A REMARK ON THE GAUSSIAN RADON TRANSFORM, PADÉ APPROXIMANTS, AND SHAPE RECONSTRUCTION

MAXIM DEREVYAGIN, AMBAR SENGUPTA, AND JASPER TOMAN–YIH

Abstract. To be done later

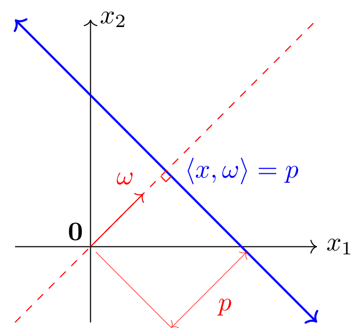
1. Introduction

To be done later.

Date: September 3, 2025.

2020 *Mathematics Subject Classification.* Primary ????; Secondary ????.

Key words and phrases. ????; ????; ????.



2. Notation

First recall the conventional multiindex notation. Let $\mathbb{N}_0$ denote the nonnegative integers. A multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is an n-tuple of nonnegative integers. The degree of a multiindex is $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Multivariate exponentiation is defined as follows. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

The multinomial formula gives a convenient expansion for multinomial powers. Let $k \in \mathbb{N}_0$ and $b = b_1 + b_2 + \cdots + b_n$. Then

$$b^k = \sum_{|\alpha|=k} \binom{k}{\alpha} b^\alpha$$

where the multinomial coefficients are defined

$$\binom{k}{\alpha} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!}.$$

Note that the multinomial expansion sums over all multiindeces $\alpha \in \mathbb{N}_0^n$ of degree k.

We denote the standard euclidean inner product

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$$

where $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. When discussing hyperplanes in $\mathbb{R}^n$ we index them by a unit normal vector, $\omega \in \mathbb{R}^n$, and distance from origin $-\infty < p < \infty$, and we write for example $\langle x, \omega \rangle = p$. Note that $\langle x, -\omega \rangle = p$ is the same hyperplane. It is perhaps more correct, in later defining the Radon and Gaussian Radon transforms, to identify these indexes and define the transforms over a projective space. We omit this discussion as it is not relevant within the scope of this work.

Unless otherwise indicated all measures are Euclidean measures, that is the Borel measure assocciated with the standard Euclidean metric on a given space. In particular the Euclidean measure on a hyperplane of $\mathbb{R}^n$ is equivalent to the Euclidean measure on $\mathbb{R}^{n-1}$. We chose for convenience to denote measures by their associated variable such as dx, dp, dz, and others. It should be stated that this notation, while uniform, is context dependent; it is a loving abuse of notation. For example in the integrals

$$\int_{\mathbb{R}^n} dx \qquad \text{and} \qquad \int_{\langle x, \omega \rangle = p} dx,$$

the measure dx is to be understood as the n-dimensional and $(n-1)$-dimensional Euclidean measure respectively.

3. Classical and Multivariate Moments

Let $f(x)$ be a measurable function on $\mathbb{R}$. For $k \in \mathbb{N}_0$, define the kth moment of f as

$$c_k = \int_{-\infty}^{\infty} f(x)x^k \, dx.$$

The sequence $(c_k)_{k\in\mathbb{N}_0}$ is called a moment sequence or a moment problem, and the function f is called a solution to the moment problem. Loosely speaking a moment problem poses the question: Under given constraints (e.g. domain, continuity, etc...), to what extent can one determine the solution f from its moments?

The classical study of moment problems is divided into three cases depending on the domain:

(1) Markov (or Haussdorf) moment problems for bounded domains (i.e. the unit interval),
(2) Stieltjes moment problems for one-sided unbounded domains (i.e. the positive real line), and
(3) Hamburger moment problems for bi-infinite domains (i.e. the real line).

There are two natural questions one can ask about a moment problem:

(1) Solvability: Does a solution f exist possessing the given moments?
(2) Determinacy: Is a solution unique? If not, what can be said about the set of solutions?

In the classical cases (Markov, Stieltjes, Hamburger) these questions have been resolved. Precise conditions for solvable and determinate moment problems exist, and the nature of solution sets to indeterminate moment problems are well understood. For instance, in the class of continuous functions on the unit interval, all solvable moments problems are determinate. However many classes of generalized moment problems remain in active study.

In particular let us discuss multivariate moment problems. Let $f(x)$ be a measureable function now on $\mathbb{R}^n$. For $\alpha \in \mathbb{N}_0^n$, define the αth multivariate moment of f as

$$c_\alpha = \int_{\mathbb{R}^n} f(x)x^\alpha \, dx.$$

The moment sequence $(c_\alpha)_{\alpha\in\mathbb{N}_0^n}$ is now multiindexed. No precise general conditions are known for the solvability or determinacy of multivariate moment problems. However some sufficient conditions have been discovered by leveraging classical moment problem theory.

