VE414 Appendix 2

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Random Variable

Definition

Let Ω be a set of all possible outcomes, then a random variable X is a function

$$X : \Omega \to \mathcal{A} \qquad \text{where} \quad \mathcal{A} \subset \mathbb{R}$$

that assigns a real value $X(o)$ to each outcome $o \in \Omega$.

- Given a random variable X is defined for a random phenomenon, each time the phenomenon gives out an outcome, we obtain a particular real value.
- These values are called the realisations of the random variable X.
- When the range of X is finite or countably infinite, the random variable X is called a discrete random variable. If the range of X is uncountably infinite, then X is called a continuous random variable.

Probability mass function

Definition

Let Ω be a sample space, and $\mathcal{A} \subset \mathbb{R}$. Suppose

$$X : \Omega \to \mathcal{A}$$

is a *discrete random variable*, then the probability mass function (PMF)

$$f_X : \mathcal{A} \to [0, 1]$$

is a function such that

$$f_X(x) = \Pr(X = x)$$

- The PMF is a way to *characterise* a discrete random variable, loosely speaking, that is, it gives a description of how the "masses" are distributed.

Probability density function

Definition

Let Ω be a sample space, and $\mathcal{A} \subset \mathbb{R}$. Suppose

$$X : \Omega \to \mathcal{A}$$

is a *continuous random variable*, then the probability density function (PDF)

$$f_X : \mathcal{A} \to [0, \infty)$$

is a non-negative function such that

$$\int_{-\infty}^{x} f_X(u)\, du = \Pr(X \leq x)$$

- Similar to the PMF that *characterises* a discrete random variable, The PDF is a way to *characterise* a continuous random variable, but they differ in...

Cumulative distribution function

Definition

Let $X\colon \Omega \to \mathcal{A}$ for $\mathcal{A} \subset \mathbb{R}$ be random variable defined on a sample space Ω, then

$$F_X(x) = \Pr(X \le x)$$

is known as the cumulative distribution function of X.

- The cumulative distribution function (CDF) is closely relative to PMF/PDF:

$$F_X(x) = \begin{cases} \displaystyle\sum_{u \le x} f_X(u) & \text{if } X \text{ is discrete,} \\ \displaystyle\int_{-\infty}^{x} f_X(u)\,du & \text{if } X \text{ is continuous.} \end{cases}$$

- One need define f_X to cover all $x \in \mathbb{R}$ by defining $f_X(x) = 0$ for all $x \notin \mathcal{A}$.

- For two random variables X and Y, the following function

$$F_{X,Y}(x,y) = \Pr\left(X \leq x, Y \leq y\right)$$

is known as the joint cumulative distribution.

- If both X and Y are discrete, then the joint probability mass function is

$$f_{X,Y}(x,y) = \Pr\left(X = x, Y = y\right) \implies F_{X,Y}(x,y) = \sum_{u \leq x} \sum_{v \leq y} f_{X,Y}(x,y)$$

- If both X and Y are continuous, then the joint probability density function is

$$f_{X,Y} = \frac{\partial^2}{\partial x \partial y} F_{X,Y}\left(x,y\right) \implies F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du\, dv$$

- If X is continuous but Y is discrete, then

$$F_{X,Y}(x,y) = \sum_{v \leq y} \int_{-\infty}^{x} f_{X,Y}(u,v)\, du$$

Marginal and Conditional

- In the context of two random variables X and Y, the distribution of X is known as the marginal distribution of X, which can be characterised by

$$f_X(x) = \begin{cases} \displaystyle\sum_y f_{X,Y}(x,y) & \text{if } Y \text{ is discrete,} \\ \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy & \text{if } Y \text{ is continuous.} \end{cases}$$

  which is known as the marginal PMF/PDF of X.

- Recall the conditional distribution of Y given $X = x$ is defined by

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad \text{where} \quad f_X(x) \neq 0$$

  which is known as the conditional PMF/PDF of Y given X.

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