VE414 Appendix 5

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- Recall we use the following notation when changing variables from x to u

$$u = g(x) \implies f(x) = f(g^{-1}(u)) \\ = \left(f \circ g^{-1}\right)(u)$$

- Often people abuse the notation in practice by writing it as

$$\left(f \circ g^{-1}\right)(u) = f(u)$$

which is not true, but we usually understand it correctly from the context.

- However, let us use the proper notation in the next few discussion

$$\left(f \circ g^{-1}\right)(u)$$

- We are considering changing random variables in a PDF, not the parameters,

$$f_X(x)$$

- When changing variables that to do with PDF, there is a small complication,

$$\int_{-\infty}^{x} f_X(t)\,dt \neq \int_{-\infty}^{u} \left(f \circ g^{-1}\right)(s)\,ds \qquad \text{where } u = g(x)$$

  which means we cannot make a simple substitution and expect

$$U \text{ and } \left(f \circ g^{-1}\right)(u)$$

  to represent the same random process that are defined X and $f_X(x)$.

- In order to make sure that the new PDF $f_U(u)$ has the property

$$\int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{u} f_U(s)\,ds$$

- We invoke the u-substitution formula from elementary calculus to define

$$f_U(u)$$

- Recall we have the following for smooth functions $f(x)$ and $g(x)$ in general,

$$\int_a^b f(x)g'(x)\,dx = \int_{g(a)}^{g(b)} \left(f \circ g^{-1}\right)(u)\,du \qquad \text{where } u = g(x)$$

- Alternatively, you should have also seen the following related formula

$$\int_a^b \frac{f(x)}{g'(x)} g'(x)\,dx = \int_{g(a)}^{g(b)} \left(\frac{f}{g'} \circ g^{-1}\right)(u)\,du$$

from which, you might be eager to jump to a conclusion, but the new PDF is

$$f_U(u) = \left(\frac{f}{|g'|} \circ g^{-1}\right)(u)$$

- For a joint distribution of several random variables, the Jacobian needs to be used instead of the first derivative, but the essential idea remain valid.

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