VE414 Lecture 2

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- The updating or learning favour of Bayesian inference is particularly strong with Thomas Bayes' original study on the subject before Laplace refined the the idea of probability and formulated mathematical form of Bayes' theorem.

Thomas Bayes' original problem

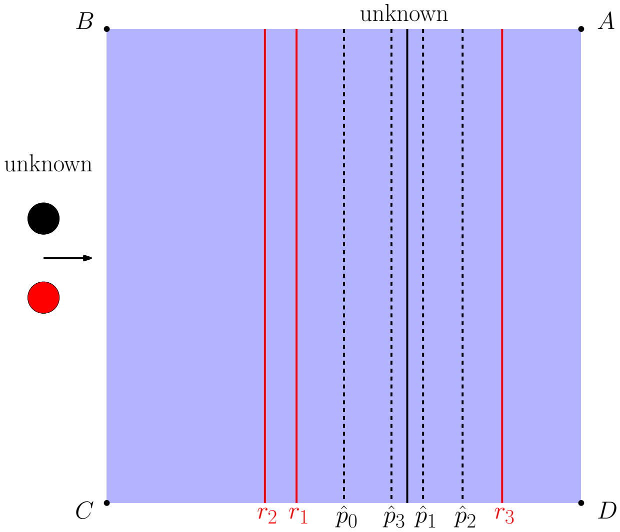
"*Given* the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named."

- In modern terms, it is about statistical inference on binomial proportion p,

$$X \sim \mathrm{Binomial}\,(k, p)$$

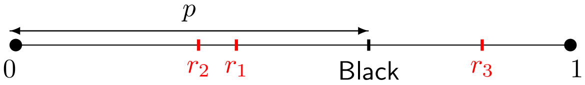
- **Bayes**[1] used a thought experiment to illustrate his process of inferencing.

[1] Thomas Bayes. "LII. An essay towards solving a problem in the doctrine of chances. By the late Rev. Mr. Bayes, FRS communicated by Mr. Price, in a letter to John Canton, AMFR S". In: *Philosophical transactions of the Royal Society of London* 53 (1763), pp. 370–418.



Q: How would you estimate where the black ball was?

- The only information available is whether the succeeding red balls are on the left of where the black ball was, but not the actual positions of the red balls.



- Notice the position of the black ball determines the chance of whether a red ball is on the left of it, this connection allows us to estimate where it was.
- If X_k denotes the number of red balls out of k number of trials that are on the left of the black ball, with the assumption of independence, then

$$X_k \sim \mathrm{Binomial}\,(k, p)$$

- In modern terms, we have identified the distribution from which the data are generated, thus the likelihood function, and the objective is to estimate p,

$$\mathcal{L}\,(p; x) = f_{X\mid p}\,(x \mid p) = \frac{k!}{x!(k-x)!}p^x(1-p)^{k-x}$$

- The maximum likelihood estimate is one of typical frequentist solutions

$$\tilde{\theta}_{\mathrm{MLE}} = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta; x)$$

$$\Longrightarrow \; \tilde{p}_k = \frac{x}{k}$$

where x denotes the number of success out of k number of trials.

- Having nice asymptotic properties is the main reason of using MLE; if there is very little data, it can be unreliable since it has no mechanism to take into account what could happen, it is entirely based on what has happened.
- Suppose $X_1 = 1$, $X_2 = 2$ and $X_3 = 2$ as depicted in the graph, then

$$\tilde{p}_1 = 1; \quad \tilde{p}_2 = 1 \quad \text{and} \quad \tilde{p}_3 = 2/3$$

Q: Do you see what I mean it does not take into account what could happen?

Q: Recall the Monty hall problem, can you guess what Bayes' estimates are?

Q: Have you ever questioned the definition of $f_{Y|X}$ when X is continuous?

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)} \qquad \text{where} \quad f_X(x) \neq 0$$

- The discrete case is clearly meaningful since PMF gives probability directly

$$f_{Y|X}(y \mid x) = \Pr(Y = y \mid X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

  and it follows directly from the modern concept of conditional probability.

- However, in the continuous case, PDF does not link to probability directly

$$f_{Y|X}(y \mid x) \neq \Pr(Y = y \mid X = x)$$

  and the event to condition on has 0 probability in the continuous case

$$\Pr(Y = y \mid X = x) = 0$$

- That said, it is meaningful and should be understood in the following sense

$$
\begin{aligned}
F_{Y|X}(y \mid x) &= \lim_{\varepsilon \to 0^+} \Pr\left(Y \le y \mid X \in (x, x+\varepsilon]\right) \\
&= \lim_{\varepsilon \to 0^+} \frac{\Pr\left(Y \le y, X \in (x, x+\varepsilon]\right)}{\Pr\left(x < X \le x+\varepsilon\right)} \\
&= \lim_{\varepsilon \to 0^+} \frac{F_{X,Y}(x+\varepsilon, y) - F_{X,Y}(x, y)}{F_X(x+\varepsilon) - F_X(x)} = \frac{\partial F_{X,Y}(x,y)/\partial x}{f_X(x)}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad f_{Y|X}(y \mid x) &= \frac{\partial}{\partial y} \frac{\partial F_{X,Y}(x,y)/\partial x}{f_X(x)} \\
&= \frac{1}{f_X(x)} \frac{\partial^2 F_{X,Y}(x,y)}{\partial y \partial x} = \frac{f_{X,Y}(x,y)}{f_X(x)}
\end{aligned}
$$

