Jasper Steinberg

On Some Harmonic and Odd Harmonic Series Identities

Master's thesis in Mathematical Sciences (MSMNFMA)
Supervisor: Andrii Bondarenko
June 2024

Master's thesis

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Abstract

In this thesis we derive some results on Euler sums of odd harmonic numbers. We do so mostly through use of the books by Cornel Ioan Vălean, namely (Almost) Impossible Integrals, Sums, and Series (2019) and More (Almost) Impossible Integrals, Sums, and Series (2023). We further discuss a binomial transform related to the Roman harmonic numbers, and connect it to the work of Khristo Boyadzhiev.

Sammendrag

I denne avhandlingen utleder vi noen resultater relatert til Euler summer av odde harmoniske tall. Dette gjør vi hovedsakelig ved å bruke bøkene til Cornel Ioan Vălean, (Almost) Impossible Integrals, Sums, and Series (2019) og More (Almost) Impossible Integrals, Sums, and Series (2023). Vi betrakter også en binomisk transformasjon relatert til de Roman harmoniske tallene, og kobler den til arbeidet til Khristo Boyadzhiev.

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I would like to thank Andrii Bondarenko for the persistent guidance and interesting discussions over the course of my time at NTNU, and especially the freedom to peruse my academic interests somewhat outside that of his own.

Sustainability

As required by the Faculty of Information Technology and Electrical Engineering at NTNU, we discuss the relevance of this thesis to the United Nations' Sustainable Development Goals. We feel that claiming any direct relevance of this thesis to the Sustainable Development Goals would be disingenuous. This thesis is concerned with some mathematical theoretical curiosities, and is meant to be viewed more through an aesthetic lens, than a practical one.

Contents

Chapter 1

Notation

We briefly define the notation used for this thesis. Define the Riemann zeta function for real $s > 1$ to be

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For aesthetic purposes, we never write out the explicit evaluations of the zeta function for even numbers s. We use the same convention for the Dirichlet eta function, now valid for real $s \geq 1$,

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

Define the polylogarithm of order $m \in \mathbb{R}$ to be

$$\mathrm{Li}_m(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^m},$$

valid for complex $|z| < 1$. We define the Catalan constant to be the number

$$G := \mathrm{Im}\{\mathrm{Li}_2(i)\}.$$

By defining the Catalan constant in this manner, the constant

$$\mathrm{Im}\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\},$$

dubbed a natural companion to the Catalan constant in [CLN21], might feel more natural when it appears later in the thesis. We define the generalized harmonic numbers for integer $m \geq 1$ as

$$H_n^{(m)} := \sum_{k=1}^{n} \frac{1}{k^m}.$$

Similarly we define for integer $m \geq 1$ the generalized skew harmonic numbers

$$\overline{H}_n^{(m)} := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m}.$$

Related numbers are the generalized odd harmonic numbers

$$O_n^{(m)} := \sum_{k=1}^{n} \frac{1}{(2k-1)^m} = H_{2n}^{(m)} - \frac{1}{2^m} H_n^{(m)}.$$

We note that some authors prefer to work with

$$h_n^{(m)} := \sum_{k=1}^{n} \frac{1}{\left(k - \frac{1}{2}\right)^m} = 2^m O_n^{(m)},$$

also called the generalized odd harmonic numbers, or the odd harmonic numbers of order m. We define the skew odd harmonic numbers, also called the alternating odd harmonic numbers, $\overline{O}_n^{(m)}$, $\overline{h}_n^{(m)}$, analogous as for the harmonic numbers. Following the convention of the seminal paper [FS98], we define the Euler sums to be

$$S_{p_1 p_2 \ldots p_k, q} := \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)} \ldots H_n^{(p_k)}}{n^q}.$$

This allows us to define the notion of weight, which is the quantity $p_1 + p_2 + \cdots + p_k + q$, and degree, which is the number k. We will denote repeated indices by powers. To illustrate, one example of a cubic Euler sum of weight 6 could read

$$S_{1^2 2, 2} = \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}.$$

Euler sums have been extended in all sorts of ways. Of special interest for us is the following variant of the classical Euler sums

$$\widetilde{S}_{p_1 p_2 \ldots p_k, q} := \sum_{n=1}^{\infty} \frac{h_n^{(p_1)} h_n^{(p_2)} \ldots h_n^{(p_k)}}{n^q}.$$

These are studied in detail along with some related variants in [XW23]. We extend the notion of weight and degree to all variants of Euler sums in the natural way. To extend these general sums to include alternating terms, we use the convention that over-lining indices results in the alternating variant corresponding to the index. For example, one instance of a weight 8 alternating $\widetilde{S}$ series might read

$$\widetilde{S}_{1\overline{1}^2\overline{2},\overline{3}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h_n \overline{h}_n^2 \overline{h}_n^{(2)}}{n^3},$$

where the the over-lining of the last index corresponds to an added $(-1)^{n-1}$ factor in the sum. Following the notation in [ZX23], we define the Bell polynomials $\Omega_k(t_1, t_2, \ldots, t_k)$ to be

$$\Omega_k(t_1, t_2, \ldots, t_k) \coloneqq \sum_{\theta(k)} \frac{k!}{a_1!a_2!\ldots a_k!} \left(\frac{t_1}{1}\right)^{a_1} \left(\frac{t_2}{2}\right)^{a_2} \cdots \left(\frac{t_k}{k}\right)^{a_k},$$

where we sum over all sets of non-negative integers $a_1, a_2, \ldots a_k$ such that $a_1 + 2a_2 + \cdots + ka_k = k$. For example, if $k = 2$, then both $\{a_1 = 0, a_2 = 1\}$ and $\{a_1 = 2, a_2 = 0\}$ satisfy $a_1 + 2a_2 = 2$, so

$$\Omega_2(t_1, t_2) = \sum_{\theta(2)} \frac{2!}{a_1!a_2!} (t_1)^{a_1} \left(\frac{t_2}{2}\right)^{a_2} = \frac{2}{0!1!} t_1^0 \left(\frac{t_2}{2}\right)^1 + \frac{2}{2!0!} t_1^2 \left(\frac{t_2}{2}\right)^0 = t_2 + t_1^2.$$

