Natural Language Models and Interfaces

BSc Artificial Intelligence

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Institute for Logic, Language, and Computation

2019, week 1, lecture b

NLMI

Random variables

Probability distributions

Discrete distributions

Maximum likelihood estimation

Variables: Deterministic vs Random

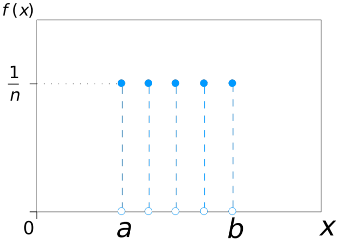
Deterministic variable: $v = 5$

Image from Wikipedia

Variables: Deterministic vs Random

Deterministic variable: $v = 5$

Random variable: $X \sim \mathcal{U}(a, b)$



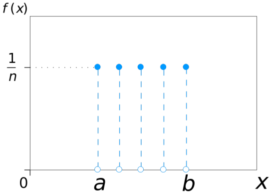
- the random variable can take on *any value* in a certain set
- here this set is the discrete interval $[a, b]$
- we don't know the value of the random variable
  we know it's distribution

Image from Wikipedia

Probability of an outcome

We cannot talk about **the exact value** of the random variable but we can reason about it's possible values

- we quantify the degree of belief we have in each *outcome*



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Uniform distribution: every outcome is **equally likely**

- if n is the size of the set of possible outcomes
  the probability that X takes on any value (e.g. a) is $\frac{1}{n}$
  $P(X = x) = \frac{1}{n}$ for all $x \in [a, b]$

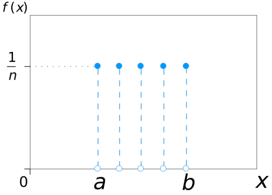


Image from Wikipedia

Let's name some things

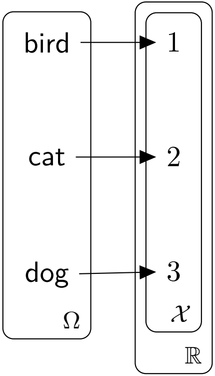
A random variable is a **function**

- it maps from a sample space Ω to $\mathbb{R}$
  $X : \Omega \to \mathbb{R}$

Example: "which pet do kids love the most?"

- Sample space: $\Omega = \{\text{bird}, \text{cat}, \text{dog}\}$

$$X(\omega) = \begin{cases} 1 & \omega = \{\text{bird}\} \\ 2 & \omega = \{\text{cat}\} \\ 3 & \omega = \{\text{dog}\} \end{cases}$$



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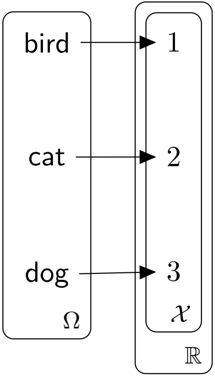

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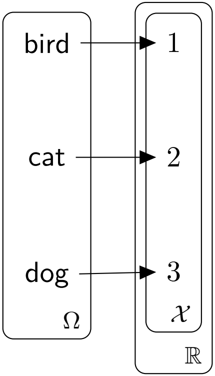
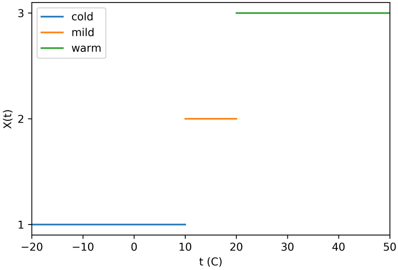
- we call $\mathcal{X}$ the support of X



Temperature example

Let's take the outside temperature as a random variable

- we might not particularly care whether it's -3 or -3.2
- but we probably care to ask
  "How does it feel outside?"



