Natural Language Models and Interfaces

BSc Artificial Intelligence

Lecturer: Wilker Aziz
Institute for Logic, Language, and Computation

2018, week 1, lecture b

NLMI

Random variables

Probability distributions

Discrete distributions

Maximum likelihood estimation

Variables: Deterministic vs Random

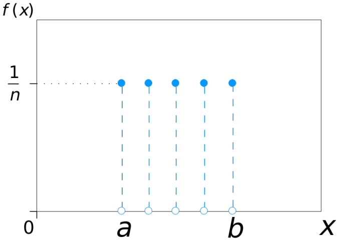
Deterministic variable: $v = 5$

Image from Wikipedia

Variables: Deterministic vs Random

Deterministic variable: $v = 5$

Random variable: $X \sim \mathcal{U}(a, b)$



- the random variable can take on *any value* in a certain set
- here this set is the discrete interval $[a, b]$
- we don't know the value of the random variable
  we know it's distribution

Image from Wikipedia

Probability of an outcome

We cannot talk about **the exact value** of the random variable but we can reason about it's possible values

- we quantify the degree of belief we have in each *outcome*

Uniform distribution: every outcome is **equally likely**

- if n is the size of the set of possible outcomes the probability that X takes on any value (e.g. a) is $\frac{1}{n}$
$P(X = x) = \frac{1}{n}$ for all $x \in [a, b]$

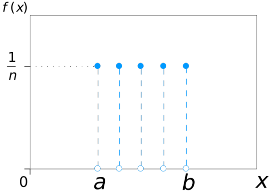


Image from Wikipedia

Let's name some things

A random variable is a **function**

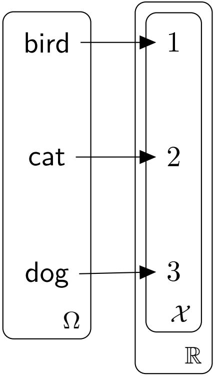
- it maps from a sample space Ω to $\mathbb{R}$
  $X : \Omega \to \mathbb{R}$

Example: "which pet do kids love the most?"

- Sample space: $\Omega = \{\text{bird}, \text{cat}, \text{dog}\}$

$$X(\omega) = \begin{cases} 1 & \omega = \{\text{bird}\} \\ 2 & \omega = \{\text{cat}\} \\ 3 & \omega = \{\text{dog}\} \end{cases}$$

- if say $X = x$ we mean the set of outcomes $\{\omega : X(\omega) = x\}$ which is called an event
- we call $\mathcal{X}$ the support of X



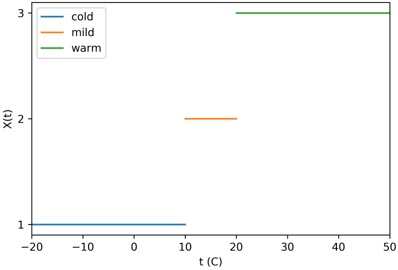
Temperature example

Let's take the outside temperature as a random variable

- we might not particularly care whether it's -3 or -3.2
- but we probably care to ask
  "How does it feel outside?"

Let's define an RV

- Sample space
  some segment of the real line
  - perhaps from -40 to 50?
  - cap on precision?
- $X(t) = \begin{cases} 1 & t < 10 \\ 2 & 10 \leq t \leq 20 \\ 3 & t > 20 \end{cases}$



Example from ▸ Basic Probability by Schulz and Schaffner (2016)

Types of random variables

Random variables are different in nature

- categorical: toss a coin
- ordinal: number of items in a bag
- continuous: height, weight

They can have finite or infinite support

- toss a coin, throw a die: finitely many outcomes
- distances: infinitely many outcomes
- number of stars: infinitely many outcomes

They can be vector-valued

- a point in a 2D-plane: e.g. (x, y) coordinates
- a point in a d-dimensional space: e.g. database records
  house: *floor area, latitude, longitude, altitude, number of rooms, age, number of past owners, market value*

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Discrete probability distribution

The discrete probability distribution of a random variable X

- assigns a probability value to each value X may take on
- probability values are *never less than 0*
  $P(X = x) \geq 0$ for all $x \in \mathcal{X}$
- and a probability distribution *sums to 1*
  $\sum_{x \in \mathcal{X}} P(X = x) = 1$
- thus we have
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Notation

- distribution: P_X, $P_X(X)$, $P(X)$
- value: $P_X(X = x)$, $P(X = x)$, $P_X(x)$, $P(x)$

Joint probability distribution

Oftentimes we care about multiple random variables
and how their outcomes co-occur

Ω		Letter (L)	
Grade	G	0	1
$[0, 6)$	1	$(1, 0)$	$(1, 1)$
$[6, 8)$	2	$(2, 0)$	$(2, 1)$
$[8, 10]$	3	$(3, 0)$	$(3, 1)$

P_{GL}		Letter (L)	
Grade	G	0	1
$[0, 6)$	1	0.16	0.04
$[6, 8)$	2	0.42	0.28
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Table : Joint sample space Ω and joint distribution P_{GL}