Of special interest to us are the numbers $\Omega_k(H_n, H_n^{(2)}, \ldots, H_n^k)$, which are related to the so called Roman harmonic numbers $c_n^{(k)} = \Omega_k(H_n, H_n^{(2)}, \ldots, H_n^k)/k!$ discussed in [Ses17].

Chapter 2

Introduction

Sparked by a correspondence with Goldbach, Euler [Eul44] derived what would later be called the symmetric Euler sums

$$\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^p} = \frac{1}{2}\left(\zeta^2(p) + \zeta(2p)\right).$$

Since then a vast amount of variants and generalizations have been explored. Some authors have stuck with the spirit of Euler, most notably Cornel Ioan Vălean, which through his books [Văl19] [Văl23] provides both powerful and beautiful elementary techniques in the study of Euler sums. Elementary approaches are in general still relevant when considering sums not covered by the more general classes of series found by modern techniques. These can for the most part be divided into either the use of complex analysis, spearheaded by Flajolet and Salvy [FS98], or by comparing coefficients of certain hypergeometric transforms as done in [Zhe07].

In [FS22], the following series are claimed as open as of October 2022

$$\sum_{n=1}^{\infty} \frac{O_n^2}{n^3}, \quad \sum_{n=1}^{\infty} \frac{O_n^3}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{O_n^3}{n^3}. \tag{2.1}$$

It should be noted that the second series in (2.1) could be readily deduced by combining what is found in [Zhe07] and [FS98]. Nevertheless, we will give our own Cornel Vălean type solution to the problem. We use that $O_n = H_{2n} - \frac{1}{2}H_n$, to rewrite the series in terms of the harmonic numbers

$$\sum_{n=1}^{\infty} \frac{O_n^3}{n^2} = \sum_{n=1}^{\infty} \frac{H_{2n}^3}{n^2} - \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_n H_{2n}^2}{n^2} + \frac{3}{4}\sum_{n=1}^{\infty} \frac{H_{2n} H_n^2}{n^2} - \frac{1}{8}\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} \tag{2.2}$$

$$= \frac{15}{8}\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} - 2\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^3}{n^2} - \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_n H_{2n}^2}{n^2} + \frac{3}{4}\sum_{n=1}^{\infty} \frac{H_{2n} H_n^2}{n^2}, \tag{2.3}$$

where we used the formula (3.6) in the second transition. The first two terms in (5.4) can be found in [Văl19, p. 294, p. 312], the last two we evaluate in chapter 3. We derived these results prior to realizing that they already exist in the wasteland of the Mathematics Stack Exchange, namely in [19]. Nevertheless, our work will result in the following theorem.

Theorem 2.1.

$$\sum_{n=1}^{\infty} \frac{O_n^3}{n^2} = \frac{21}{8}\zeta(2)\zeta(3).$$

We just mention that by the same process outlined above, one can obtain the first series in (2.1), we state the result without proof below.

Theorem 2.2.

$$\sum_{n=1}^{\infty} \frac{O_n^2}{n^3} = -\frac{31}{16}\zeta(5) + \frac{7}{4}\zeta(2)\zeta(3).$$

In [Văl23], lots of series with $H_n H_{2n}$ terms are considered. One might wonder what happens if we instead consider series with $O_n O_{2n}$ terms. Chapter 4 is thus devoted to proving the following.

Theorem 2.3.

$$\sum_{n=1}^{\infty} \frac{O_n O_{2n}}{n^2} = 2\pi \operatorname{Im}\left\{\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{45}{32}\zeta(4) + \pi\log(2)G - \frac{3}{8}\log^2(2)\zeta(2)$$

In essence the idea behind how we derive the series in both Chapter 3 and 4 is simple. We always begin with the following classical Cauchy product.

Lemma 2.4. *For* $|x| < 1$,

$$\operatorname{arctanh}^2(x) = \sum_{n=1}^{\infty} x^{2n} \frac{O_n}{n}. \tag{2.4}$$

We then multiply by suitable functions and integrate to obtain integral representations for our desired series.

In the last chapter, we study a binomial transform with a wonderful amount of symmetry, namely

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{k^m} = \frac{1}{m!}\Omega_m\left(H_n, H_n^{(2)}, \ldots, H_n^{(m)}\right). \tag{2.5}$$

By taking the viewpoint of the binomial transform we can lean on the theory of Boyadzhiev [Boy18] to derive some results.

Chapter 3

Deriving two cubic weight five series

The goal of this chapter is to give Cornel Vălean - type solutions for two of the sums needed to show Theorem 2.1. Our starting point will be the a useful integral.

Lemma 3.1. *For any positive integer n,*

$$\int_0^1 x^{n-1}\log^2(1-x)\,dx = \frac{H_n^2 + H_n^{(2)}}{n}. \tag{3.1}$$

We give a somewhat different proof than what is found in [Văl19, p. 60], relevant to chapter 5.