4. Basics of the Radon and Gaussian Radon transforms

Let f be a multivariable function on the n-dimensional Euclidean space $\mathbb{R}^n$. We imagine taking "slices" of f by restricting it to a $(n-1)$-dimensional hyperplane Λ. These hyperplanes form the domain of the Radon Transform. More precisely, the Radon Transform associates each slice with a corresponding integral

$$R_f(\Lambda) = \int_\Lambda f(x)\ dx,$$

which can be thought of as a $(n-1)$-dimensional measurement of f. To be more precise we parametrize the collection of hyperplanes Λ by unit normal vector, $\omega \in \mathbb{R}^n$, and (signed) distance from the origin $-\infty < p < \infty$. Indeed any hyperplane can be described in the form $\Lambda = \{x \in \mathbb{R}^n : \langle x, \omega \rangle = p\}$. As mentioned, there is a slight inconsistency in these definitions where a hyperplane Λ can be indexed by both $\langle \omega, x \rangle = p$ and $\langle -\omega, x \rangle = -p$. This difference is inconsequential for our purposes so we choose the latter for clarity.

Definition 4.1. *Let f be a positive measurable function on $\mathbb{R}^n$. The **Radon transform (RT)** of f is a function which, given a unit vector $\omega \in \mathbb{R}^n$ and $-\infty < p < \infty$, is defined as*

$$R_f(\omega, p) = \int\limits_{\langle x, \omega \rangle = p} f(x)\ dx,$$

provided the integral converges.

Our main addition to previous work will be the use of a modified RT, the Gaussian Radon transform. This transform is very similar to the RT, but the inclusion of a Gaussian weight in the integral allows for convergence on a larger class of functions f. In a broader context the Gaussian Radon transform also has the advantages of generalizng to infite dimensional Hilbert spaces (on which the Lebesgue measure is not defined), as well as having a natural probabilistic interpretation.

Definition 4.2. *The **Gaussian Radon transform (GRT)** of f is defined similarly to the RT. Given a unit vector $\omega \in \mathbb{R}^n$ and $-\infty < p < \infty$, the GRT is*

$$GR_f(\omega, p) = \int\limits_{\langle x, \omega \rangle = p} f(x) e^{-\frac{\|x - p\omega\|^2}{2}}\ dx.$$

provided the integral converges. Note the Gaussian function $e^{-\|x - p\omega\|^2/2}$ is centered on the point of the hyperplane closest to the origin.

Remark 4.3. It may be helpful to understand the GRT as a simple modification of the RT with respect to a Gaussian measure on $\mathbb{R}^n$. Let $g(x) := f(x)e^{-\|x\|^2/2}$. The RT of g can be rewritten

$$\int\limits_{\langle x, \omega \rangle = p} f(x) e^{-\frac{\|x\|^2}{2}}\ dx = \int\limits_{\langle x, \omega \rangle = p} f(x) e^{-\frac{\|x - p\omega\|^2}{2}}\ dx\ e^{-\frac{\|p\omega\|^2}{2}},$$

where we used the Pythagorean relation $\|x\|^2 = \|x - p\omega\|^2 + \|p\omega\|^2$. Since ω is a unit vector, $\|p\omega\|^2 = p^2$. In short, we have proved the formula

$$R_g(\omega, p) = GR_f(\omega, p) e^{-\frac{p^2}{2}}, \qquad g(x) := f(x) e^{-\frac{\|x\|^2}{2}}. \tag{4.1}$$

The relation above provides decent intuition for the GRT, and is also a useful tool proving some basic properties of the transform.

Now imagine sweeping a hyperplanar "slice" across $\mathbb{R}^n$. As a function of p, the RT $R(\omega, p)$ can be seen as a projection of f onto the linear subspace spanned by ω. It is not surprising that integrating this projection over $-\infty < p < \infty$ we get the same result as the n-fold integral of f over $\mathbb{R}^n$.

$$\int_{-\infty}^{\infty} R(\omega, p)\ dp = \int_{\mathbb{R}^n} f(x)\ dx$$

The so called "slice theorem" further generalizes this observation.