Example from ▸ Basic Probability by Schulz and Schaffner (2016)

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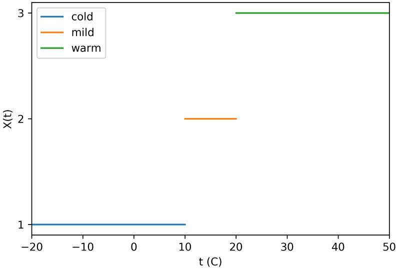
Let's define an RV

- Sample space
  some segment of the real line
  - perhaps from -40 to 50?
  - cap on precision?
- $X(t) = \begin{cases} 1 & t < 10 \\ 2 & 10 \leq t \leq 20 \\ 3 & t > 20 \end{cases}$



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Types of random variables

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They can be vector-valued

- a point in a 2D-plane: e.g. (x, y) coordinates
- a point in a d-dimensional space: e.g. database records house: *floor area, latitude, longitude, altitude, number of rooms, age, number of past owners, market value*

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Notation

- distribution: P_X, $P_X(X)$, $P(X)$
- value: $P_X(X = x)$, $P(X = x)$, $P_X(x)$, $P(x)$

Joint probability distribution

Oftentimes we care about multiple random variables
and how their outcomes co-occur

Ω		Letter (L)	
Grade	G	0	1
$[0, 6)$	1	$(1, 0)$	$(1, 1)$
$[6, 8)$	2	$(2, 0)$	$(2, 1)$
$[8, 10]$	3	$(3, 0)$	$(3, 1)$

P_{GL}		Letter (L)	
Grade	G	0	1
$[0, 6)$	1	0.16	0.04
$[6, 8)$	2	0.42	0.28
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Joint probability $P(G = g, L = l)$

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Properties

- $0 \leq P(G = g, L = l) \leq 1$ for all $(g, l) \in \mathcal{G} \times \mathcal{L}$
- $\sum_{g \in \mathcal{G}} \sum_{l \in \mathcal{L}} P(G = g, L = l) = 1$

Marginal probability

Recover the distribution of each RV

P_{GL}		Letter (L)		
Grade	G	0	1	P_G
$[0, 6)$	1	0.16	0.04	0.2
$[6, 8)$	2	0.42	0.28	0.7
$[8, 10]$	3	0.01	0.09	0.1
	P_L	0.59	0.41	

Table: Joint distribution P_{GL} and marginals P_G and P_L

Sum over all values of one of the RVs

- $P(G = g) = \sum_{l \in \mathcal{L}} P(G = g, L = l)$
- $P(L = l) = \sum_{g \in \mathcal{G}} P(G = g, L = l)$

Conditional probability

If we know the value of one of the RVs
we can rescale to get a distribution

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$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

$P_{L\|G=g}$		Letter (L)		
Grade	G	0	1	$\rightarrow$
$[0,6)$	1	0.8	0.2	1.0
$[6,8)$	2	0.6	0.4	1.0
$[8,10]$	3	0.1	0.9	1.0

$P_{G\|L=l}$		Letter (L)	
Grade	G	0	1
$[0,6)$	1	0.27	0.10
$[6,8)$	2	0.71	0.68
$[8,10]$	3	0.02	0.22
	$\downarrow$	1.00	1.00

Table: Conditional distributions $P_{L|G=g}$ and $P_{G|L=l}$

Rules of probability

Chain rule

- Two RVs
  $P(X = x, Y = y) = P(X = x)P(Y = y|X = x)$

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$$P_{X|Y}(x|y) = \frac{P_X(x)P_{Y|X}(y|x)}{P_Y(y)}$$

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If X does not depend on Y
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And in general

$$P_{X_1^n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P_{X_i}(x_i)$$

NLMI

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Bernoulli

A Bernoulli variable is a binary random variable

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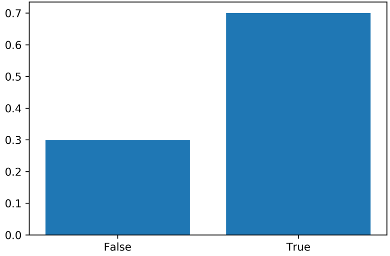