Q: What does this modern understanding mean in terms of Bayes' theorem?

$$
\Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)}
$$

- In additional to the Bayes' theorem for events, we have the following form

$$\Pr(Y = y \mid X = x) = \frac{\Pr(X = x \mid Y = y)\Pr(Y = y)}{\Pr(X = x)}$$

for discrete random variables, and the following form

$$f_{Y|X=x}(y) = \frac{f_{X|Y}(x \mid y)\, f_Y(y)}{f_X(x)} \propto \mathcal{L}(y; x)\, f_Y(y)$$

for PDF as well as PMF given our understanding of conditional distributions.

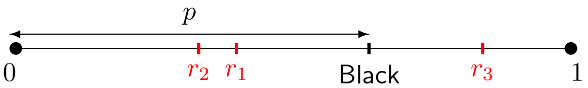
- Therefore, we often summarise various forms of Bayes' theorem simply as

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$ ✪

since $f_X(x)$ depends only on the observed $X = x$, not on the unknown y.

- It is clear from this form, identifying the prior and the likelihood is crucial.

- Back to the Bayes' original problem,



for which we have identified that the following should be the likelihood

$$\mathcal{L}\left(p;x\right) = f_{X|P}\left(x \mid p\right) = \frac{k!}{x!(k-x)!} p^x (1-p)^{k-x}$$

Q: What should we use as our prior distribution before seeing any data?

- Although not in the following modern formulation, Bayes essentially used

uniform distribution $\mathrm{Unif}\left(0,1\right)$: $\quad f_P(p) = \begin{cases} 1 & \text{for} \quad 0 \le p \le 1, \\ 0 & \text{otherwise.} \end{cases}$

and argued the prior should not prefer any particular $p \in [0,1]$ over another.

Q: Given the data $X_3 = 2$, the uniform prior and the likelihood

$$\mathcal{L}(p; x) = \frac{k!}{x!(k-x)!} p^x (1-p)^{k-x}$$

what is the posterior density function based on the data $X_3 = 2$?

- As you have leaned from any elementary courses on estimation/prediction, if a single value is required to describe or predict a random variable, the mean of the random variable is one of the best choices. Without any data,

$$\hat{p}_0 = \mathbb{E}[P] = 1/2$$

is used as our estimate since the uniform distribution is our prior in this case.

Q: Given the arrival of new information in the form of each data point

$$X_1 = 1, \quad X_2 = 2, \quad X_3 = 2$$

what should we use as our estimate of p?

- For point estimates, Bayes naturally suggested the mean of the posterior

$$\hat{p} = \mathbb{E}\left[p \mid X = x\right] = \int_{-\infty}^{\infty} p f_{P|X=x}(p)\, dp$$

- Given the data $X_1 = 1$, $X_2 = 2$ and $X_3 = 2$, the corresponding estimate

$$\hat{p}_0 = \frac{1}{2}; \quad \hat{p}_1 = \frac{2}{3}; \quad \hat{p}_2 = \frac{3}{4} \quad \text{and} \quad \hat{p}_3 = \frac{3}{5}$$

  alters with the information of the second ball in contrast to MLE

$$\tilde{p}_1 = 1; \quad \tilde{p}_2 = 1 \quad \text{and} \quad \tilde{p}_3 = 2/3$$

- Notice the Bayesian point estimator of p can be broken into two components

$$\hat{p}_k = w\frac{1}{2} + (1-w)\frac{x}{k} = w\hat{p}_0 + (1-w)\tilde{p}_k \qquad \text{where} \quad w = \frac{2}{2+k}$$

  which reflect the prior knowledge and the information from the data.

- No surprise in seeing the two types of point estimates converge in some sense

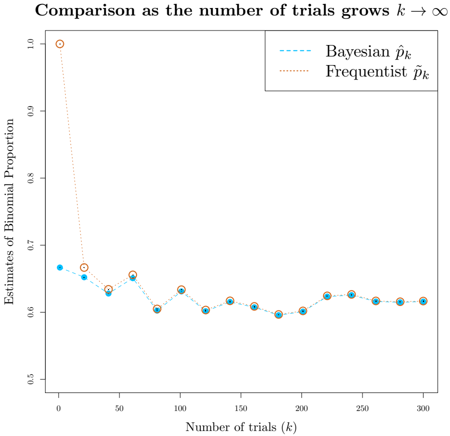


Figure: R Code: `freq_ci_414.R`