Proof. First we set $u = 1 - x$ in the integral, then use the binomial theorem in combination with the elementary integral

$$\int_0^1 t^n \log^m(t)\,dt = \frac{(-1)^m m!}{(n+1)^{m+1}} \tag{3.2}$$

to obtain

$$\begin{aligned}\int_0^1 x^{n-1}\log^2(1-x)\,dx &= \int_0^1 (1-u)^{n-1}\log^2(u)\,du \\ &= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\int_0^1 u^k\log^2(u)\,du \\ &= 2\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{1}{(k+1)^3}.\end{aligned}$$

Now we shift the index, and use that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ to clean up the expression

$$2\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\frac{1}{(k+1)^3} = \frac{2}{n}\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{k^2}. \tag{3.3}$$

The latter expression is a known binomial transform, for example found in [Boy18], which we state below

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{k^2} = \frac{H_n^2 + H_n^{(2)}}{2}. \tag{3.4}$$

Thus in combining (3.3) and (3.4) we obtain the result. □

The following family of integrals are not found in [Văl23], probably because they are treated quite easily. We will need them nevertheless.

Lemma 3.2. *For any integer $m \geq 1$,*

$$\int_0^1 \frac{\log^m(t)\log(1+t)}{1+t}\, dt = (-1)^{m-1} m! \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^{m+1}} + (-1)^m m! \eta(m+2).$$

Moreover, if m is odd, then the result fully reduces to zeta values as follows

$$(-1)^{m-1} m! \frac{m}{2} \eta(m+2) + (-1)^m \frac{m!}{2} \zeta(m+2) + (-1)^m m! \sum_{i=1}^{(m-1)/2} \eta(2i)\zeta(m-2i+2).$$

Proof. Using the generating function in [Văl23, p. 398], we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^n H_n = \frac{\log(1+x)}{1+x},$$

which we us in the integral to obtain

$$\begin{aligned}
\int_0^1 \frac{\log^m(t)\log(1+t)}{1+t}\, dt &= \int_0^1 \log^m(t) \sum_{n=1}^{\infty} (-1)^{n-1} t^n H_n \, dt \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} H_n \int_0^1 t^n \log^m(t)\, dt \\
&= (-1)^m m! \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{(n+1)^{m+1}},
\end{aligned}$$

where we again used the integral (3.2). We now use the recursive nature of the harmonic numbers to reduce the sum to the more standard form

$$(-1)^m m! \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{(n+1)^{m+1}} = (-1)^{m-1} m! \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^{m+1}} + (-1)^m m! \eta(m+2),$$

which proves the first part of the lemma. To get the other evaluation, use

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k^{2p}} = \left(p + \frac{1}{2}\right) \eta(2p+1) - \frac{1}{2}\zeta(2p+1) - \sum_{i=1}^{p-1} \eta(2i)\zeta(2p-2i+1),$$

found in [Văl23, p. 421]. □

Example 3.3. Setting $m = 2$ and $m = 3$ in Lemma 3.2 gives

$$\int_0^1 \frac{\log^2(t)\log(1+t)}{1+t}\,dt = -\frac{15}{4}\zeta(4) - \frac{7}{2}\log(2)\zeta(3) - \log^2(2)\zeta(2) + \frac{1}{6}\log^4(2) + 4\mathrm{Li}_4\left(\frac{1}{2}\right)$$

and

$$\int_0^1 \frac{\log^3(t)\log(1+t)}{1+t}\,dt = \frac{87}{16}\zeta(5) - 3\zeta(2)\zeta(3),$$

respectively. In the $m = 2$ case we used the evaluation

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3} = \frac{11}{4}\zeta(4) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2) - 2\mathrm{Li}_4\left(\frac{1}{2}\right),$$

found in [Văl23, p. 421]. ♢

We can now use the above lemma to evaluate the following integral.

Lemma 3.4.

$$\begin{aligned}\int_0^1 \frac{\log^2(1-x)\,\mathrm{arctanh}^2(x)}{x}\,dx = {} & \frac{31}{4}\zeta(5) - \frac{7}{8}\log^2(2)\zeta(3) + \frac{21}{16}\zeta(2)\zeta(3) + \frac{1}{3}\log^3(2)\zeta(2) \\ & - \frac{1}{15}\log^5(2) - 2\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - 2\mathrm{Li}_5\left(\frac{1}{2}\right).\end{aligned}$$

Proof. We first turn to the logarithmic form of the inverse hyperbolic tangent, then do the substitution $t = (1-x)/(1+x)$. As a rule of thumb, this has the effect of transforming integrals in our context into the framework of what is found in [Văl23]. We obtain

$$\frac{1}{4}\int_0^1 \log^2(1-x)\log^2\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \frac{1}{2}\int_0^1 \frac{\log^2(t)\log^2\left(\frac{2t}{1+t}\right)}{1-t^2}\,dt.$$

Now we use that

$$\frac{1}{1-t^2} = \frac{1}{2}\frac{1}{1+t} + \frac{1}{2}\frac{1}{1-t}$$

to split the integral. Then we fully expand the $\log^2$-factor as follows

$$\begin{aligned}\log^2\left(\frac{2t}{1+t}\right) = {} & \log^2(2) + 2\log(2)\log(t) + \log^2(t) - 2\log(t)\log(1+t) \\ & - 2\log(2)\log(1+t) + \log^2(1+t).\end{aligned}$$