Theorem 4.4 (Slice Theorem)**.** *If* $\int_{\mathbb{R}^n} |f(x)F(\langle x, \omega\rangle)|dx < \infty$ *then*

$$\int_{-\infty}^{\infty} R_f(\omega, p)F(p)dp = \int_{\mathbb{R}^n} f(x)F(\langle x, \omega\rangle)dx. \tag{4.2}$$

Proof. Inserting the definition of the RT, the left side is

$$\int_{-\infty}^{\infty} \int_{\langle x,\omega\rangle = p} f(x)\ dx\ F(p)\ dp = \int_{-\infty}^{\infty} \int_{\langle x,\omega\rangle = p} f(x)F(\langle x, \omega\rangle)\ dxdp.$$

Up to a rigid transformation (under which the Euclidean measures are invariant) this is essentially an itterated integral over $\mathbb{R}$ and $\mathbb{R}^{n-1}$. Thus given the integrability requirement, Fubini's theorem applies and the theorem is proved. □

If $F(p) = e^{-ip}$ and $f(x)$ is such that $\int_{-\infty}^{\infty} R_f(\omega, p)dp < \infty$ then (4.2) becomes the well known Fourier slice theorem

$$\int_{-\infty}^{\infty} R_f(\omega, p)e^{-ip}\ dp = \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\omega\rangle}\ dx,$$

which is often articulated as saying that the 1-D Fourier transform of the Radon transform is the N-D Fourier transform of f.

By way of the relation (4.1) we can prove an analagous slice theorem for the GRT.

Proposition 4.5 (Gaussian Slice Theorem)**.** *If* $\int_{\mathbb{R}^n} |f(x)F(\langle x, \omega\rangle)e^{-\|x\|^2/2}dx < \infty$ *then*

$$\int_{-\infty}^{\infty} GR_f(\omega, p)F(p)e^{-\frac{p^2}{2}}\ dp = \int_{\mathbb{R}^n} f(x)F(\langle x, \omega\rangle)e^{-\frac{\|x\|^2}{2}}\ dx. \tag{4.3}$$

Proof. From (4.1)

$$\int_{-\infty}^{\infty} GR_f(\omega, p)F(p)e^{-\frac{p^2}{2}}\ dp = \int_{-\infty}^{\infty} R_g(\omega, p)F(p)\ dp$$

where $g(x) = f(x)e^{-\|x\|^2/2}$. Then applying the slice theorem:

$$\int_{-\infty}^{\infty} R_g(\omega, p)F(p)\ dp = \int_{\mathbb{R}^n} f(x)F(\langle x, \omega\rangle)e^{-\frac{\|x\|^2}{2}}\ dx,$$

completing the proof. □

We will make use of two related applications of the slice theorems going forward. First we show that, fixing ω, the kth projection moment at ω can be written as a weighted sum of the degree k multivariate moments of f.

Proposition 4.6. *Let $c_\alpha(\omega) = \int_{-\infty}^{\infty} R_f(\omega, p)p^k dp$ be the projection moments of f at a fixed ω, and c_α the multivariate moments of f. Then*

$$c_k(\omega) = \sum_{|\alpha|=k} \binom{k}{\alpha} \omega^\alpha c_\alpha$$

where $\binom{k}{\alpha} = \frac{k!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$ are multinomial coefficients.

Proof. By the slice theorem (4.2) with $F(p) = p^k$,

$$\int_{-\infty}^{\infty} R_f(\omega, p)p^k \, dp = \int_{\mathbb{R}^n} f(x)\langle x, \omega\rangle^k \, dx.$$

Now $\langle x, \omega\rangle^k = (x_1\omega_1 + \cdots + x_n\omega_n)^k$ has the multinomial expansion

$$\langle x, \omega\rangle^k = \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha \omega^\alpha.$$

Thus after a bit of rearranging we get

$$\begin{aligned} \int_{\mathbb{R}^n} f(x)\langle x, \omega\rangle^k \, dx &= \int_{\mathbb{R}^n} f(x) \sum_{|\alpha|=k} \binom{k}{\alpha} x^\alpha \omega^\alpha \, dx \\ &= \sum_{|\alpha|=k} \binom{k}{\alpha} \omega^\alpha \int_{\mathbb{R}^n} f(x) x^\alpha \, dx, \end{aligned}$$

where the integrands are precisely the kth degree multivariate moments of f. □

Similarly, moments of the GRT (Gaussian projection moments) can be expressed in terms of multivariate gaussian moments.

Proposition 4.7. *Let $c_k^G(\omega) = \int_{-\infty}^{\infty} GR_f(\omega, p)p^k e^{-p^2/2} dp$ be the Gaussian weighted moments of the GRT of f at a fixed ω. Let $c_\alpha^G = \int_{\mathbb{R}^n} f(x)e^{-\|x\|^2/2} x^\alpha dx$ be the Gaussian weighted multivariate moments of f. Then*

$$c_k^G(\omega) = \sum_{|\alpha|=k} \binom{k}{\alpha} \omega^\alpha c_\alpha^G.$$

Proof. The proof follows as it did for the RT. This time we apply the GRT slice theorem (4.3) with $F(p) = p^k$,

$$\int_{-\infty}^{\infty} GR_f(\omega, p)p^k e^{-\frac{p^2}{2}} \, dp = \int_{\mathbb{R}^n} f(x)\langle x, \omega\rangle^k e^{-\frac{\|x\|^2}{2}} \, dx$$

