
Question1 (6 points)

Consider the following model

$$\begin{aligned} X_i \mid Y_i &\overset{i.i.d}{\sim} \text{Poisson}\,(y_i) \\ Y_i \mid \alpha, \beta &\overset{i.i.d}{\sim} \text{Gamma}\,(\alpha, \beta) \\ \alpha &\sim \text{Exp}\,(a) \\ \beta &\sim \text{Gamma}\,(b, c) \end{aligned}$$

where a, b and c are treated as fixed constants.

(a) (2 points) Let $\mathbf{Y}$ and $\mathbf{X}$ denote the following random vectors, respectively

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Write down the joint posterior $f_{\{\mathbf{Y},\alpha,\beta\}|\mathbf{X}}$ up to an multiplicative constant.

(b) (1 point) State the assumption(s) of independence that you used in part (a).

(c) (2 points) Find the conditional posterior $f_{\{\mathbf{Y},\beta\}|\{\mathbf{X},\alpha\}}$.

(d) (1 point) Find the marginal posterior $f_{\{\alpha,\beta\}|\mathbf{X}}$ up to an multiplicative constant.

Question2 (4 points)

Suppose tiger groupers' length and weight follow a normal distribution with

$$\boldsymbol{\mu}_t = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \boldsymbol{\Sigma}_t = \begin{bmatrix} 12 & 1 \\ 1 & 2 \end{bmatrix}$$

while greasy groupers' length and weight also follow a normal distributions but with

$$\boldsymbol{\mu}_g = \begin{bmatrix} 13 \\ 10 \end{bmatrix}; \quad \boldsymbol{\Sigma}_g = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Given there are twice as many tiger groupers in the tank, and the loss function is

$$C = \begin{cases} 0.1, & \text{greasy grouper classified as tiger grouper,} \\ 0.9 & \text{tiger grouper classified as greasy grouper,} \\ 0 & \text{correctly classified.} \end{cases}$$

(a) (2 points) Write down the optimal decision rule in terms of a pair of observed length and weight.

(b) (1 point) Plot the decision boundary of the optimal decision rule under this loss function.

(c) (1 point) Clearly state inside which region you would classify the fish as a tiger grouper and vice versa. Note the decision boundary is no longer a point in the real line.