The process culminates in the problem of evaluating the the integrals below

$$\begin{aligned}\int_0^1 \frac{\log^2(t)\log^2\left(\frac{2t}{1+t}\right)}{1\pm t}\,dt &= \log^2(2)\int_0^1 \frac{\log^2(t)}{1\pm t}\,dt + 2\log(2)\int_0^1 \frac{\log^3(t)}{1\pm t}\,dt \\ &\quad + \int_0^1 \frac{\log^4(t)}{1\pm t}\,dt - 2\int_0^1 \frac{\log^3(t)\log(1+t)}{1\pm t}\,dt \\ &\quad - 2\log(2)\int_0^1 \frac{\log^2(t)\log(1+t)}{1\pm t}\,dt + \int_0^1 \frac{\log^2(t)\log^2(1+t)}{1\pm t}\,dt.\end{aligned}$$

The first three integrals are elementary. The fourth and fifth are found in [Văl23, p. 36] and in example 3.3. The last integral pair is found in [Văl23, p. 56], where we use that

$$\int_0^1 \frac{\log^2(t)\log^2(1+t)}{1+t}\,dt = -\frac{2}{3}\int_0^1 \frac{\log(t)\log^3(1+t)}{t}\,dt.$$

The proof is then concluded upon doing the summation. □

We exploit the above integral to find a closely related integral.

Lemma 3.5.

$$\begin{aligned}\int_0^1 \frac{\log^2(1-x)\operatorname{arctanh}^2(\sqrt{x})}{x}\,dx &= \frac{217}{8}\zeta(5) - 7\log^2(2)\zeta(3) + \frac{8}{3}\log^3(2)\zeta(2) \\ &\quad - \frac{8}{15}\log^5(2) - 16\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - 16\mathrm{Li}_5\left(\frac{1}{2}\right).\end{aligned}$$

Proof. We first set $x = ((1-t)/(1+t))^2$, which gives

$$\int_0^1 \frac{\log^2(1-x)\operatorname{arctanh}^2(\sqrt{x})}{x}\,dx = \int_0^1 \frac{\log^2\left(\frac{4t}{(1+t)^2}\right)\log^2(t)}{1-t^2}\,dt,$$

then expand in terms of

$$\begin{aligned}\int_0^1 \frac{\log^2\left(\frac{4t}{(1+t)^2}\right)\log^2(t)}{1-t^2}\,dt &= 4\int_0^1 \frac{\log^2\left(\frac{2t}{1+t}\right)\log^2(t)}{1-t^2}\,dt - 4\int_0^1 \frac{\log\left(\frac{2t}{1+t}\right)\log^3(t)}{1-t^2}\,dt \\ &\quad + \int_0^1 \frac{\log^4(t)}{1-t^2}\,dt.\end{aligned}$$

The result is then obtained by re-using the results in the proof of Lemma 3.4. □

We now have what we need to find our first series.

Theorem 3.6.

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}^2}{n^2}=\frac{917}{32}\zeta(5)+\frac{23}{8}\zeta(2)\zeta(3)-7\log^2(2)\zeta(3)+\frac{8}{3}\log^3(2)\zeta(2)$$
$$-\frac{8}{15}\log^5(2)-16\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right)-16\mathrm{Li}_5\left(\frac{1}{2}\right).$$

Proof. We multiply both sides of the Cauchy product in (2.4) with $\log^2(1-x)$, to obtain

$$\log^2(1-x)\operatorname{arctanh}^2(x)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2H_{2n}-H_n}{n}x^{2n}\log^2(1-x). \tag{3.5}$$

Now we divide (3.5) by x, and integrate from 0 to 1,

$$\int_0^1\frac{\log^2(1-x)\operatorname{arctanh}^2(x)}{x}\,dx=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2H_{2n}-H_n}{n}\int_0^1 x^{2n-1}\log^2(1-x)\,dx.$$

At this point we use Lemma 3.1, and get

$$\int_0^1\frac{\log^2(1-x)\operatorname{arctanh}^2(x)}{x}\,dx=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2H_{2n}-H_n}{n}\right)\left(\frac{H_{2n}^2+H_{2n}^{(2)}}{2n}\right).$$

We now want to isolate the desired series, and use the classical formula

$$\sum_{n=1}^{\infty}a_{2n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}(-1)^{n-1}a_n\right) \tag{3.6}$$

whenever possible. This results in

$$\sum_{n=1}^{\infty}\frac{H_nH_{2n}^2}{n^2}=4\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}-4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}+4\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2}$$
$$-4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}-\sum_{n=1}^{\infty}\frac{H_nH_{2n}^{(2)}}{n^2}-4\int_0^1\frac{\log^2(1-x)\operatorname{arctanh}^2(x)}{x}\,dx.$$

The integral was evaluated in Lemma 3.4, and the rest of the series can be found with evaluation in [Văl19, p. 293-294, p. 312] and in [Văl23, p. 439]. Using the aforementioned evaluations we obtain the result after a tedious calculation. □

Almost identically to the process above, we evaluate the second sum.

