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**Question1** (6 points)

**Julia should be the only computer language you use in this question.**

$$X \mid \{\mu, \sigma^2\} \sim \text{Normal}\left(\mu, \sigma^2\right)$$
$$\mu \sim \text{Normal}\,(10, 1)$$
$$\ln \sigma \sim \text{Uniform}\,(-100, 100)$$

Suppose $x = 0.5$ is observed.

(a) (2 points) Choose a conditional proposal distribution and implement a Metropolis-Hasting algorithm to obtain the mode of the marginal posterior $f_{\mu|x}$.

(b) (2 points) Work out the set of conditional posterior and implement a Gibbs sampling scheme to obtain the mode of the marginal posterior $f_{\mu|x}$.

(c) (2 points) Implement an EM algorithm to obtain the mode of the marginal posterior $f_{\mu|x}$.

**Question2** (4 points)

Consider the following standard linear regression model

$$\mathbf{Y} \mid \{\boldsymbol{\beta}, \boldsymbol{\sigma}^2\} \sim \text{Normal}\left(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I}\right)$$
$$\boldsymbol{\beta} \sim \text{Normal}\,(\boldsymbol{\beta}_0, \boldsymbol{\Sigma}_0)$$
$$\frac{1}{\sigma^2} \sim \text{Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$

(a) (2 points) Show the conditional posterior of $\sigma^2$ is given by

$$\sigma^2 \mid \{\boldsymbol{\beta}, \mathbf{Y}, \mathbf{X}\} \sim \text{Inverse-Gamma}\,(\alpha, \beta)$$

where

$$\alpha = \frac{\nu_0 + n}{2}; \qquad \beta = \frac{\nu_0\sigma_0^2 + \text{RSS}\,(\boldsymbol{\beta})}{2}$$

(b) (2 points) Describe in detail how to obtain a point estimate of $\boldsymbol{\beta} \mid \{\mathbf{Y}, \mathbf{X}\}$.

(c) (1 point (bonus)) Describe in detail how to obtain a prediction interval on

$$Y^*$$

given $X_1 = x_1^*$, $X_2 = x_2^*$, $\ldots$, $X_k = x_k^*$ and having observed $\mathbf{Y}$ and $\mathbf{X}$, where the prediction interval here is defined in a similar fashion to the central credible interval. Roughly speaking, the central credible interval is for an unobservable random variable, while the prediction interval is for a random variable that we are yet to observe.