VE414 Lecture 13

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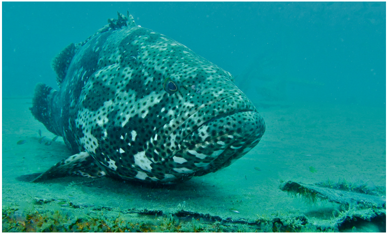
- Not to intentionally scrutinise a certain brand, but let us revisit our fish again



- Recall our analysis used the following made-up likelihoods

$$X_t \sim \mathrm{Normal}\,(5, 1) \qquad \text{and} \qquad X_g \sim \mathrm{Normal}\,(13, 9)$$

having observed $x = 7$, we pick the y value either maximises the likelihood

$$\mathcal{L}\,(y; 7) = (1 - y) \cdot f_{X_t}\,(7) + y \cdot f_{X_g}\,(7) \qquad \text{over} \quad y \in \{0, 1\}$$

or maximises the posterior $\mathcal{L}\,(y; 7) \cdot f_Y\,(y)$ over $y \in \{0, 1\}$.

- In practice, our understanding on the likelihood, e.g.

$$X \sim \mathrm{Normal}\left(\mu, \sigma^2\right)$$

are obtained previously using some other data which are not available now.

- Suppose someone in the seafood market measures the length of each fish

$$\{x_1, \ldots, x_n\}$$

which we assume to be independent. For simplicity, let us assume

$$\mu \sim \mathrm{Normal}\left(\nu, \omega^2\right)$$

where ν and ω^2 are treated as knowns. If σ^2 is also known, then

$$\mu \mid \bar{x} \sim \mathrm{Normal}\left(\frac{\omega^2\bar{x} + \nu\sigma^2/n}{\omega^2 + \sigma^2/n}, \frac{\omega^2\sigma^2/n}{\omega^2 + \sigma^2/n}\right), \quad \text{where} \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

can be obtained as we have done so in our discussion of conjugate priors.

- Of course, σ^2 is unavailable in practice, so let us model it as well as μ.
- Recall Jeffreys prior for unobserved Y and observed X is given by

$$f_Y(y) \propto \sqrt{I(y)}$$

where $I(y)$ is the variance of $\dfrac{\partial \ln \mathcal{L}(y; X)}{\partial y}$ conditional on y

$$I(y) = \mathrm{Var}\left[\left. \frac{\partial \ln \mathcal{L}(y; X)}{\partial y} \right| y \right] = -\mathbb{E}\left[\left. \frac{\partial^2 \ln \mathcal{L}(y; X)}{\partial y^2} \right| y \right]$$

- Given μ and $\{x_1, \ldots, x_n\}$, the Jeffreys prior of $\sigma \in (0, \infty)$ is given by

$$\varphi_{\sigma^2} \propto \sigma^{-2}$$

which is an improper prior proposed by Jeffreys

$$\int_0^\infty \frac{1}{\sigma^2} \, d\sigma^2 \to \infty$$

- Therefore, we consider the following model

$$\begin{aligned} X \mid \{\mu, \sigma^2\} &\sim \text{Normal}\left(\mu, \sigma^2\right) \\ \mu &\sim \text{Normal}\left(\nu, \omega^2\right) \\ \sigma^2 &\sim \varphi_{\sigma^2} \end{aligned}$$

for which the joint posterior of μ and σ^2 is given by

$$\begin{aligned} f_{\{\mu,\sigma^2\}|\{x_1,\ldots x_2\}} \propto& \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \\ & \cdot \frac{1}{\sqrt{2\pi\omega^2}} \exp\left(-\frac{(\mu-\nu)^2}{2\omega^2}\right) \cdot \frac{1}{\sigma^2} \\ \propto& \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2\right) - \frac{(\mu-\nu)^2}{2\omega^2}\right) \end{aligned}$$

Q: How can we sample from this posterior?

- Notice the sample mean $\bar{x}$ and sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

are sufficient to capture everything in the data for this model since

$$\begin{aligned}
f_{\{\mu,\sigma^2\}|\{x_1,\ldots x_2\}} &\propto \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2\right) - \frac{(\mu-\nu)^2}{2\omega^2}\right) \\
&= \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right) \\
&= \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) \\
&\qquad \cdot \exp\left(-\frac{(\mu-\nu)^2}{2\omega^2}\right)
\end{aligned}$$

- The prior μ and σ^2 are independent, but the posterior μ and σ are NOT.

- Since we are using an improper prior, we should double check whether

$$q_{\mu,\sigma^2} = \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right)$$

leads to any proper distribution before proceeding any further.