Theorem 3.7.

$$\sum_{n=1}^{\infty}\frac{H_{2n}H_n^2}{n^2}=\frac{421}{16}\zeta(5)+\frac{9}{4}\zeta(2)\zeta(3)-7\log^2(2)\zeta(3)+\frac{8}{3}\log^3(2)\zeta(2)$$
$$-\frac{8}{15}\log^5(2)-16\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right)-16\mathrm{Li}_5\left(\frac{1}{2}\right).$$

Proof. Since the result is so similar to that of Theorem 3.6, we just outline the procedure. We first use (2.4) to get the series expansion of $\operatorname{arctanh}^2(\sqrt{x})$. This in conjunction with Lemma 3.1 is used to obtain

$$\int_0^1 \frac{\log^2(1-x)\operatorname{arctanh}^2(\sqrt{x})}{x}\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n^2 + H_n^{(2)}}{n}\right)\left(\frac{2H_{2n} - H_n}{n}\right).$$

Then we again isolate for the desired series, and use the evaluation of the integral in Lemma 3.5, together with the evaluations of the remaining series found in [Văl19, p. 293, p. 294] and [Văl23, p. 439] to conclude. □

Chapter 4

Deriving a weight four odd harmonic series

We now turn to the problem of proving Theorem 2.3. The computation is somewhat involved, so we break up parts of the proof in the form of lemmas. In this case, we were not able to evaluate the integral representation of our series directly. We will instead transform the resulting integral into a few different series, which through the use of some recent results are possible to evaluate. At the heart of our proof lies the simple form of the integral below.

Lemma 4.1. *For any non-negative integer n,*

$$\int_0^1 t^{2n+1}(\mathrm{Li}_2(t) - \mathrm{Li}_2(-t) + \log(t)\log(1+t) - \log(t)\log(1-t))\,dt = -\frac{H^{(2)}_{2n+2}}{n+1} + \frac{1}{4}\frac{H^{(2)}_{n+1}}{n+1}.$$

Proof. We split the integral and consider the pieces by themselves. Integration by parts gives

$$\begin{aligned}\int_0^1 t^{2n+1}\mathrm{Li}_2(-t)\,dt &= \frac{1}{2}\mathrm{Li}_2(-1)\frac{1}{n+1} + \frac{1}{2}\frac{1}{n+1}\int_0^1 t^{2n+1}\log(1+t)\,dt \\ &= -\frac{1}{4}\zeta(2)\frac{1}{n+1} + \frac{1}{4}\frac{H_{2n+2}}{(n+1)^2} - \frac{1}{4}\frac{H_{n+1}}{(n+1)^2},\end{aligned}$$

where we used the evaluation in [Văl23, p. 26]. Similarly we have

$$\int_0^1 t^{2n+1}\mathrm{Li}_2(t) = \frac{1}{2}\zeta(2)\frac{1}{n+1} - \frac{1}{4}\frac{H_{2n+2}}{(n+1)^2}.$$

The remaining integrals we state below for convenience, found in [Văl23, p. 33-34]

$$\int_0^1 t^{2n+1}\log(t)\log(1-t)\,dt = \frac{1}{4}\frac{H_{2n+2}}{(n+1)^2} + \frac{1}{2}\frac{H_{2n+2}^{(2)}}{n+1} - \frac{1}{2}\zeta(2)\frac{1}{n+1}$$

$$\int_0^1 t^{2n+1}\log(t)\log(1+t)\,dt = \frac{1}{4}\zeta(2)\frac{1}{n+1} - \frac{1}{4}\frac{H_{2n+2}}{(n+1)^2} + \frac{1}{4}\frac{H_{n+1}}{(n+1)^2} - \frac{1}{2}\frac{H_{2n+2}^{(2)}}{n+1} + \frac{1}{4}\frac{H_{n+1}^{(2)}}{n+1}.$$

Thus we find, after some marvelous cancellation, that

$$\int_0^1 t^{2n+1}(\mathrm{Li}_2(t) - \mathrm{Li}_2(-t) + \log(t)\log(1+t) - \log(t)\log(1-t))\,dt = -\frac{H_{2n+2}^{(2)}}{n+1} + \frac{1}{4}\frac{H_{n+1}^{(2)}}{n+1}.$$

□

The above result allows us to reduce a hard integral into manageable sums.

Lemma 4.2.

$$\int_0^1 \frac{\log(t)\log^2\left(\frac{1+t^2}{2}\right)}{1-t^2}\,dt = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_nH_{2n+2}^{(2)}}{n+1} - \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_nH_{n+1}^{(2)}}{n+1} - 2\log(2)\sum_{n=0}^{\infty}(-1)^n\frac{H_{2n+2}^{(2)}}{n+1} + \frac{1}{2}\log(2)\sum_{n=0}^{\infty}(-1)^n\frac{H_{n+1}^{(2)}}{n+1}.$$

Proof. First define

$$I := \int_0^1 \frac{\log(t)\log^2\left(\frac{1+t^2}{2}\right)}{1-t^2}\,dt$$

for convenience. Then we note that the least workable term here is the $\log^2$-term, which we target via integration by parts to obtain

$$I = -\frac{1}{2}\int_0^1 \frac{\log(1+t)\log^2\left(\frac{1+t^2}{2}\right)}{t}\,dt - 2\int_0^1 \frac{t\log(t)\log(1+t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt + \frac{1}{2}\int_0^1 \frac{\log(1-t)\log^2\left(\frac{1+t^2}{2}\right)}{t}\,dt + 2\int_0^1 \frac{t\log(t)\log(1-t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt.$$

We can now target the final $\log^2$-terms via integration by parts again to obtain

$$\begin{aligned} I =& -2\int_0^1 \frac{t\mathrm{Li}_2(-t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt - 2\int_0^1 \frac{t\log(t)\log(1+t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt \\ &+ 2\int_0^1 \frac{t\mathrm{Li}_2(t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt + 2\int_0^1 \frac{t\log(t)\log(1-t)\log\left(\frac{1+t^2}{2}\right)}{1+t^2}\,dt. \end{aligned}$$