Categorical

A Categorical variable can model 1 of k categories

$$X \sim \text{Cat}(\theta_1, \ldots, \theta_k)$$

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- the categorical parameter is a probability vector
  - $0 \leq \theta_x \leq 1$ for $x \in [1, k]$
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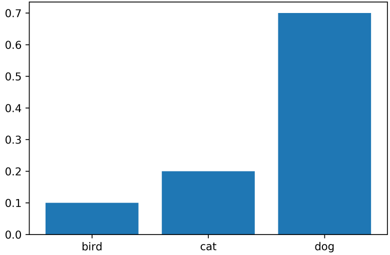

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Statistical estimation

We investigate problems

- we hypothesise interactions between variables
- we assume variables have a certain nature
- we choose probability distributions
- we try to estimate parameters for these distributions as to reproduce “natural” observations

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and with *iid* observations $\prod_{i=1}^n P_{X_i}(x_i) = \prod_{i=1}^n P_X(x_i)$

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The maximum likelihood principle is about

- picking α to give maximum probability to observations
- where the probability of observations (or *likelihood*) is
  $P_{X_1^n}(x_1, \ldots, x_n; \alpha) = \prod_{i=1}^n P_X(x_i; \alpha)$
  due to the *idd* assumption

Optimisation

We start with our likelihood function

$$P_{X_1^n}(x_1, \ldots, x_n; \alpha) = \prod_{i=1}^{n} P_X(x_i; \alpha)$$

and proceed to optimise the parameter α

$$\alpha^\star = \operatorname*{argmax}_{\alpha} \; P_{X_1^n}(x_1, \ldots, x_n; \alpha) \quad \alpha \text{ such that likelihood is maximised}$$

We assume argmax to return a point (not a set). Want to know more about argmax? Check this out

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$$\begin{aligned} \alpha^\star &= \operatorname*{argmax}_{\alpha} \; P_{X_1^n}(x_1, \ldots, x_n; \alpha) && \alpha \text{ such that likelihood is maximised} \\ &= \operatorname*{argmax}_{\alpha} \; \prod_{i=1}^{n} P_X(x_i; \alpha) && \text{iid observations} \end{aligned}$$

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$$\begin{aligned}
\alpha^\star &= \operatorname*{argmax}_{\alpha} \; P_{X_1^n}(x_1, \ldots, x_n; \alpha) && \alpha \text{ such that likelihood is maximised} \\
&= \operatorname*{argmax}_{\alpha} \; \prod_{i=1}^{n} P_X(x_i; \alpha) && \text{iid observations} \\
&= \operatorname*{argmax}_{\alpha} \; \log \prod_{i=1}^{n} P_X(x_i; \alpha) && \log \text{ is monotonic}
\end{aligned}$$

We assume argmax to return a point (not a set). Want to know more about argmax? Check this out

Optimisation

We start with our likelihood function

$$P_{X_1^n}(x_1, \ldots, x_n; \alpha) = \prod_{i=1}^{n} P_X(x_i; \alpha)$$

and proceed to optimise the parameter α

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&= \operatorname*{argmax}_{\alpha} \; \prod_{i=1}^{n} P_X(x_i; \alpha) && \text{iid observations} \\
&= \operatorname*{argmax}_{\alpha} \; \log \prod_{i=1}^{n} P_X(x_i; \alpha) && \log \text{ is monotonic} \\
&= \operatorname*{argmax}_{\alpha} \; \sum_{i=1}^{n} \log P_X(x_i; \alpha) && \text{numerically convenient}
\end{aligned}$$

We assume argmax to return a point (not a set). Want to know more about argmax? Check this out

MLE solutions

Bernoulli

- $p = \frac{n_1}{n}$ where $n_1 = \sum_{i=1}^{n} x_i$

δ is the *Kronecker delta*

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Categorical

- $\theta_x = \frac{\text{count}(x)}{n}$ where $\text{count}(x) = \sum_{i=1}^{n} \delta_{x_i x}$
  for all $x \in \mathcal{X} = \{1, \ldots, k\}$