Again we use the multinomial expansion and rearange:

$$\int_{\mathbb{R}^n} f(x)\langle x, \omega\rangle^k e^{-\frac{\|x\|^2}{2}} \, dx = \sum_{|\alpha|=k} \binom{k}{\alpha} \omega^\alpha \int_{\mathbb{R}^n} f(x) e^{-\frac{\|x\|^2}{2}} x^\alpha dx.$$

Thus

$$c^G(\omega) = \sum_{|\alpha|=k} \binom{k}{\alpha} \omega^\alpha c_\alpha^G.$$

□

Finally we prove two formulas regarding the Hamburger transform of a projection, defined more preceisely in the next section. For now we note that the transform is closely related to the projection moments and has the integral representation

$$g(z) = \int_{\infty}^{\infty} \frac{R(\omega, p)}{1 + zp} dp$$

By applying the slice theorem, with $F(p) = (1 + zp)^{-1}$ we see that the Hamburger transform of a projection can be represented by a similar multivariable integral of f over $\mathbb{R}^n$.

$$\int_{-\infty}^{\infty} \frac{R_f(\omega, p)}{1 + zp} dp = \int_{\mathbb{R}^n} \frac{f(x)}{1 + z\langle x, \omega \rangle} dx$$

Similarly the GRT slice theorem with $F(p) = (1 + zp)^{-1}$ is

$$\int_{-\infty}^{\infty} \frac{GR_f(\omega, p) e^{-p^2/2}}{1 + zp} dp = \int_{\mathbb{R}^n} \frac{f(x) e^{-\|x\|^2/2}}{1 + z\langle x, \omega \rangle} dx.$$

5. The Markov and Hamburger Transforms

Let μ be a positive Borel measure with infinite support and finite moments $c_k = \int_{-\infty}^{\infty} p^k d\mu$.

Definition 5.1. *The function on $\mathbb{C}$ defined by*

$$g(z) = \int_{\infty}^{\infty} \frac{d\mu}{1+zp}$$

is called the Markov (Hamburger resp.) transform of μ when μ has bounded (unbounded resp.) support.

Let us consider the domain of definition for $g(z)$. Since p is a real number, the denominator $1+zp$ can only vanish if Im $z=0$ In fact:

Proposition 5.2. *The function $g(z)$ is holomorphic on the upper half plane Im $z>0$.*

Proof. Let γ be a closed piecewise C^1 curve in the upper half plane.

$$\begin{aligned} |zg(z)| &\leq \int_{-\infty}^{\infty} \frac{d\mu}{|\frac{1}{z}+p|} \\ &\leq \int_{-\infty}^{\infty} \frac{d\mu}{|\mathrm{Im}\ \frac{1}{z}|} \\ &= \frac{c_0}{|\mathrm{Im}\ \frac{1}{z}|}. \end{aligned}$$

Thus

$$|g(z)| \leq \frac{c_0}{|z\mathrm{Im}\ \frac{1}{z}|} = c_0 \left| \frac{z}{\mathrm{Im}\ z} \right|.$$

Note that as a compact subset of the upper half plane γ must have positive distance from the real line. As a consequence $g(z)$ is bounded on γ, and by Fubini's theorem

$$\begin{aligned} \oint_\gamma g(z)dz &= \oint_\gamma \int_{-\infty}^{\infty} \frac{d\mu}{1+zp}\ dz \\ &= \int_{-\infty}^{\infty} \oint_\gamma \frac{dz}{1+zp}\ d\mu \\ &= \int_{-\infty}^{\infty} 0\ d\mu = 0. \end{aligned}$$

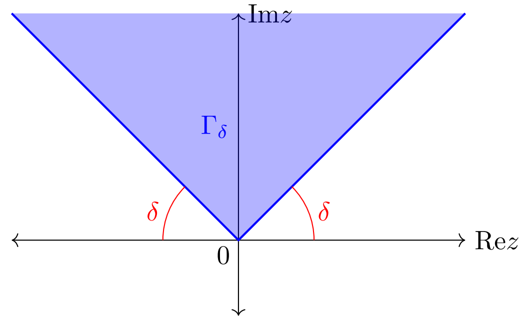
By Morera's theorem $g(z)$ is holomorphic on the upper half plane. □

Remark 5.3. Similarly one can show $g(z)$ is holomorphic on the lower half plane Im $z<0$. Moreover $g(z)$ commutes with conjugation:

$$g(\bar{z}) = \int_{\infty}^{\infty} \frac{d\mu}{1+\bar{z}p} = \overline{\int_{\infty}^{\infty} \frac{d\mu}{1+zp}} = \overline{g(z)}$$

since μ is a real measure.