We now group terms to factor out some known generating functions

$$\begin{aligned} I =& -2\int_0^1 t\frac{\log(1+t^2)}{1+t^2}\left(\mathrm{Li}_2(-t) - \mathrm{Li}_2(t) + \log(t)\log(1+t) - \log(t)\log(1-t)\right)\,dt \\ &+ 2\log(2)\int_0^1 \frac{t}{1+t^2}\left(\mathrm{Li}_2(-t) - \mathrm{Li}_2(t) + \log(t)\log(1+t) - \log(t)\log(1-t)\right)dt. \end{aligned}$$

For the first integral we use

$$\sum_{n=1}^{\infty}(-1)^{n+1}H_n x^{2n} = \frac{\log(1+x^2)}{1+x^2},$$

and for the second we expand the geometric series to obtain

$$\begin{aligned} I =& -2\sum_{n=1}^{\infty}(-1)^{n+1}H_n\int_0^1 t^{2n+1}\left(\mathrm{Li}_2(t) - \mathrm{Li}_2(-t) + \log(t)\log(1+t) - \log(t)\log(1-t)\right)dt \\ &+ 2\log(2)\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n+1}\left(\mathrm{Li}_2(t) - \mathrm{Li}_2(-t) + \log(t)\log(1+t) - \log(t)\log(1-t)\right)dt. \end{aligned}$$

The integrals were evaluated in Lemma 4.2, which concludes the proof. □

We will need one weight 4 sum which we were not able to find in the literature, however it is easily deducible by some recent results.

Lemma 4.3.

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_nH_{2n}^{(2)}}{n} =& \pi\,\mathrm{Im}\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{363}{64}\zeta(4) - \frac{5}{4}\log^2(2)\zeta(2) + \frac{63}{32}\log(2)\zeta(3) \\ &+ \frac{9}{2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \pi\log(2)G + \frac{3}{16}\log^4(2). \end{aligned}$$

Proof. We will use the evaluation of $\widetilde{S}_{12,\bar{1}}$ found in [XW23], together with the relation

$O_n^{(r)} = \frac{1}{2^r} h_n^{(r)}$ to find

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{O_n O_n^{(2)}}{n} &= \frac{1}{8}\widetilde{S}_{12,\bar{1}} \\ &= -\frac{1}{2}\pi \operatorname{Im}\left\{\operatorname{Li}_3\left(\frac{1+i}{2}\right)\right\} - \frac{1}{4}\pi\log(2)G \\ &+ \frac{3}{32}\log^2(2)\zeta(2) + \frac{225}{128}\zeta(4).\end{aligned}$$

Now we use $O_n^{(r)} = H_{2n} - \frac{1}{2^r}H_n$ to find

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{O_n O_n^{(2)}}{n} &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}H_{2n}^{(2)}}{n} - \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(2)}H_{2n}}{n} \\ &- \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{2n}^{(2)}}{n} + \frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_n^{(2)}}{n}.\end{aligned}$$

We isolate for the desired series, and use the evaluations found in [XC18a] and [Văl23, p. 451, p. 453] to conclude. □

With these results in mind, we can prove Theorem 2.3.

Proof of Theorem 2.3. We begin by finding an integral representation for the series. Combine the integral

$$\int_0^1 x^{4n-1}\operatorname{arctanh}(x)\,dx = \frac{O_{2n}}{4n}$$

found in [Văl23, p. 28] with our usual Cauchy product

$$\operatorname{arctanh}^2(x^2) = \sum_{n=1}^{\infty} x^{4n}\frac{O_n}{n}$$

to obtain

$$\sum_{n=1}^{\infty}\frac{O_n O_{2n}}{n^2} = 4\int_0^1 \frac{\operatorname{arctanh}(x)\operatorname{arctanh}^2(x^2)}{x}\,dx.$$

The standard substitution $x = (1-t)/(1+t)$ gives

$$J := 4\int_0^1 \frac{\operatorname{arctanh}(x)\operatorname{arctanh}^2(x^2)}{x}\,dx = -\int_0^1 \frac{\log(t)\log^2\left(\frac{1+t^2}{2t}\right)}{1-t^2}\,dt.$$

We further reduce the integral to a more workable form, by extracting the t^{-1}-factor in the $\log^2\left(\frac{1+t^2}{2t}\right)$ term. The resulting extra terms have evaluation in [Văl23, p. 39-40], and we obtain

$$J = -\int_0^1 \frac{\log(t)\log^2\left(\frac{1+t^2}{2}\right)}{1-t^2}\,dt + 4G^2$$

Now use Lemma 4.2 to get

$$J = 4G^2 - 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{2n+2}^{(2)}}{n+1} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{n+1}^{(2)}}{n+1} \tag{4.1}$$

$$+ 2\log(2)\sum_{n=0}^{\infty}(-1)^{n}\frac{H_{2n+2}^{(2)}}{n+1} - \frac{1}{2}\log(2)\sum_{n=0}^{\infty}(-1)^{n}\frac{H_{n+1}^{(2)}}{n+1}. \tag{4.2}$$

