Project 2: Jasper, Henrik, and Maxi

Problem 1

a)

There are four main criteria for using an $M/M/1$-queuing model with arrival rate $\lambda > 0$, and expected service time $\frac{1}{\mu} > 0$.

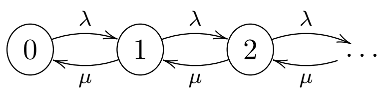
i) Inter-arrival times are iid. $\mathrm{Exp}(\lambda)$,

ii) Service times are iid. $\mathrm{Exp}(\mu)$,

iii) The service times and the inter-arrival times are independent,

iv) The queue has only one server.

As to the first criterion, we are told that the arrival times are distributed $\mathrm{Pois}(\lambda)$; from class, we know that the inter-arrival times of a Poisson are exponentially distributed with parameter λ. We are given that the service time is exponentially distributed with parameter μ, so the second criterion is satisfied. Furthermore, we are given that the inter-arrival times and the service times are independent of each other. Lastly, we are also told that there is only one worker at the clinic, so all criteria for using an $M/M/1$-model are satisfied.
We can interpret the model as a Birth-&-death process with constant birth rate $\lambda_i := \lambda$ for all $i \in \mathbb{N}_0$, as well as constant death rate $\mu_0 := 0$, and $\mu_i = \mu$ for all $i \in \mathbb{N}$. Thus we can model the predicament using a B&D-process illustrated with the following diagram:



By Little's law, we know that the average time a patient will spend in the UCC, will be given by

$$W = \frac{L}{\lambda} = \frac{1}{\lambda}\left(\frac{\lambda}{\mu - \lambda}\right) = \frac{1}{\mu - \lambda},$$

as we know from class that $X + 1 \sim \mathrm{Geom}\left(1 - \dfrac{\lambda}{\mu}\right)$, where X denotes the amount of customers, and L is its expected value.

b)

Figure 1 shows a realization of $X(t)$. The 95 percent confidence interval is $[0.8636529, 1.1322805]$, and the error is 0.4356667.

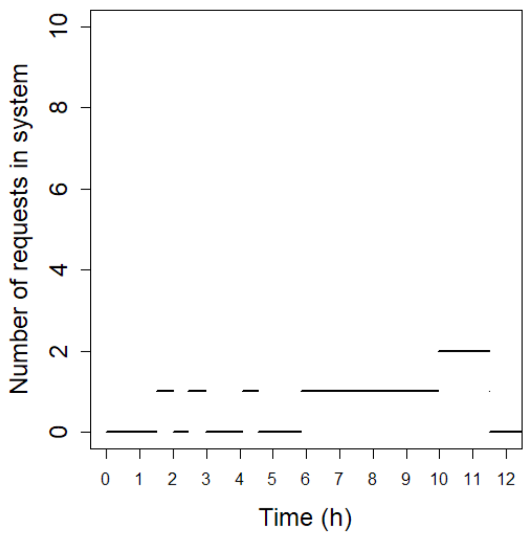


Figure 1: A realization

c)

The process $\{U(t)\}_{t\geqslant 0}$ is an $M/M/1$-queue, because as soon as an urgent patient arrives at the UCC, they will immediately receive care, and therefore they do not have to take into consideration how many normal patients are present; i.e. an urgent patient gets care as soon as he enters the queue, independent of the process $\{N(t)\}_{t\geqslant 0}$. Therefore, by precisely the same arguments as in task **a**), we have that $\{U(t)\}$ is an $M/M/1$-queue. Patients arrive with rate λ, the probability that such a patient is urgent is p, thus urgent patient arrive with rate $\lambda_U := p\lambda$. As $\{U(t)\}$ is an $M/M/1$-queue, we can use the same formula as in **a**), so that

$$L_U = L_U(p, \lambda, \mu) = \frac{\lambda_U}{\mu - \lambda_U} = \frac{p\lambda}{\mu - p\lambda}.$$

d)