- The corresponding conditional distribution of μ is fine

$$\begin{aligned} f_{\mu|\{\sigma^2,\bar{x},s\}} = \frac{f_{\{\mu,\sigma^2\}|\{\bar{x},s\}}}{f_{\sigma^2|\{\bar{x},s\}}} &\propto f_{\{\mu,\sigma^2\}|\{\bar{x},s\}} \\ &\propto q_{\mu,\sigma^2} \\ &\propto \exp\left(-\frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right) \end{aligned}$$

since it is identical to what we had in our discussion of conjugate priors,

$$\mu \mid \{\sigma^2, \bar{x}, s^2\} \sim \mathrm{Normal}\left(\frac{\omega^2\bar{x} + \nu\sigma^2/n}{\omega^2 + \sigma^2/n}, \frac{\omega^2\sigma^2/n}{\omega^2 + \sigma^2/n}\right)$$

- Thus, if the marginal

$$
\begin{aligned}
f_{\sigma^2|\{\bar{x},s^2\}}\left(\sigma^2 \mid \bar{x}, s^2\right) &= \int_{-\infty}^{\infty} f_{\{\mu,\sigma^2\}|\{\bar{x},s^2\}}\left(\mu, \sigma^2 \mid \bar{x}, s^2\right) \, d\mu \\
&\propto \int_{-\infty}^{\infty} q_{\mu,\sigma^2}\left(\mu, \sigma^2\right) \, d\mu = J(\sigma^2)
\end{aligned}
$$

is a proper distribution, that is, $\int_0^{\infty} J(\sigma^2) \, d\sigma^2 < \infty$, then the joint is proper

$$
f_{\{\mu,\sigma^2\}|\{\bar{x},s^2\}} = f_{\mu|\{\sigma^2,\bar{x},s^2\}} \cdot f_{\sigma^2|\{\bar{x},s^2\}}
$$

- Integrating over μ boils down to the following

$$
\begin{aligned}
J(\sigma^2) &= \int_{-\infty}^{\infty} \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right) d\mu \\
&= \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\beta\right) \, d\mu
\end{aligned}
$$

- Completing the following square

$$\beta = -\frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2} = a\mu^2 + b\mu + c$$

where

$$a = -\frac{n}{2\sigma^2} - \frac{1}{2\omega^2}; \qquad b = \frac{n\bar{x}}{\sigma^2} + \frac{\nu}{\omega^2}; \qquad c = -\frac{n\bar{x}^2}{2\sigma^2} - \frac{\nu^2}{2\omega^2}$$

we have

$$\beta = a\cdot(\mu-h)^2 + k \qquad \text{where} \quad h = -\frac{b}{2a} \quad \text{and} \quad k = c - \frac{b^2}{4a}$$

which gives the following

$$\begin{aligned} J(\sigma^2) &= \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} + k\right) \int_{-\infty}^{\infty} \exp\left(-\frac{(\mu-h)^2}{2\cdot(-1/(2a))}\right) d\mu \\ &= \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} + k\right) \cdot \sqrt{-\frac{\pi}{a}} \end{aligned}$$

- Rearranging in the following way,

$$J(\sigma^2) = \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} + k\right) \cdot \sqrt{-\frac{\pi}{a}}$$
$$= \left(\sigma^2\right)^{-(1+n/2)} \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n\left(\nu - \bar{x}\right)^2}{2\left(n\omega^2 + \sigma^2\right)}\right) \sqrt{\frac{2\pi\omega^2\sigma^2}{n\omega^2 + \sigma^2}}$$

  and applying the comparison test with the following

$$f_Z\left(z\right) \propto \left(z\right)^{-(1+n/2)} \exp\left(-\frac{ns^2}{2z}\right)$$

  which is a proper distribution, namely, scaled inverse chi-squared distribution

$$Z \sim \text{Scaled Inverse}\, \chi^2\left(n, s^2\right)$$

- Therefore, $\int_0^\infty J(\sigma^2)\, d\sigma^2 < \infty$, and the marginal thus the joint must exist.

