Exam-presentation

Apéry's proof of the irrationality of $\zeta(3)$

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Introduction

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What about $\zeta(2k+1)$? So far only we only know $\zeta(3)$ is irrational.

Introduction

Each step he wrote on the blackboard appeared to be a remarkable identity that his audience considered unlikely to be true. When someone asked him "where do these identities come from?" he replied "They grow in my garden." Obviously this did not boost anyone's confidence.

Irrationality criterion

If there exists $\delta > 0$, $p_n, q_n \in \mathbb{Z}$ such that $\frac{p_n}{q_n} \neq \beta$ and

$$\left|\beta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$$

for $n \in \mathbb{N}$, then β is irrational.

Finding a new sequence

We need to find a better sequence. Motivated by a derivation of a series representation for $\zeta(3)$, we define

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

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$$\frac{1}{q_n^{1+\delta}} \approx \frac{1}{(e^{3n}4^n)^{1+\delta}}$$

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$$a_{n,k} = \sum_{k_2=0}^{k} \binom{k}{k_2}\binom{n}{k_2} \sum_{k_1=0}^{k_2} \binom{k_2}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n} c_{n,n-k_1}$$

$$b_{n,k} = \sum_{k_2=0}^{k} \binom{k}{k_2}\binom{n}{k_2} \sum_{k_1=0}^{k_2} \binom{k_2}{k_1}\binom{n}{k_1}\binom{2n-k_1}{n}.$$

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The key is that our sequence now has much better convergence relative to denominator.

Recurrence relation

Set $\{a_n\} = \{a_{n,n}\}, \{b_n\} = \{b_{n,n}\}$, then both sequences satisfy

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5)u_{n-1},$$
$$2 \leq n.$$

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> *We were quite unable to proove that the sequences did satisfy the recurrence (Apéry rather tartly pointed out to me in Helsinki that he regarded this more a compliment than a criticism of his method).*

Recurrence relation

This means that we have

$$n^3 a_n + (n-1)^3 a_{n-2} = P(n) a_{n-1}$$
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Collapse recurrences to get

$$n^3(a_n b_{n-1} - a_{n-1} b_n) = (n-1)^3(a_{n-1} b_{n-2} - a_{n-2} b_{n-1}).$$

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We can now iterate down to initial values

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3},$$

which we use to estimate convergence.

Recurrence relation

To that end, set

$$x_n = \zeta(3) - \frac{a_n}{b_n},$$

$$x_n - x_{n+1} = \frac{a_n b_{n+1} - b_n a_{n+1}}{b_n b_{n+1}} = \frac{6}{(n+1)^3 b_n b_{n+1}}.$$

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Now iterate:

$$\begin{aligned}
&\zeta(3) - \frac{a_n}{b_n} \\
&= x_n - x_{n+1} + x_{n+1} \\
&= x_{n+1} + \frac{6}{(n+1)^3 b_n b_{n+1}} \\
&= \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}} = O(b_n^{-2}).
\end{aligned}$$

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with roots $(1 \pm \sqrt{2})^4$. Thus with $\alpha = (1 + \sqrt{2})^4$, we have

$$b_n = O(\alpha^n).$$

Final touch

Recall, a_n and b_n are not quite integers, thus set

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$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = O(\alpha^{-2n}) = O(q_n^{-1+\delta})$$