When z is a real number $g(z)$ may not converge. In particular $g(z)$ does not converge when $z=-1/p$ for some p in the support of μ. Thus if μ has bounded support g converges in a neighborhood of 0. On the other hand if μ has unbounded support $g(0)$ is not defined. However — as we will see — even in the Hamburger transform case, a formal series expansion at $z=0$ holds valuable information.



By a simple geometric series expansion,

$$\frac{1}{1+zp} = \sum_{k=0}^{\infty} (-z)^k p^k$$

when $|zp| < 1$. This suggests the connection between $g(z)$ and the moment sequence $(c_k)_{k\in\mathbb{N}_0}$. Indeed, at least formally,

$$\begin{aligned}\int_{\infty}^{\infty} \frac{d\mu}{1+zp} &= \int_{\infty}^{\infty} \sum_{k=0}^{\infty} (-z)^k p^k \; d\mu \\ &= \sum_{k=0}^{\infty} (-z)^k \int_{\infty}^{\infty} p^k \; d\mu \\ &= \sum_{k=0}^{\infty} c_k (-z)^k.\end{aligned}$$

In the Markov case we have a positive radius of convergence. But even in the Hamburger case, there is a sense in which this assymptotic expansion holds "non-tangentially".

Proposition 5.4. *For all* $n \in \mathbb{N}_0$

$$g(z) = c_0 - c_1 z + c_2 z^2 - \cdots + c_n(-z)^n + z^n h_n(z) \tag{5.1}$$

where $\lim_{z\to 0} h_n(z) = 0$ *non-tangentially.*

We first note that

$$\sum_{k=0}^{n} c_k(-z)^k = \sum_{k=0}^{n} \int_{-\infty}^{\infty} (-zp)^k d\mu = \int_{-\infty}^{\infty} \frac{1-(-zp)^{n+1}}{1+zp} d\mu$$

and so

$$h_n(z) = \frac{1}{z^n}\left\{ g(z) - \sum_{k=0}^{n} c_k(-z)^k \right\} = \int_{-\infty}^{\infty} \frac{zp^{n+1}}{1+zp} d\mu.$$

It remains to determine if and in what sense this trailing term $h_n(z)$ vanishes at the origin.

In the case when μ has bounded support, $h_n(z)$ exists in a neighborhood of 0 and vanishes as $z \to 0$ in this neighboorhood. Thus the assymptotic expansion 5.1 is the Taylor expansion: The Markov transform is analytic at the origin.

When μ has unbounded support $h_n(z)$ may not be well defined on the real line. We turn to a weaker notion of convergence, namely *non-tangential* limits. For any

$0 < \delta < \pi/2$ let Γ_δ be the *wedge* (or *cone*) domain, $\Gamma_\delta = \{\delta < \arg z < \pi - \delta\}$. Restricting z to Γ_δ gives us the following inequalities for any $p \in \mathbb{R}$,

$$|p - z| \geq |p| \sin\delta \quad \text{and} \quad |p - z| \geq |p| \sin\delta.$$

We will show that

$$\lim_{z\to 0} h_{2n}(z) = 0, \quad z \in \Gamma_\delta$$

from which the corresponding limit for odd orders will immediately follow from the relation

$$h_{2n-1}(z) = z h_{2n}(z) + c_{2n} z^{2n}.$$

For the sake of convenience we define $w = -\frac{1}{z}$. Note that $|w| = \frac{1}{|z|}$ and if z is in the wedge Γ_δ then so is w. Now

$$\begin{aligned}
|h_{2n}(z)| &\leq \int_{-\infty}^{\infty} \frac{|p|^{2n+1}}{|p-w|} d\mu \\
&\leq \frac{|z|}{\sin\delta} \int_{|p|\leq A} |p|^{2n+1} d\mu + \frac{1}{\sin\delta} \int_{|p|\geq A} p^{2n} d\mu \\
&\leq \frac{2|z|A^{2n+2}}{\sin\delta} + \frac{1}{\sin\delta} \int_{|p|\geq A} p^{2n} d\mu
\end{aligned}$$

Thus

$$\lim_{z\to 0} |h_{2n}(z)| \leq \frac{1}{\sin\delta} \int_{|p|\geq A} p^{2n} d\mu, \quad z \in \Gamma_\delta.$$

Since A is arbitrary and the integral $\int p^{2n} d\mu = c_{2n}$ is convergent we are done.

Evidently the assymptotic series expansion $g(z) \simeq \sum_{k=0}^{\infty} c_k(-z)^k$ is generally formal, having zero radius of convergence except in the Markov case. However as we will see in the next section, rational functions constructed from this formal series exist which approximate $g(z)$ on the upper half plane $\{\mathrm{Im} z > 0\}$.

