VE414 Appendix 4

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Independence

Definition

Two event $\mathcal{A}$ and $\mathcal{B}$ are independent if and only if

$$\Pr(\mathcal{A} \cap \mathcal{B}) = \Pr(\mathcal{A}) \Pr(\mathcal{B})$$

- The above definition leads to the following in terms of random variables:

Definition

Jointly distributed random variables X and Y are independent if their joint CDF

$$F_{X,Y}(x, y)$$

is the product of the marginal CDFs

$$F_{X,Y}(x, y) = F_X(x) F_Y(y)$$

- For discrete random variables, this definition is equivalent to

$$f_{X,Y}(x_i, y_j) = f_X(x_i) f_Y(y_j)$$

that is, the joint PMF being the product of the marginal PMFs,

$$\begin{aligned}
f_{X,Y}(x_i, y_j) &= F_{X,Y}(x_i, y_j) - F_{X,Y}(x_i, y_{j-1}) \\
&\quad - F_{X,Y}(x_{i-1}, y_j) + F_{X,Y}(x_{i-1}, y_{j-1}) \\
&= F_X(x_i)F_Y(y_j) - F_X(x_i)F_Y(y_{j-1}) \\
&\quad - F_X(x_{i-1})F_Y(y_j) + F_X(x_{i-1})F_Y(y_{j-1}) \\
&= \Big(f_X(x_i) + F_X(x_{i-1})\Big)\Big(f_Y(y_j) + F_Y(y_{j-1})\Big) \\
&\quad - \Big(f_X(x_i) + F_X(x_{i-1})\Big)F_Y(y_{j-1}) \\
&\qquad - F_X(x_{i-1})\Big(f_Y(y_j) + F_Y(y_{j-1})\Big) + F(x_{i-1})F(y_{j-1}) \\
&= f_X(x_i) f_Y(y_j)
\end{aligned}$$

- For continuous random variables, this definition is equivalent to

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

that is, the joint PDF being the product of the marginal PDFs,

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\Big(F_X(x)F_Y(y)\Big) = f_X(x)f_Y(y)$$

- When there are three or more events/random variables, say n of them,

$$\Pr\left(\bigcap_{i=1}^{n} \mathcal{A}_i\right) = \prod_{i=1}^{n} \Pr(\mathcal{A}_i) \quad \text{or} \quad F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

holds if and only if they are mutually independent, or simply as independent

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

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