CS281A/STAT241A Homework 5

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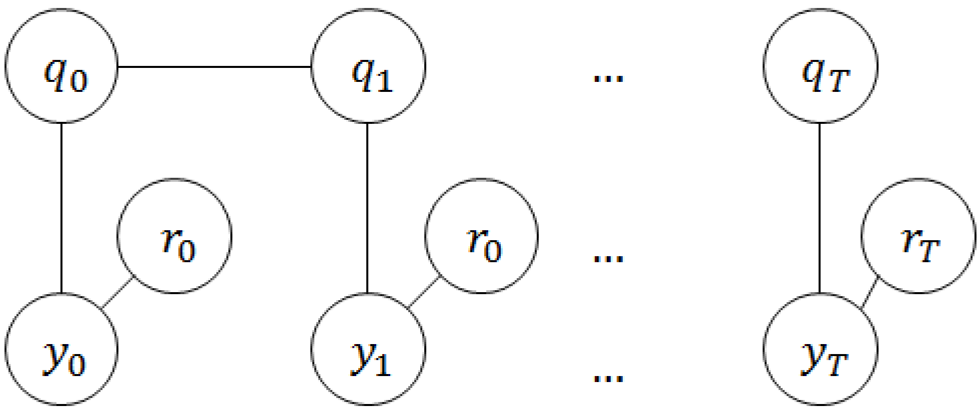
HMMs with mixtures of Poissons

a. Graphical model

The graphical model for HMM with mixtures of Poissons looks like this:



b. Expected complete log likelihood

$$\log p(q,r,y) = \sum_{i=1}^{m} q_0^i \log \pi_i + \sum_{t=0}^{T-1}\sum_{i=1}^{m}\sum_{j=1}^{m} q_t^i q_{t+1}^j \log a_{ij} + \sum_{t=0}^{T}\sum_{i=1}^{m}\sum_{j=1}^{k} y_t q_t^i r_t^j \log \lambda_{ij} + [\lambda_{ij}]^{q_t^i r_t^j} - \log y_t!$$

Sufficient statistic for a_{ij} is $m_{ij} \triangleq \sum_{t=0}^{T-1} q_t^i q_{t+1}^j$; sufficient statistic for λ_{ij} is $n_{ij} \triangleq \sum_{t=0}^{T} y_t q_t^i r_t^j$; sufficient statistic for π_i is q_0^i.

The maximum likelihood estimates for the case of complete data are therefore given by:

$$\hat{a_{ij}} = \frac{m_{ij}}{\sum_{s=1}^{m} m_{is}}$$
$$\hat{\pi_{ij}} = q_0^i$$
$$\hat{\lambda_{ij}} = \frac{n_{ij}}{\sum_{s=1}^{k} n_{is}}$$

Parameters of the mdoel are defined as $\theta \triangleq (\pi, A, \lambda)$. The expectations for E step is given by:

$$E(m_{ij}|y,\theta^{(p)}) = \sum_{t=0}^{T-1} p(q_t^i q_{t+1}^j|y,\theta^{(p)}) \triangleq \sum_{t=0}^{T-1} \xi_{t,t+1}^{ij}$$
$$E(n_{ij}|y,\theta^{(p)}) = \sum_{t=0}^{T} y_t r_t^j p(q_t^i|y,\theta^{(p)}) \triangleq \sum_{t=0}^{T} \gamma_t^i r_t^j y_t$$

c. Algorithm for E step

- Forward recursion

$$\alpha(q_{t+1}) = \sum_{q_t} \alpha(q_t) a_{q_t,q_{t+1}} p(y_{t+1}|q_{t+1})$$

Initialization of $\alpha(q_0)$ is given by:

$$\alpha(q_0) = p(y_0|q_0)\pi_{q_0}$$

- Backward recursion

$$\beta(q_t) = \sum_{q_{t+1}} \beta(q_{t+1}) a_{q_t,q_{t+1}} p(y_{t+1}|q_{t+1})$$

Initialization of $\beta(q_T)$ is no more complex than a vector of ones.

Thus, we have everything we need for the E step:

$$\gamma(q_t) = \frac{\alpha(q_t)\beta(q_t)}{\sum_{q_t} \alpha(q_t)\beta(q_t)}$$
$$\xi(q_t, q_{t+1}) = \frac{\alpha(q_t)p(y_{t+1}|q_{t+1})\gamma(q_{t+1})a_{q_t,q_{t+1}}}{\alpha(q_{t+1})}$$

γ_t^i is equal to $\gamma(q_t)$ evaluated at $q_t^i = 1$; $\xi_{t,t+1}^{ij}$ is equal to $\xi(q_t, q_{t+1})$ evaluated at $q_t^i = 1, q_{t+1}^j = 1$.

d. Algorithm for M step

After the calculation of E step, updates in M step are given by:

$$\hat{\lambda}_{ij}^{(p+1)} = \frac{\sum_{t=0}^{T} \gamma_t^i r_t^j y_t}{\sum_{t=0}^{T} \gamma_t^i y_t}$$
$$\hat{a}_{ij}^{(p+1)} = \frac{\sum_{t=0}^{T-1} \xi_{t,t+1}^{ij}}{\sum_{t=0}^{T-1} \gamma_t^i}$$
$$\hat{\pi}_i^{(p+1)} = \gamma_0^i$$