The first series in (4.1) we find by noting that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{2n+2}^{(2)}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_{2n}^{(2)}}{n^2} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{2n}^{(2)}}{n},$$

which upon using Lemma 4.3 and the evaluation [Văl23, p. 450] gives

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{2n+2}^{(2)}}{n+1} = 2G^2 - \pi\log(2)G - \pi\,\mathrm{Im}\left\{\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right\}$$
$$+ \frac{5}{32}\zeta(4) + \frac{77}{32}\log(2)\zeta(3) + \frac{1}{48}\log^4(2) + \frac{1}{2}\mathrm{Li}_4\left(\frac{1}{2}\right).$$

The second sum in (4.1) is found by using the evaluations in [XC18a] and [Văl19, p. 310] to obtain

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_{n+1}^{(2)}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(2)}}{n^2} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n H_n^{(2)}}{n}$$
$$= \frac{1}{12}\log^4(2) - \frac{3}{4}\log^2(2)\zeta(2) + \frac{21}{8}\log(2)\zeta(3) - \frac{35}{16}\zeta(4) + 2\mathrm{Li}_4\left(\frac{1}{2}\right)$$

Now we just need the two last weight 3 series in (4.1) by using the evaluations found in [XW23], [Văl23, p. 424]

$$\sum_{n=0}^{\infty}\frac{H_{2n+2}^{(2)}}{n+1} = \frac{1}{4}\widetilde{S}_{2,\bar{1}} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n^{(2)}}{n}$$
$$= 2\zeta(3) - \frac{1}{2}\pi G - \frac{1}{8}\log(2)\zeta(2)$$

$$\sum_{n=0}^{\infty}(-1)^{n}\frac{H_{n+1}^{(2)}}{n+1} = \zeta(2) - \frac{1}{2}\log(2)\zeta(2).$$

The result then follows by using the above evaluations in (4.1) and carrying out the summation.

∎

Chapter 5

The binomial transform and the harmonic numbers

We know want to explore some properties of the harmonic numbers in relation to the symmetric binomial transform. For a sequence $\{a_n\}_{n\geq 0}$, we define the symmetric binomial transform to be the new sequence $\{b_n\}_{n\geq 0}$, where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} a_k.$$

We begin with an integral result from [Ses17], also given in [ZX23]. It states that for $m, n \in \mathbb{N}$,

$$\int_0^1 x^{n-1} \log^m(1-x)\, dx = \frac{(-1)^m}{n} \Omega_m \left(H_n, H_n^{(2)}, \ldots, H_n^{(m)} \right). \tag{5.1}$$

Using the exact same idea as in the proof of Lemma 3.1, we can show

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{1}{k^m} = \frac{1}{m!} \Omega_m \left(H_n, H_n^{(2)}, \ldots, H_n^{(m)} \right). \tag{5.2}$$

This result is also stated in [Ses17], a recommended read for the properties of the equation (5.2). Other articles delve into infinite series expansions of the numbers $\Omega_m \left(H_n, H_n^{(2)}, \ldots, H_n^{(m)} \right)$, one nice example is [XC18b]

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Omega_{m-1} \left(H_n, H_n^{(2)}, \ldots, H_n^{(m-1)} \right) \overline{H}_n}{n} = m! \left(\zeta(m+1) - \mathrm{Li}_{m+1}\left(\frac{1}{2}\right) \right).$$

We derive a formula similar to the above equation, by using the binomial transform (5.2). It arises from considering the modified Euler transform found in [Boy18], which states that for

$f(z) = \sum_{k=0}^{\infty} a_k z^k$ with z small enough,

$$\sum_{n=0}^{\infty} H_n a_n z^n + \log(1+z) f(z) = \frac{1}{1+z} \sum_{n=0}^{\infty} \left(\frac{z}{z+1}\right)^n H_n \left\{ \sum_{k=0}^{n} \binom{n}{k} a_k \right\}.$$

Thus if we set $a_0 = 0$, $a_k = \frac{(-1)^{k-1}}{k^{2m}}$, and $z = 1$, we obtain by use of the evaluation in [Văl23, p. 421] the following interesting series.

Theorem 5.1. *For any integer $m \geq 1$,*

$$\sum_{n=1}^{\infty} \frac{H_n}{2^n} \Omega_{2m}\left(H_n, H_n^{(2)}, \ldots, H_n^{(2m)}\right) =$$
$$2(2m)!\Big(\log(2)\eta(2m) + \left(m + \frac{1}{2}\right)\eta(2m+1) - \frac{1}{2}\zeta(2m+1) - \sum_{i=1}^{m-1} \eta(2i)\zeta(2m-2i+1)\Big).$$

We have not seen the above in the literature, but with the vast amount of articles on related topics it may very well exist already. Before proceeding, we give the case of $m = 2$ in Theorem 5.1.

Example 5.2.

$$\sum_{n=1}^{\infty} \frac{H_n^5 + 8H_n^2 H_n^{(3)} + 6H_n^3 H_n^{(2)} + 3H_n\left(H_n^{(2)}\right)^2 + 6H_n H_n^{(4)}}{2^n}$$
$$= \frac{177}{2}\zeta(5) - 24\zeta(2)\zeta(3) + 42\log(2)\zeta(4).$$

♢

Finally, we want to see if we can improve on the family of transforms

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k^m}.$$

These currently only enjoy explicit evaluations in terms of the harmonic numbers up to $k = 2$ [Boy18],

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k} = H_n^{(2)}$$
$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} \frac{H_k}{k^2} = \sum_{k=1}^{n} \frac{H_k^{(2)}}{k}.$$

We begin by exploiting another transform given by [Boy18]. It states that for sequences $\{d_n\}, \{c_k\}$ related by $d_n = (-1)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}c_k.$, we have

$$\sum_{k=0}^{n}(-1)^{k-1}H_n c_k = (-1)^{n-1}H_n d_n + \sum_{k=0}^{n-1}\frac{(-1)^k d_k}{n-k}.$$

Then upon setting $c_k = 1/k^m$, we find the following theorem.