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Properties

- $0 \leq P(G = g, L = l) \leq 1$ for all $(g, l) \in \mathcal{G} \times \mathcal{L}$
- $\sum_{g \in \mathcal{G}} \sum_{l \in \mathcal{L}} P(G = g, L = l) = 1$

Marginal probability

Recover the distribution of each RV

P_{GL}		Letter (L)		
Grade	G	0	1	P_G
$[0, 6)$	1	0.16	0.04	0.2
$[6, 8)$	2	0.42	0.28	0.7
$[8, 10]$	3	0.01	0.09	0.1
	P_L	0.59	0.41	

Table : Joint distribution P_{GL} and marginals P_G and P_L

Sum over all values of one of the RVs

- $P(G = g) = \sum_{l \in \mathcal{L}} P(G = g, L = l)$
- $P(L = l) = \sum_{g \in \mathcal{G}} P(G = g, L = l)$

Conditional probability

If we know the value of one of the RVs we can rescale to get a distribution

P_{GL} Grade	G	Letter (L) 0	1	P_G
$[0,6)$	1	0.16	0.04	0.2
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$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

$P_{L\|G=g}$ Grade	G	Letter (L) 0	1	$\rightarrow$
$[0,6)$	1	0.8	0.2	1.0
$[6,8)$	2	0.6	0.4	1.0
$[8,10]$	3	0.1	0.9	1.0

$P_{G\|L=l}$ Grade	G	Letter (L) 0	1
$[0,6)$	1	0.27	0.10
$[6,8)$	2	0.71	0.68
$[8,10]$	3	0.02	0.22
	$\downarrow$	1.00	1.00

Table : Conditional distributions $P_{L|G=g}$ and $P_{G|L=l}$

Rules of probability

Chain rule

- Two RVs
  $P(X = x, Y = y) = P(X = x)P(Y = y|X = x)$

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$$P_{X|Y}(x|y) = \frac{P_X(x)P_{Y|X}(y|x)}{P_Y(y)}$$

Independence

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we say X is independent of Y or $X \perp Y$
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$$P_{XY}(x, y) = P_X(x)P_Y(y)$$

And in general

$$P_{X_1^n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} P_{X_i}(x_i)$$

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Bernoulli

A Bernoulli variable is a binary random variable

$$X \sim \mathrm{Bern}(p)$$

- $\mathcal{X} = \{0, 1\}$
- p is the **Bernoulli parameter**
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Quiz

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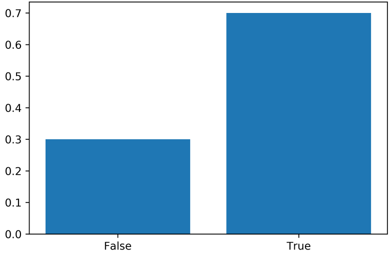

Categorical

A Categorical variable can model 1 of k categories

$$X \sim \mathrm{Cat}(\theta_1, \ldots, \theta_k)$$

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- the categorical parameter is a probability vector
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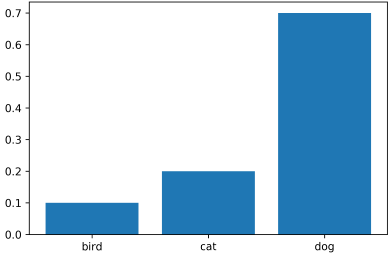

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Statistical estimation

We investigate problems

- we hypothesise interactions between variables
- we assume variables have a certain nature
- we choose probability distributions
- we try to estimate parameters for these distributions as to reproduce "natural" observations

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and with *iid* observations $\prod_{i=1}^{n} P_X(x_i)$

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The maximum likelihood principle is about

- picking parameter values that give maximum probability to observations
- where the probability of observations is
  $P(x_1, \ldots, x_n; \alpha) = \prod_{i=1}^{n} P_{X;\alpha}(x_i)$
  due to the *idd* assumption

Optimisation

We start with our likelihood function

$$P(x_1, \ldots, x_n; \alpha) = \prod_{i=1}^{n} P_{X;\alpha}(x_i)$$

which depends on a choice of α

And we proceed by optimising this choice

$$\begin{aligned}
\alpha^\star &= \underset{\alpha}{\operatorname{argmax}} \; P(x_1, \ldots, x_n; \alpha) \\
&= \underset{\alpha}{\operatorname{argmax}} \; \prod_{i=1}^{n} P_X(x_i; \alpha) \\
&= \underset{\alpha}{\operatorname{argmax}} \; \log \prod_{i=1}^{n} P_X(x_i; \alpha) \\
&= \underset{\alpha}{\operatorname{argmax}} \; \sum_{i=1}^{n} \log P_X(x_i; \alpha)
\end{aligned}$$

MLE solutions

Bernoulli

- $p = \frac{1}{n} \sum_{i=1}^{n} x_i$

Categorical

- $\theta_x = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}[x_i = x]$

Quiz

Next week

Lab2

- probability theory
- MLE for Bernoulli and Categorical

Next lecture we will discuss sequence prediction

- we will model with Categorical distributions
- and obtain maximum likelihood estimates from text

References I