- Given the joint exists,

$$f_{\{\mu,\sigma^2\}|\{\bar{x},s^2\}} \propto \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right)$$

we need to check the conditionals in order to use Gibbs sampling,

$$\mu \mid \{\sigma^2, \bar{x}, s\} \sim \text{Normal}\left(\frac{\omega^2\bar{x} + \nu\sigma^2/n}{\omega^2 + \sigma^2/n}, \frac{\omega^2\sigma^2/n}{\omega^2 + \sigma^2/n}\right)$$

$$\sigma^2 \mid \{\mu, \bar{x}, s\} \sim \text{Scaled Inverse}\,\chi^2\left(n, \frac{(n-1)s^2}{n} + (\bar{x}-\mu)^2\right)$$

- The conditional is also scaled inverse χ^2, which is nonzero for $\sigma^2 \in (0, \infty)$

$$\begin{aligned} f_{\sigma^2|\{\sigma,\bar{x},s\}} = \frac{f_{\{\mu,\sigma^2\}|\{\bar{x},s\}}}{f_{\mu|\{\bar{x},s\}}} &\propto f_{\{\mu,\sigma^2\}|\{\bar{x},s\}} \\ &\propto \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2}\right) \end{aligned}$$

- Therefore, we can reply on Gibbs sampling to sample the joint.

$$\mu^{(t)} \mid \{{\sigma^2}^{(t-1)}, \bar{x}, s\} \sim \text{Normal}\left(\frac{\omega^2\bar{x} + \nu{\sigma^2}^{(t-1)}/n}{\omega^2 + {\sigma^2}^{(t-1)}/n}, \frac{\omega^2{\sigma^2}^{(t-1)}/n}{\omega^2 + {\sigma^2}^{(t-1)}/n}\right)$$
$${\sigma^2}^{(t)} \mid \{\mu^{(t)}, \bar{x}, s\} \sim \text{Scaled Inverse}\,\chi^2\left(n, \frac{(n-1)s^2}{n} + (\bar{x} - \mu^{(t)})^2\right)$$

Q: Given some initial value ${\sigma^2}^{(0)}$, prior parameters ν, ω^2, and data n, $\bar{x}$, s^2, we can easily sample from the normal, how about the scaled inverse χ^2?

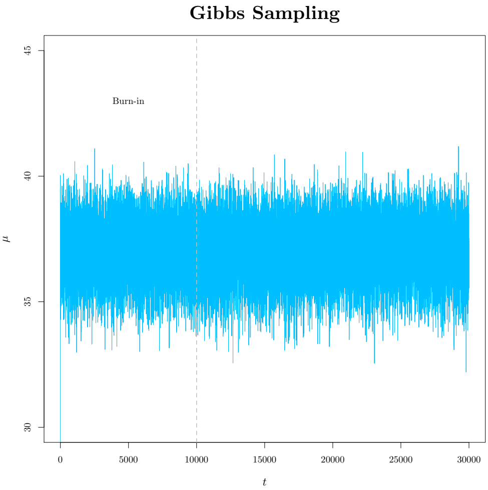
Q: You have seen what we need to do when uses Gibbs sampling, how about using Metropolis-Hastings algorithm? What do we need to do?

Q: Will the following proposal be a valid proposal?

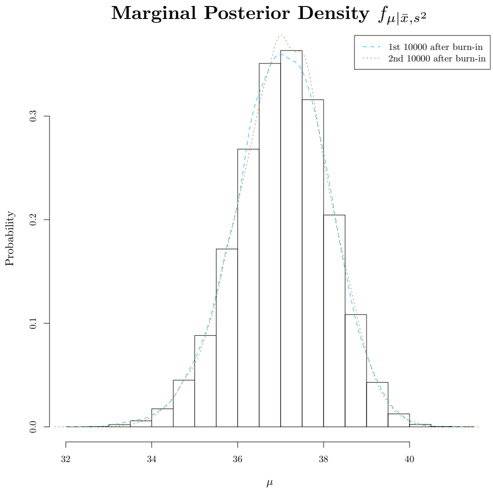
$$\begin{bmatrix}\mu^{(t)} & {\sigma^2}^{(t)}\end{bmatrix}^{\mathrm{T}} \sim \text{Normal}\left(\begin{bmatrix}\mu^{(t-1)} & {\sigma^2}^{(t-1)}\end{bmatrix}^{\mathrm{T}}, \begin{bmatrix}\gamma^2 & 0\\ 0 & \gamma^2\end{bmatrix}\right)$$

- The proposal will work, however, the parameter γ^2 needs to tuned at least.