In the process $\{U(t)\}_{t\geqslant 0}$, the service times are no longer independent of the arrival process. Given one normal patient in the queue, the service time will change depending on if an urgent patient will arrive during his treatment or not. The long-run mean number of patients in the UCC that are normal, will be equal to the total long-run mean amount of patients, minus the mean-run amount of urgent patients; thus, we have

$$\begin{aligned} L_N = L - L_U &= \frac{\lambda}{\mu - \lambda} - \frac{p\lambda}{\mu - p\lambda} \\ &= \mu\lambda \left[\frac{1-p}{(\mu-\lambda)(\lambda - \mu p)} \right]. \end{aligned}$$

e)

Recall Little's law: $L = \lambda W$. It implies that

$$W_U = \frac{1}{p\lambda} L_U = \frac{1}{\mu - p\lambda}.$$

Little's law can be used in any system, thus in the case of normal patients, where the rate is $\lambda_N = (1-p)\lambda$, we have

$$W_N = \frac{1}{\lambda_N} L_N = \frac{1}{(1-p)\lambda} \mu\lambda \left[\frac{1-p}{(\mu-\lambda)(\lambda-\mu p)} \right] = \frac{\mu}{(\mu-\lambda)(\mu-\lambda p)}.$$

f)

As $p \searrow 0$, we get that the probability that a patient is urgent vanishes, and we get the expected limit for $W_U \to \frac{1}{\mu}$; similarly, we get that $W_N \to \frac{1}{\mu - \lambda}$, showcasing that we do not have an $M/M/1$-queue structure for $N = N(t)$.
As $p \nearrow 1$, we get that $W_U \to \frac{1}{\mu - \lambda}$; similarly $W_N \to \frac{\mu}{(\mu-\lambda)^2}$.
Let us interpret these results: the limit $p \searrow 0$ is interpreted as the limiting case where we have very few urgent patients. The average waiting time will then converge to the same waiting time we had before we introduced the concepts of urgent and normal patients, because the urgent patients are prioritized. W_N will now approach the same value we calculated for W in **a)**, as we now regard every patient as normal, and therefore equal.
As $p \nearrow 1$, we get that most patients are urgent, and since we showed that urgent patients follow an $M/M/1$-structure, so the urgent patients more or less follow a first-come, first-serve principle. This results in a long waiting time for the normal patients. We see that $\lim_{p \nearrow 1} W_N = \frac{\mu}{(\mu-\lambda)^2}$, which makes sense, as the waiting times increase for the normal patients, and the inequality $\frac{\mu}{(\mu-\lambda)^2} \geqslant \frac{1}{\mu-\lambda} \iff \lambda \geqslant 0$ is trivially satisfied as long as $\mu \neq \lambda$.

Lastly, if the expected waiting time for a normal patients is 2 hours, then

$$\begin{aligned} W_n = 2 &\iff \frac{\mu}{(\mu-\lambda)(\mu-\lambda p)} = 2 \\ &\iff p = \frac{3}{5}. \end{aligned}$$

