VE414 Appendix 6

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Law of total expectation

Let $f_{X,Y}(x,y)$ be the joint probability mass/density function of X and Y, and

$$g(x,y)$$

be a smooth function of x and y, and the following expectations exist, then

$$\mathbb{E}\big[g\left(X,Y\right)\big] = \mathbb{E}\big[\mathbb{E}\left[g\left(X,Y\right) \mid Y\right]\big] = \mathbb{E}\big[\mathbb{E}\left[g\left(X,Y\right) \mid X\right]\big]$$

Proof

- Suppose X and Y are continuous, then by definition of expectation, we have

$$\begin{aligned}\mathbb{E}\big[g\left(X,Y\right)\big] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x,y\right) f_{X,Y}(x,y)\, dx\, dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g\left(x,y\right) f_{X|Y}(x \mid y) \cdot f_Y(y)\, dx\, dy\end{aligned}$$

  where the definition of condition PDF, and marginal PDF are used.

Proof

- Since f_Y does not depend on y,

$$\begin{aligned}\mathbb{E}\big[g(X,Y)\big] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x \mid y)\cdot f_Y(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x \mid y)\,dx\right)\cdot f_Y(y)\,dy\end{aligned}$$

- By the definition of conditional expectation, we have

$$\mathbb{E}\big[g(X,Y)\big] = \int_{-\infty}^{\infty} \mathbb{E}\left[g(X,Y) \mid Y\right]\cdot f_Y(y)\,dy = \mathbb{E}\big[\mathbb{E}\left[g(X,Y) \mid Y\right]\big]$$

- Of course, we could condition on X, and reach the following

$$\mathbb{E}\big[g(X,Y)\big] = \mathbb{E}\big[\mathbb{E}\left[g(X,Y) \mid X\right]\big]$$

- If X or Y is discrete, we would have summations instead of integrations in the above steps, but the idea is essentially the same. **Back**