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Quiz

δ is the *Kronecker delta*

MLE: Bernoulli

Probability mass function

- $\text{Bern}(X = a|p) = p^a(1-p)^{1-a}$
  $0 < p < 1$

Problem: optimisation of the $\log$-likelihood function $\mathcal{L}(p)$

$$p^\star = \underset{p \in (0,1)}{\text{argmax}} \; \underbrace{\sum_{i=1}^{n} \log \text{Bern}(x_i|p)}_{\mathcal{L}(p)}$$

Strategy

1. set first derivative of $\mathcal{L}(p)$ to 0
2. solve for p

Bernoulli: MLE derivation

Derivative

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{L}(p)}{\mathrm{d}p} &= \frac{\mathrm{d}}{\mathrm{d}p}\left[\sum_{i=1}^{n} x_i \log p + (1-x_i)\log(1-p)\right] \\
&= \sum_{i=1}^{n} x_i \frac{\mathrm{d}}{\mathrm{d}p}\log p + (1-x_i)\frac{\mathrm{d}}{\mathrm{d}p}\log(1-p) \\
&= \sum_{i=1}^{n} \frac{x_i}{p} + \frac{1-x_i}{1-p}(-1) \\
&= \sum_{i=1}^{n} \frac{x_i(1-p) - (1-x_i)p}{p(1-p)} \\
&= \frac{(1-p)}{p(1-p)} \underbrace{\sum_{i=1}^{n} x_i}_{n_1} - \frac{p}{p(1-p)} \underbrace{\sum_{i=1}^{n} 1 - x_i}_{n_0} \\
&= \frac{(1-p)}{p(1-p)} n_1 - \frac{p}{p(1-p)} n_0
\end{aligned}
$$

Set to 0 and solve for p

$$
\begin{aligned}
0 &= \frac{(1-p)}{p(1-p)} n_1 - \frac{p}{p(1-p)} n_0 \\
&= (1-p)n_1 - pn_0 \\
&= n_1 - p_n 1 - pn_0 \\
&= n_1 - p(n_1 + n_0) \\
n_1 &= p(n_1 + n_0) \\
p &= \frac{n_1}{n_1 + n_0} \\
p &= \frac{n_1}{n}
\end{aligned}
$$

Note

- $n_1 = \sum_{i=1}^{n} x_i$
- $n_0 = \sum_{i=1}^{n} (1 - x_i)$
- $n = n_1 + n_0$

MLE: Categorical

Probability mass function

- $\mathrm{Cat}(X = a|\theta_1, \ldots, \theta_k) = \prod_{x=1}^{k} {\theta_x}^{\delta_{xa}}$
  $\sum_{x=1}^{k} \theta_x = 1$ with $\theta_x \in \mathbb{R}_{>0}$ for all $x \in [1, k]$

Problem: optimisation of the log-likelihood function $\mathcal{L}(\theta_1^k)$

$$p^\star = \operatorname*{argmax}_{\theta_1^k \in \mathbb{R}_{>0}^k} \underbrace{\sum_{i=1}^{n} \log \mathrm{Cat}(x_i|\theta_1^k)}_{\mathcal{L}(\theta_1, \ldots, \theta_k)} \qquad \text{s.t.} \sum_{x=1}^{k} \theta_x = 1$$

Strategy

1. introduce Lagrange multiplier λ for the constraint $\sum_{x=1}^{k} \theta_x = 1$
2. set partial derivatives to 0
3. solve for λ and θ_1^k

Check the complete derivation

Next steps

Lab2

- probability theory
- MLE for Bernoulli and Categorical

Next lecture we will discuss sequence prediction

- we will model with Categorical distributions
- and obtain maximum likelihood estimates from text

References I