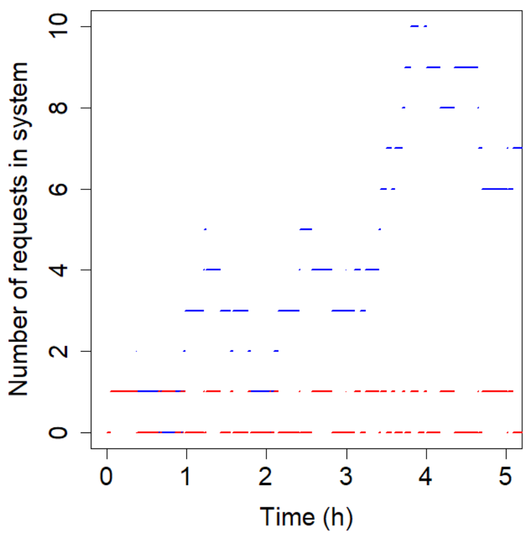


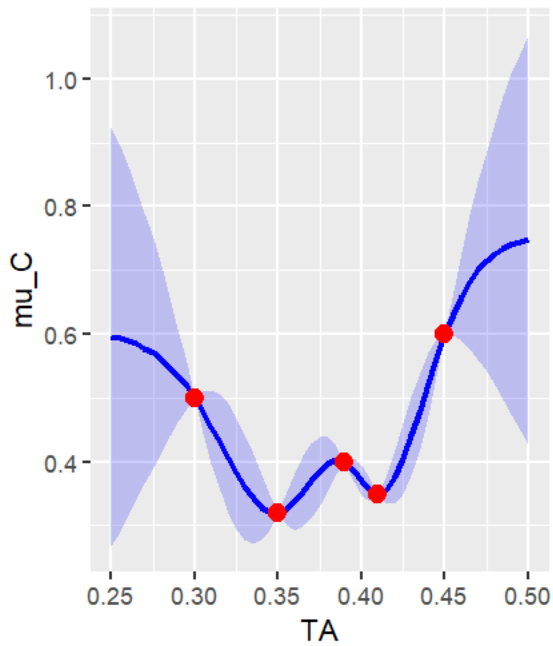
Figure 2: A joint realization

g)

In Figure 2 we can see a joint realization (shortened to 5 hours for legibility). Here urgent patients are colored in red and normal patients in blue. The CI for urgent patients is $[0.02226688, 0.02425534]$, while the one for normal patients is $[0.2902906, 0.3247206]$. From 1e), W_U is $1/6$ and W_N is $1/3$. Note how the blue lines are always above the red, since we prioritize the urgent patients.

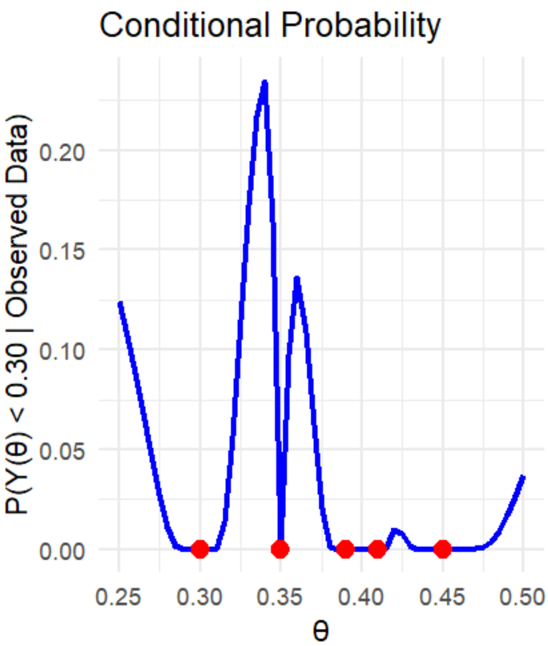
2

a)



mu_C
1.0
0.8
0.6
0.4
0.25
0.30
0.35
0.40
0.45
0.50
TA

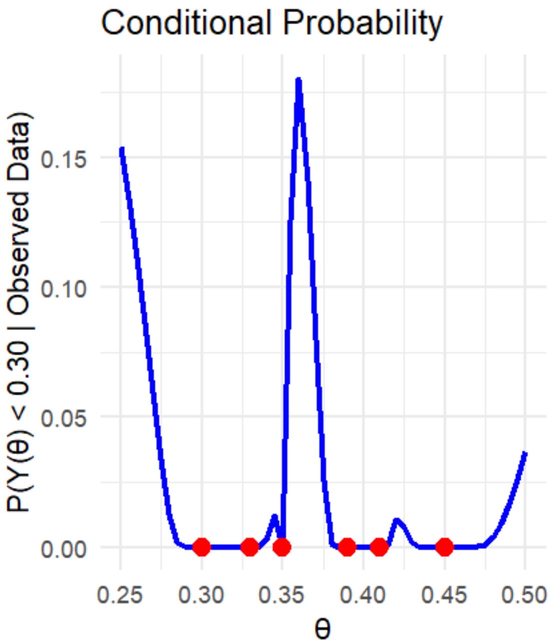
b)



Conditional Probability
P(Y(θ) < 0.30 | Observed Data)
0.20
0.15
0.10
0.05
0.00
0.25
0.30
0.35
0.40
0.45
0.50
θ

c)

Our choice of θ as to maximize the conditional probability $P(y(\theta) < 0.30 \mid x_{\mathbf{B}})$ is



$\theta = 0.36$.