- For this simple model, the Markov Chain converges fairly quickly,



- Again, the following shows the Markov Chain has converged,



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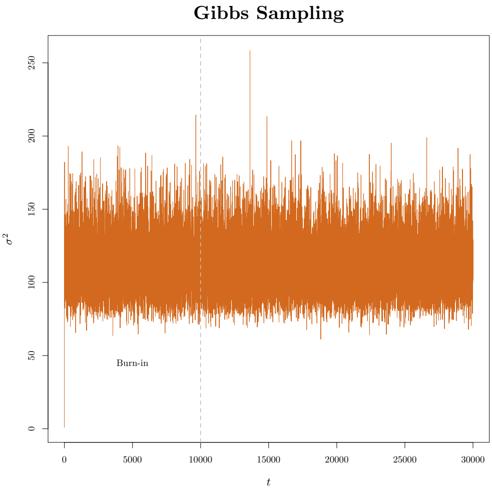

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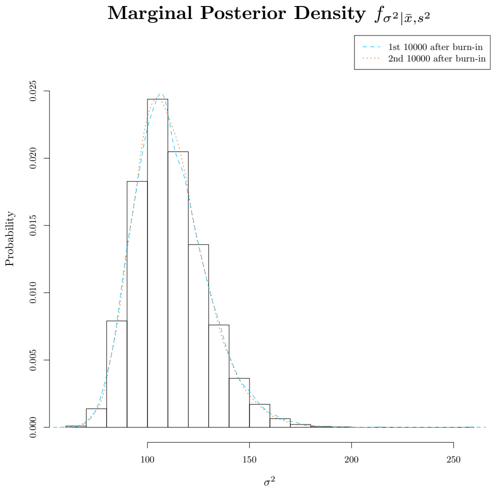
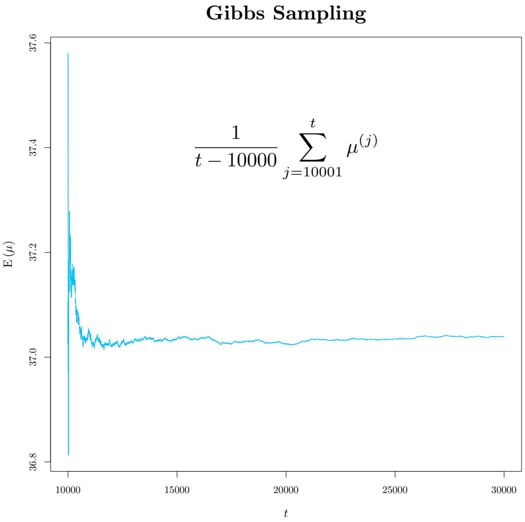
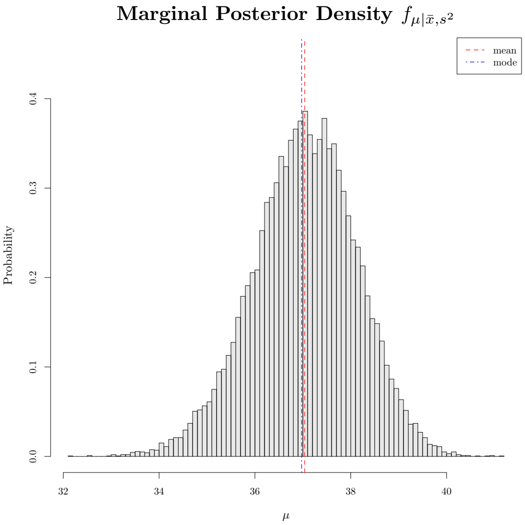


- Suppose we are primarily interested in μ, specifically, $f_{\mu|\{\bar{x},s^2\}}$



Q: Is there any way other than MCMC we can use to obtain a point estimate



- Note we could use numerical optimisations to find the mode in this case,

$$f_{\mu|\{\bar{x},s^2\}} = \frac{f_{\{\mu,\sigma^2\}|\{\bar{x},s^2\}}}{f_{\sigma^2|\{\mu,\bar{x},s^2\}}}$$

since we have the joint up to a constant A depends only on $\bar{x}$ and s^2,

$$\begin{aligned} Af_{\{\mu,\sigma^2\}|\{\bar{x},s^2\}} &= q_{\mu,\sigma^2} \\ &= \left(\sigma^2\right)^{-(1+n/2)} \cdot \exp\left(-\frac{(n-1)s^2}{2\sigma^2} - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{(\mu-\nu)^2}{2\omega^2}\right) \end{aligned}$$

and we have worked the full conditional

$$\sigma^2 \mid \{\mu,\bar{x},s\} \sim \text{Scaled Inverse}\,\chi^2\left(n, \frac{(n-1)s^2}{n} + (\bar{x}-\mu)^2\right)$$

- In general, the marginal is intractable, or difficult to obtain for the very least

$$f_{\phi|X} = \int_{-\infty}^{\infty} f_{\{\phi,\gamma\}|X}\, d\gamma$$