6. Padé approximants

In this section we recall the necessary definitions and results. Then we will prove some new results to be applied to the Gaussian Radon transform later.

Definition 6.1 (Classical definition of Padé approximants)**.** *The Padé approximant to a (possibly formal) power series*

$$R^{[L/M]}(z) \simeq \sum_{k=0}^{\infty} c_k z^k$$

is a rational function with nummerator (denominator resp.) degree at most L (M resp.), with series equal to $\sum_{k=0}^{N} c_k z^k + O(z^{N+1})$ up to as high an order N as possible.

Let

$$R^{[L/M]}(z) = \frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)} = \frac{a_L z^L + \cdots + a_1 z + a_0}{b_M z^M + \cdots + b_1 z + b_0}.$$

Notice that in general there is a negligable constant common factor between the numerator and denominator, so that with the remaining $L+M+1$ free parameters, we expect an order of accuracy of up to $L+M+1$ constraints, $c_0, c_1, \dots, c_{L+M}$. Thus we define the $[L/M]$ Padé approximant by the condition,

$$\frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)} = \sum_{k=0}^{L+M} c_k z^k + O\left(z^{L+M+1}\right). \tag{6.1}$$

It is helpful to consider the related, and necessary condition

$$P^{[L/M]}(z) = Q^{[L/M]}(z) \left(\sum_{k=0}^{L+M} c_k z^k \right) + O\left(z^{L+M+1}\right). \tag{6.2}$$

In the classical theory of Padé approximants (6.2) was often taken as a definition. It is always possible to find polynomials of the required degree satisfying this second condition, however they do not necessarily attain the degree of accuracy required by the first. We follow Baker, defining $R^{[L/M]}(z)$ by (6.1), provided such a rational function exists.

Definition 6.2. *The Padé approximant $R^{[L/M]}$ to a (possibly formal) power series is a the unique rational function with nummerator (denominator resp.) degree at most L (M resp.) satisfying the condition (6.1). If no such rational function exists we say the Padé approximant does not exist.*

Here we note that a sufficient condition for the equivalence of the two definitions, and hence for the existence of the Padé approximant by Baker's definition is that $b_0 = Q^{L/M}(0) \neq 0$.

Equating coeffictients of z^k, $k = 0, 1, \dots, L+M$ in (6.2) gives two linear systems

$$\begin{aligned} a_0 &= b_0 c_0 \\ a_1 &= b_1 c_0 + b_0 c_1 \\ &\vdots \\ a_L &= b_L c_0 + b_{L-1} c_1 + \cdots + b_0 c_L \end{aligned}$$

and,

$$\begin{aligned} 0 &= b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \cdots + b_0 c_{L+1} \\ 0 &= b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \cdots + b_0 c_{L+2} \\ &\vdots \\ 0 &= b_M c_L + b_{M-1} c_{L+1} + \cdots + b_0 c_{L+M} \end{aligned}$$

where for convenience we set $c_k = 0$ for $k < 0$. The first systems shows the numerator $P^{[L/M]}$ is determined by the denominator $Q^{[L/M]}$. From the second system we can derive a determinantal formula for $Q^{[L/M]}$. Augmented with the desired definition,

$$Q^{[L/M]}(z) = b_M z^M + b_{M-1} z^{M-1} + \cdots + b_0$$

the system can be written in the form

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q^{[L/M]}(z) \end{pmatrix} = \begin{pmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ z^M & z^{M-1} & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix}$$

Solving for b_0 by Cramer's Rule we get,

$$b_0 \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M} \\ z^M & z^{M-1} & \cdots & 1 \end{vmatrix} = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} \end{vmatrix} Q^{[L/M]}(z)$$

The LHS is clearly a polynomial, and nonzero if the RHS minor is nonzero. Thus if the Hankel determinant

$$\begin{vmatrix} c_{L-M+1} & \cdots & c_L \\ \vdots & \ddots & \vdots \\ c_L & \cdots & c_{L+M-1} \end{vmatrix}$$

is nonzero, then up to the afformentioned constant factor,

$$Q^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & \cdots & c_{L+1} \\ \vdots & \ddots & \vdots \\ c_L & \cdots & c_{L+M} \\ z^M & \cdots & 1 \end{vmatrix} \tag{6.3}$$

By convention we normalize $R^{[L/M]}(z)$ so that $b_0 = Q^{[L/M]}(0) = 1$.

