VE414 Appendix 8

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Invariant distribution of Gibbs Sampling

Proof

Assume that $\mathbf{Y}^{(t-1)} \sim f_{\mathbf{Y}}$, then

$$\Pr\left(\mathbf{Y}^{(t)} \in \mathcal{Y}\right) = \int_{\mathcal{Y}} \underbrace{\int_{\mathcal{D}} \kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right) \cdot f_{\mathbf{Y}}\left(\mathbf{y}^{(t-1)}\right)\, d\mathbf{y}^{(t-1)}}_{I}\, d\mathbf{y}^{(t)}$$

Notice I is a p-dimensional integral, and can be iteratively presented

$$I = \int_{\mathcal{D}_p} \int_{\mathcal{D}_{p-1}} \cdots \int_{\mathcal{D}_1} \kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right) \cdot f_{\mathbf{Y}}\left(\mathbf{y}^{(t-1)}\right)\, dy_1^{(t-1)}\, dy_2^{(t-1)} \cdots dy_p^{(t-1)}$$

Recall the transition kernel $\kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right)$ actual does not depend on $y_1^{(t-1)}$,

$$f_{Y_1|Y_{-1}}\left(y_1^{(t)} \mid y_2^{(t-1)}, \ldots y_p^{(t-1)}\right) \cdot f_{Y_2|Y_{-2}}\left(y_2^{(t)} \mid y_1^{(t)}, y_3^{(t-1)} \ldots y_p^{(t-1)}\right) \cdots$$
$$\cdots \cdot f_{Y_p|Y_{-p}}\left(y_p^{(t)} \mid y_1^{(t)}, \ldots y_{p-1}^{(t)}\right)$$

Proof

So we start the integral with respect to $y_1^{(t-1)}$, and obtain the marginal

$$
\begin{aligned}
I &= \cdots \int_{\mathcal{D}_1} \kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right) f_{\mathbf{Y}}\left(\mathbf{y}^{(t-1)}\right) \, dy_1^{(t-1)} \cdots \\
&= \cdots \kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right) \int_{\mathcal{D}_1} f_{\{Y_1, Y_2, \ldots, Y_p\}}\left(\mathbf{y}^{(t-1)}\right) \, dy_1^{(t-1)} \cdots \\
&= \cdots \int_{\mathcal{D}_2} \kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right) f_{\{Y_2, \ldots, Y_p\}}\left(y_2^{(t-1)}, \ldots, y_p^{(t-1)}\right) \, dy_2^{(t-1)} \cdots
\end{aligned}
$$

Only the 1st term in the transition kernel $\kappa\left(\mathbf{y}^{(t-1)}, \mathbf{y}^{(t)}\right)$ depends on $y_2^{(t-1)}$,

$$
\begin{aligned}
f_{Y_1|Y_{-1}}\left(y_1^{(t)} \mid y_2^{(t-1)}, \ldots y_p^{(t-1)}\right) \cdot f_{Y_2|Y_{-2}}\left(y_2^{(t)} \mid y_1^{(t)}, y_3^{(t-1)} \ldots y_p^{(t-1)}\right) \cdots \\
\cdots \cdot f_{Y_p|Y_{-p}}\left(y_p^{(t)} \mid y_1^{(t)}, \ldots y_{p-1}^{(t)}\right)
\end{aligned}
$$

$$
\Longrightarrow I = \cdots \int_{\mathcal{D}_3} \kappa^* \cdot f_{\{Y_1, Y_3, \ldots, Y_p\}}\left(y_1^{(t)}, y_3^{(t-1)}, \ldots, y_p^{(t-1)}\right) \, dy_3^{(t-1)} \cdots
$$

Proof

Only the 1st term of κ^* depends on $y_3^{(t-1)}$,

$$f_{Y_2|Y_{-2}}\left(y_2^{(t)} \mid y_1^{(t)}, y_3^{(t-1)} \dots y_p^{(t-1)}\right) \cdot \underbrace{f_{Y_3|Y_{-3}}\left(y_3^{(t)} \mid y_1^{(t)}, y_2^{(t)}, y_4^{(t-1)} \dots y_p^{(t-1)}\right) \cdots f_{Y_p|Y_{-p}}\left(y_p^{(t)} \mid y_1^{(t)}, \dots y_{p-1}^{(t)}\right)}_{\kappa^{**}}$$

Combining this first term of κ^* with the marginal, we have the following joint

$$f_{Y_2|Y_{-2}} \cdot f_{\{Y_1,Y_3,\dots,Y_p\}} = f_{\{Y_1,Y_2,Y_3,\dots,Y_p\}}$$

from which, the integration can be continued in the same fashion as before,

$$\begin{aligned} I &= \cdots \int_{\mathcal{D}_3} \kappa^* \cdot f_{\{Y_1,Y_3,\dots,Y_p\}}\left(y_1^{(t)}, y_3^{(t-1)}, \dots, y_p^{(t-1)}\right) dy_3^{(t-1)} \cdots \\ &= \cdots \kappa^{**} \int_{\mathcal{D}_3} f_{\{Y_1,Y_2,Y_3,\dots,Y_p\}}\left(y_1^{(t)}, y_2^{(t)}, y_3^{(t-1)}, \dots, y_p^{(t-1)}\right) dy_3^{(t-1)} \cdots \end{aligned}$$

Proof

Continuing in this fashion,

$$I = \cdots \int_{\mathcal{D}_4} \kappa^{**} f_{\{Y_1, Y_2, Y_4, \ldots, Y_p\}} \left(y_1^{(t)}, y_2^{(2)}, y_4^{(t-1)}, \ldots, y_p^{(t-1)} \right) \, dy_4^{(t-1)} \cdots$$

we can successively compute one component at a time, and reach the following

$$I = f_{\{Y_1, Y_2, \ldots, Y_p\}} \left(y_1^{(t)}, y_2^{(t)}, \ldots, y_p^{(t)} \right) = f_{\mathbf{Y}} \left(\mathbf{y}^{(t)} \right)$$

Therefore,

$$\begin{aligned} \Pr\left(\mathbf{Y}^{(t)} \in \mathcal{Y} \right) &= \int_{\mathcal{Y}} I \, d\mathbf{y}^{(t)} \\ &= \int_{\mathcal{Y}} f_{\mathbf{Y}} \left(\mathbf{y}^{(t)} \right) \, d\mathbf{y}^{(t)} \end{aligned}$$

from which we conclude $\mathbf{Y}^{(t)} \sim f_{\mathbf{Y}}$.

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