VE414 Appendix 7

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Hammersley-Clifford

Proof

The general idea of the proof is to reduce the dimension by fixing one at time,

$$\underbrace{f_{Y_1 \ldots, Y_p}(y_1, \cdots, y_p)}_{\text{of } p \text{ scalars}} = f_{Y_p \mid Y_{-p}}(y_p \mid y_1, y_2, \cdots, y_{p-1}) \cdot \underbrace{f_{Y_{-p}}(y_1, y_2, \cdots, y_{p-1})}_{\text{of } (p-1) \text{ scalars}}$$

The above marginal can also be written for some value $(y_1, \cdots, \xi_p) \in \mathcal{D}$ as

$$f_{Y_{-p}}(y_1, y_2, \cdots, y_{p-1}) = \frac{f_{Y_1 \ldots, Y_p}(y_1, \cdots, \xi_p)}{f_{Y_p \mid Y_{-p}}(\xi_p \mid y_1, y_2, \cdots, y_{p-1})}$$

where ξ_p is a fixed value, which means the joint can be written as

$$f_{Y_1 \ldots, Y_p}(y_1, \cdots, y_p) = \frac{f_{Y_p \mid Y_{-p}}(y_p \mid y_1, y_2, \cdots, y_{p-1})}{f_{Y_p \mid Y_{-p}}(\xi_p \mid y_1, y_2, \cdots, y_{p-1})} \cdot f_{Y_1 \ldots, Y_p}(y_1, \cdots, \xi_p)$$

We then use the same trick on the joint density given we have fixed $y_p = \xi_p$.

Proof

The joint can be written as

$$f_{Y_1 \dots, Y_p}(y_1, \cdots, \xi_p) = f_{Y_{p-1} \mid Y_{-(p-1)}}(y_{p-1} \mid y_1, y_2 \cdots y_{p-2}, \xi_p) \cdot \underbrace{f_{Y_{-(p-1)}}(y_1, y_2, \cdots y_{p-2}, \xi_p)}_{\text{marginal}}$$

where the marginal here can be written for some value $(y_1, \cdots, \xi_{p-1}, \xi_p) \in \mathcal{D}$ as

$$f_{Y_{-(p-1)}}(y_1, y_2, \cdots y_{p-2}, \xi_p) = \frac{f_{Y_1 \dots, Y_p}(y_1, \cdots, \xi_{p-1}, \xi_p)}{f_{Y_{p-1} \mid Y_{-(p-1)}}(\xi_{p-1} \mid y_1, y_2 \cdots y_{p-2}, \xi_p)}$$

again the ξ_2 is some fixed value, which means the joint can be written as

$$f_{Y_1 \dots, Y_p}(y_1, \cdots, y_p) = f_{Y_1 \dots, Y_p}(y_1, \cdots, \xi_{p-1}, \xi_p) \cdot \frac{f_{Y_p \mid Y_{-p}}(y_p \mid y_1, y_2, \cdots, y_{p-1})}{f_{Y_p \mid Y_{-p}}(\xi_p \mid y_1, y_2, \cdots, y_{p-1})} \cdot \frac{f_{Y_{p-1} \mid Y_{-(p-1)}}(y_{p-1} \mid y_1, y_2, \cdots, y_{p-2}, \xi_p)}{f_{Y_{p-1} \mid Y_{-(p-1)}}(\xi_{p-1} \mid y_1, y_2, \cdots, y_{p-2}, \xi_p)}$$

Proof

Successively introducing ξ_{p-2}, ξ_{p-3}, $\cdots$, ξ_1 in addition to ξ_p and ξ_{p-1}, we have

$$f_{Y_1 \ldots, Y_p}(y_1, \cdots, y_p) = f_{Y_1 \ldots, Y_p}(\xi_1, \cdots, \xi_p) \cdot \frac{f_{Y_1 \mid Y_{-1}}(y_1 \mid \xi_2, \cdots, \xi_p)}{f_{Y_1 \mid Y_{-1}}(\xi_1 \mid \xi_2, \cdots, \xi_p)} \cdots \\ \cdots \cdot \frac{f_{Y_p \mid Y_{-p}}(y_p \mid y_1, y_2, \cdots, y_{p-1})}{f_{Y_p \mid Y_{-p}}(\xi_p \mid y_1, y_2, \cdots, y_{p-1})}$$

Since for any arbitrary $(y_1, \cdots, y_p) \in \mathcal{D}$, the joint and the marginals are nonzero, the conditionals are nonzero, hence the above holds for all $(y_1, \cdots, y_p) \in \mathcal{D}$, and

$$f_{\{Y_1, \ldots, Y_p\}}(y_1, \ldots y_p) \propto \prod_{j=1}^{p} \frac{f_{Y_j \mid Y_{-j}}(y_j \mid y_1, \ldots, y_{j-1}, \xi_{j+1}, \ldots, \xi_p)}{f_{Y_j \mid Y_{-j}}(\xi_j \mid y_1, \ldots, y_{j-1}, \xi_{j+1}, \ldots, \xi_p)}$$

because $f_{Y_1 \ldots, Y_p}(\xi_1, \cdots, \xi_p)$ is a constant.

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