (with spectrum $\phi_e(\omega) = 1$) and F is stable and minimum phase. Determine F. [2 pts]
(ii) Propose an optimization-based control framework to design a controller which rejects disturbances and is robust to measurement noise, according to some specified weight functions W_S and W_T, respectively, at all frequencies. [1 pt]

Problem 4

Consider the following linear system:

$$\dot{x} = Ax + Bu + Nv_1,$$

$$y = Cx + v_2, \quad z = Mx,$$

where v_1, v_2 are Gaussian white noises.

a) We compute controllers solving the following LQG problem:

$$\min_u E\left[\int_0^\infty \left(Q_1|z(t)|^2 + Q_2|u(t)|^2\right) \, \mathrm{d}t\right],$$

for some real numbers $Q_1, Q_2 > 0$, and where $E[\cdot]$ denotes the expectation. We solve the above problem with three sets of parameters.
Set 1: $Q_1 = 10.2$, $Q_2 = 5.1$.
Set 2: $Q_1 = 1$, $Q_2 = 10$.
Set 3: $Q_1 = 2$, $Q_2 = 1$.
For which set of parameters does the output z have minimum energy? [2 pts]

b) Assume that the system is of order one, and that $A = a$, $B = b$, $N = 1$, $C = 3$, and $M = 1$. Further assume that the covariance matrix of $\binom{v_1}{v_2}$ is:

$$\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}$$

with $\gamma \geq 0$. We wish to find the controller that solves the following optimization problem (P):

$$\min_u E\left[\int_0^\infty \left(z(t)^2 + \rho u(t)^2\right) \, \mathrm{d}t\right],$$

where $\rho > 0$.
(i) Determine the conditions under which the Kalman filter provides an optimal observer. Derive this observer. [3 pts]
(ii) Determine the optimal state-feedback controller and solve the problem (P). [3 pts]
(iii) Explain the impact of the choice of ρ. What happens when ρ becomes very large? Discuss the impact of the noise correlation γ. [2 pts]

Problem 5

a) Consider the continuous-time system:

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} x.$$

Construct the discrete time system

$$x_{k+1} = Fx_k + Gu_k$$
$$y_k = Hx_k$$

by sampling the original system with interval $T = 0.5$ s. [2 pts]

b) Is the original continuous-time system stable? Is the sampled system stable? [1 pt]

c) Consider the optimal control problem:

$$\min_{u_0,u_1,u_2} \sum_{k=0}^{2} (x_k^T Q x_k + u_k^T R u_k) + x_3^T P x_3$$

subject to

$$x_{k+1} = Ax_k + Bu_k, \quad \forall k.$$

Let $\mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}$. Rewrite the above problem in the form

$$\min_{\mathbf{u}} \mathbf{u}^T S \mathbf{u} + h^T \mathbf{u} + c,$$

i.e., determine S, h and c. [3 pts]

d) With $A = 1$, $B = 2$, $Q = 2$, $R = 5$, $P = 4$, solve the optimal control problem. [3 pts]

e) By applying just the first element of $\mathbf{u}$, and then recomputing the optimal control problem, will the closed loop system be stable? [1 pt]