Since the Hankel determinants give a useful form for expressing some conditions of interest we will difine $H(n, m)$ to be the determinant of the $(m+1) \times (m+1)$ Hankel matrix starting with c_n,

$$H(n, m) := \begin{vmatrix} c_n & \cdots & c_{n+m} \\ \vdots & \ddots & \vdots \\ c_{n+m} & \cdots & c_{n+2m} \end{vmatrix}$$

Note in particular that the sufficient condition $Q^{[L/M]}(0) \neq 0$ for the existence of $R^{[L/M]}(z)$ is equivalent to $H(L-M+1, M-1) \neq 0$

We now narrow our focus to Padé approximants to a Humburger series. Let μ be a positive Borel measure on $\mathbb{R}$ with infinite support and finite moments $c_k = \int p^k \, d\mu$. Recall that the Hamburger transform of μ,

$$g(z) := \int_{\mathbb{R}} \frac{d\mu}{1 + pz}$$

has the assymptotic expansion, in the sense of non-tangential limits,

$$g(z) \simeq \sum_{k=0}^{\infty} c_k(-z^k).$$

We call this formal power series a Hamburger series.

Remark 6.3. To simplify the rest of this maybe we redefine

$$c_k = (-1)^k \int p^k d\mu$$

Lemma 6.4. *The hamburger moments c_k satisfy the determinental condition $H(2n, m) \neq 0$ for $n, m = 0, 1, \ldots$.*

Proof. Consider the quadratic form given by

$$G(\mathbf{x}, \mathbf{y}) := \mathbf{x}^\top \begin{pmatrix} c_{2n} & \cdots & c_{2n+m} \\ \vdots & \ddots & \vdots \\ c_{2n+m} & \cdots & c_{2n+2m} \end{pmatrix} \mathbf{y}$$

If $\mathbf{x} = (x_0, x_1, \ldots, x_m)^\top$ then

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}) &= \sum_{i,j=0}^{m} x_i x_j c_{2n+i+j} \\ &= \sum_{i,j=0}^{m} x_i x_j \int p^{2n+i+j}\ d\mu \\ &= \int \sum_{i,j=0}^{m} x_i x_j p^{2n+i+j}\ d\mu \\ &= \int p^{2n} \left(\sum_{k=0}^{m} x_k p^k \right)^2\ d\mu \geq 0. \end{aligned}$$

Thus $G(\mathbf{x}, \mathbf{y})$ is positive semi-definite. Furthermore, equality holds

$$\int p^{2n} \left(\sum_{k=0}^{m} x_k p^k \right)^2\ d\mu = 0$$

if and only if μ is supported entirely on the zeros of the polynomial $p^n \sum_{k=0}^{m} x_k p^k$. However by assumption μ has infinite support. So in fact G is strictly positive definite. We conclude, by Sylvester's criterion for example, that assocciated hankel matrix has positive determinant. That is, $D(2n, m) > 0$. $\square$

The previous lemma guarentees the existence of certain Padé approximants, specifically those with $L - M + 1$ even.

Theorem 6.5. *If J is odd then the Padé approximant $R^{[L/L+J]}(z)$ to the Hamburger series exists in Baker's sense.*

With our particular application in mind, we will assume the measure μ is absolutely continuous with respect to the Lebesgue measure and denote it $f(x)dx$ where $f(x)$ is some non-negative Lebesgue measureable function of $\mathbb{R}$. Let $R_M(z)$ be the "offdiagonal" approximant $R^{[M/M+1]}(z)$ to the Hamburger series of μ,

$$R_M(z) = \frac{P_M(z)}{Q_M(z)} \approx \sum_{k=0}^{\infty} c_k (-z)^k \simeq \int_{-\infty}^{\infty} \frac{f(p)}{1 + zp}\ dp$$

where $c_k = \int p^k f(p) dp$. In this section we will prove

Theorem 6.6. *The off-diagonal Padé approximants $R_M(z)$ to the a determinate Hamburger series $\sum_{k=0}^{\infty} c_k(-z)^k$ exist and converge to $g(z)$ uniformly on compact subsets of the upper half plane $\{Im\ z > 0\}$.*

An outline of the proof is as follows: We first justify that the approximants exist in Baker's sense. Then it can be shown that limit of any convergent subsequence of R_M must have a representation as the Hamburger transform of some Borel measure μ which is a solution to our moment problem, and that furthermore such convergent subsequences exist. Thus in order for the sequence R_M to converge to $g(z)$ it will be necessary and sufficient that the moment problem be determinate.