Theorem 5.3. *For any integer $m \geq 0$,*

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k^m} = \frac{H_n}{m!}\Omega_m\left(H_n,\ldots,H_n^{(m)}\right) - \frac{1}{m!}\sum_{k=1}^{n-1}\frac{\Omega_m\left(H_k,\ldots,H_k^{(m)}\right)}{n-k}.$$

The above in combination with the theory developed in [Văl19] gives us the tools to find a new binomial transform explicitly in terms of the harmonic numbers, given in non-explicit form in [Cop21].

Theorem 5.4.

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k^3} &= \frac{1}{2}H_n^2H_n^{(2)} + H_nH_n^{(3)} + \frac{3}{4}H_n^{(4)} + \frac{1}{4}\left(H_n^{(2)}\right)^2 \\ &\quad - \sum_{k=1}^{n}\frac{H_k}{k^3} - H_n\sum_{k=1}^{n}\frac{H_k}{k^2} + \frac{1}{2}\sum_{k=1}^{n}\frac{H_k^2}{k^2}\end{aligned}$$

.

Proof. By setting $m = 3$ in Theorem 5.3, we find

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{H_k}{k^3} \tag{5.3}$$

$$= \frac{1}{6}\left(H_n\left(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}\right) - \sum_{k=1}^{n-1}\frac{H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)}}{n-k}\right). \tag{5.4}$$

We need only find

$$S(n) := \sum_{k=1}^{n-1}\frac{H_kH_k^{(2)}}{n-k} = \sum_{k=1}^{n-1}\frac{H_{k-n}H_{k-n}^{(2)}}{k},$$

since the rest of the sums on the right-hand side of (5.4) can be found in [Văl19, p. 287-288]. To find $S(n)$, we first consider

$$S(n) - S(n-1) = \sum_{k=1}^{n-1}\frac{H_{k-n}H_{n-k}^{(2)} - H_{n-k-1}H_{n-k-1}^{(2)}}{k} \tag{5.5}$$

$$= \sum_{k=1}^{n-1}\frac{H_{n-k}}{k(n-k)^2} + \sum_{k=1}^{n-1}\frac{H_{n-k}^{(2)}}{k(n-k)} - \sum_{k=1}^{n-1}\frac{1}{k(n-k)^3}, \tag{5.6}$$

where we used that $H_{n-1}H_{n-1}^{(2)} = H_nH_n^{(2)} - H_n/n^2 - H_n^{(2)}/n + 1/n^3$. We set out to find the three sums in (5.6). From [Văl19, p. 368] we have the first and the last sum

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{H_{n-k}}{k(n-k)^2} &= \frac{3}{2}\frac{H_n^2}{n^2} - 2\frac{H_n}{n^3} - \frac{1}{2}\frac{H_n^{(2)}}{n^2} + \frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k^2} \\
\sum_{k=1}^{n-1} \frac{1}{k(n-k)^3} &= 2\frac{H_n}{n^3} + \frac{H_n^{(2)}}{n^2} + \frac{H_n^{(3)}}{n} - \frac{4}{n^4}.
\end{aligned}$$

We thus only need to show the middle sum of (5.6), namely

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{H_{n-k}^{(2)}}{k(n-k)} &= \sum_{k=1}^{n-1} \frac{H_k^{(2)}}{k(n-k)} \\
&= \frac{1}{n}\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{k} + \frac{1}{n}\sum_{k=1}^{n-1}\frac{H_n^{(2)}}{n-k} \\
&= \frac{1}{n}\left(H_{n-1}H_{n-1}^2 + H_nH_n^{(2)} + \frac{H_n}{n^2} + H_{n-1}^{(3)} - 2H_n^3\right),
\end{aligned}$$

where in the last transition we used the evaluations found in [Văl19, p. 287, p. 364]. We thus find

$$\begin{aligned}
S(n) - S(n-1) = &-\frac{3}{4}\left(H_n^{(2)}\right)^2 + \frac{1}{4}H_n^{(4)} + H_n^2H_n^{(2)} - 2H_nH_n^{(3)} \\
&- \frac{1}{2}\sum_{k=1}^{n}\frac{H_k^2}{k^2} + \sum_{k=1}^{n}\frac{H_k}{k^3} + H_n\sum_{k=1}^{n}\frac{H_k}{k^2}.
\end{aligned}$$

Upon telescoping the above, we find

$$\begin{aligned}
S(n) = &\frac{3}{2}\sum_{j=1}^{n}\frac{H_j^2}{j^2} - 4\sum_{j=1}^{n}\frac{H_j}{j^3} - \frac{5}{2}\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^2} + 2\sum_{j=1}^{n}\frac{H_jH_j^{(2)}}{j} \\
&- 2\sum_{j=1}^{n}\frac{H_j^{(3)}}{j} + 4H_n^{(4)} + \sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j}\frac{H_j}{j^2}.
\end{aligned}$$

Finally we obtain the results by plugging inn the simpler forms of $\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^2}, \sum_{j=1}^{n}\frac{H_jH_j^{(2)}}{j}, \sum_{j=1}^{n}\frac{H_j^{(3)}}{j}$ and $\sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j}\frac{H_i}{i^2}$ found in [Văl19, p. 286, p. 366]. ☐

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