Proposition 6.7. *If the sequence c_k gives a determinate moment problem then the off-diagonal Padé approximants $R_M(z)$ converge to $g(z)$ locally uniformly.*

Proof. The Padé approximant $R_M(z)$ has a convenient representation as an inner product in terms of a finite Jacobi matrix,

$$R_M(z) = \langle \delta_0, (1 + zA_M)^{-1}\delta_0 \rangle.$$

Now since A_M is a real matrix and $\frac{1}{|w|} \geq \frac{1}{|\mathrm{Im}(w)|}$ for any $w \in \mathbb{C}$, we see that

$$|zR_M(z)| = |\langle \delta_0, (z^{-1} + A_M)^{-1}\delta_0 \rangle| \leq \frac{1}{|\mathrm{Im}(1/z)|}$$

and thus

$$|R_M(z)| \leq \frac{1}{|z||\mathrm{Im}(1/z)|} = \frac{|z|}{|\mathrm{Im}(z)|}.$$

Since this bound is independent of R_M, Montel's theorem implies that the off-diagonal Pade approximants form a normal family. It can be shown that the limit of any convergent subsequence has a representation $\int(1 + xz)d\sigma(x)$ where σ is a solution to the Hamburger moment problem. Since the moment problem is determinate, the sequence R_M must converge to $g(z)$ uniformly on compact sets. □

It remains to discuss the determinacy of the Hamburger moment problem. Here we need to add an additional constraint on the measure $d\mu = f(p)dp$, which is that $f(p)$ is L^2 integrable with respect to the Gaussian weight.

Proposition 6.8. *A function $f(p) \in L^2(\mathbb{R}, e^{-p^2}dp)$ such that the moments*

$$c_k = \int_{-\infty}^{\infty} p^k f(p) e^{-p^2}\ dp,\ \ k = 0, 1 \ldots \tag{6.4}$$

are finite, is uniquely determined by those moments.

Proof. It is sufficient to show that if $c_k = 0$ for all $k \geq 0$, then $f \equiv 0$ a.e. The Hermite moments are just linear combinations of 0,

$$\int_{-\infty}^{\infty} H_k(p) f(p) e^{-p^2}\ dp = 0,\ \ k \geq 0$$

where $H_k(p)$ is the Hermite polynomial of order k. Since the Hermite polynomials are complete in the space $L^2(\mathbb{R}, e^{-p^2}dp)$ then $f \equiv 0$. □

7. Convergence results for the Gaussian Radon transform

Now we are in the position to assemble all the previous results into an approximation scheme to be used for reconstruction shapes of unbounded domains.

Take a non-negative Lebesgue measureable function $f(x)$ on $\mathbb{R}^n$ with finite multivariate Gaussian moments

$$c_\alpha^G = \int_{\mathbb{R}^n} f(x) e^{-\|x\|/2} x^\alpha dx$$

We have seen that the multivariate Stieltjes transform of f,

$$g(z) = \int_{\mathbb{R}^n} \frac{f(x)e^{-\|x\|/2}}{1+\langle x,\omega\rangle z} dx,$$

is equal to the Hamburger transform of $GR_f(\omega, p)$, whose moments in turn are easily computed from the moments of f, for fixed ω. Thus for z in the one dimensional subspace spanned by ω, $g(z)$ can be approximated well by Padé approximants, under the integrability condition $GR_f(\omega,p) \in L^2(\mathbb{R}, e^{-p^2}\ dp)$. On the other hand as an integral on $\mathbb{R}^n$, we can also approximate $g(z)$ by cubature formula,

$$\int_{\mathbb{R}^n} \frac{f(x)e^{-\|x\|^2/2}}{1+\langle x,\omega\rangle z} dx \approx \sum_{\ell=L} \frac{e^{-\|x_\ell\|^2/2}}{1+\langle x_\ell,\omega\rangle z} w_\ell f(x_\ell) =: \sum_{\ell=L} C_\ell(z) f(x_\ell).$$

By equating these parallel approximations,

$$\sum_{\ell=L} C_\ell(z) f(x_\ell) \approx R_M(z),$$

with a sufficient quantity of sample points z_j we have arrived at a linear system

$$\sum_{\ell=L} C_\ell(z_j) f(x_\ell) = R_M(z_j), \qquad j = 1, 2, \ldots, J$$

from which we should be able to recover the value of f on our cubature nodes x_ℓ.

8. An example

For a simple example let $n = 2$ and let $f(x)$ be the charactaristic function of an infinite "strip" of unit width centered on the origin and perpendicular to $\varphi \in S^1$:

$$f(x) = \begin{cases} 1, & \langle x, \varphi\rangle \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

Unit ball in $\mathbb{R}^3$? Annulus in $\mathbb{R}^2$? Some assymmetric or off-center examples?

MD, Department of Mathematics, University of Connecticut, 341 Mansfield Road, U-1009, Storrs, CT 06269–1009, USA
Email address: maksym.derevyagin@uconn.edu

AS, Department of Mathematics, University of Connecticut, 341 Mansfield Road, U–1009, Storrs, CT 0626–91009, USA
Email address: ambar.sengupta@uconn.edu

JT, Department of Mathematics, University of Connecticut, 341 Mansfield Road, U–1009, Storrs, CT 06269–1009, USA
Email address: jasper.toman{yih@uconn.edu