Limits of Computation

10 - Partial evaluation & self-referencing programs

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So far ...

- ... we have seen how programs can be encoded as objects to be used as input to other programs.
- Example: self-interpreter to show semi-decidability of Halting Problem.

Question:

Can we write a Java program that prints itself or a `While` program that returns its own AST (without using its input)?

Other uses of programs-as-data

- **Partial Evaluation** (Kleene's S-m-n Theorem)
  - Use of partial evaluation for Optimisation and Compiler Generation
- **Self-referencing Programs** (Kleene's Recursion Theorem)
  - Use of Recursion Theorem for Recursion Elimination

THIS TIME

```java
public class Quine
{
  public static void main(String[] args)
  {
    char q = 34;      // Quotation mark character
    String[] l = {    // Array of source code
    "public class Quine",
    "{",
    "  public static void main(String[] args)",
    "  {",
    "    char q = 34;      // Quotation mark character",
    "    String[] l = {    // Array of source code",
    "    ",
    "    };",
    "    for(int i = 0; i < 6; i++)           // Print opening code",
    "        System.out.println(l[i]);",
    "    for(int i = 0; i < l.length; i++)    // Print string array",
    "        System.out.println(l[6] + q + l[i] + q + ',');",
    "    for(int i = 7; i < l.length; i++)    // Print this code",
    "        System.out.println(l[i]);",
    "  }",
    "}",
    };
    for(int i = 0; i < 6; i++)           // Print opening code
        System.out.println(l[i]);
    for(int i = 0; i < l.length; i++)    // Print string array
        System.out.println(l[6] + q + l[i] + q + ',');
    for(int i = 7; i < l.length; i++)    // Print this code
        System.out.println(l[i]);
  }
}
```

an interesting Java program: what does it do?

Simple Version of S-m-n Theorem

Theorem **(S-1-1 Theorem).** *For any programming language* L with pairing *and* programs-as-data *there exists a program* spec *such that*

$$\left[\!\left[[\![\mathtt{spec}]\!]^{\mathtt{L}}(p,s) \right]\!\right]^{\mathtt{L}}(d) = [\![p]\!]^{\mathtt{L}}(s,d)$$

(_,_) notation for generic pairing

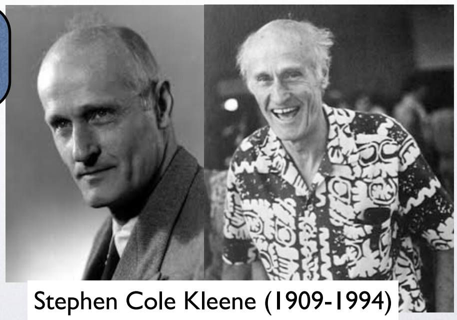
both inputs as pair

Program with 2 inputs (as pair)

partial (1st) input

remaining (2nd) input

- It can be generalised to programs with $m+1$ (not just 2) input and providing n concrete values (not just 1) thus the name (Kleene's) S-m-n Theorem.



Stephen Cole Kleene (1909-1994)
American mathematician
(also did regular expressions!)

S-m-n describes Specialiser

- The resulting program spec is a specialiser (that takes programs as input and returns a new program).
- The concrete code for spec is relatively straightforward and will be given in notes / as program on Canvas site.
- The partial input provides opportunities for optimisations in the code of program p.

Optimisation Example

```
p read XY {
 X  := hd XY;
 Y  := hd tl XY;
 A  := <add>[X,4]
}
write A
```

p

⟦spec⟧ [p,3]

```
newp read XY
XY:= [3,XY];
X:= hd XY;
Y:= hd tl XY;
A:= <add>[X,4]
write A
```

```
X:= 3;
```

```
A:=<add>[3,4]
```

```
A := 7
```

```
… read XY{
A:=7
}
write A
```


"Efficient program specialisation = partial evaluation"

Self-referencing Programs

- Can we write programs that refer to their own source or Abstract syntax tree?
- Can we give them a well-defined semantics?
- For instance, a program that prints its own source or returns its own AST?
- Try it, it's not that simple!! What is the problem here?

Acceptable Prog. Language

A *programming language* as discussed before, defined by syntax, data type, and semantics, having also programs-as-data and pairing, is called *"acceptable"* iff

- it has a universal program
- it has specialisers (S-m-n theorem holds)
- all functions computed by Turing machines or `While`-programs are also computed by one of its programs.

Recursion Theorem

Theorem (Kleene's Recursion Theorem). *Let* L *be an* acceptable programming language. *For any* L*-program p, there is a* L*-program q such that for all input d* $\in$ L*-data we have*

$$[\![q]\!]^{\mathtt{L}}(d) = [\![p]\!]^{\mathtt{L}}(q,d)$$

- q is the self-referential program
- defined by program p that has extra parameter q
- before we prove it, look at example usage

Eliminate Recursion

Factorial recursively:
0! = 1
n! = n*(n-1)!

factorial2 has argument [q,n] so extra argument q

```
fac2 read qn {
  q := hd qn;
  n := hd tl qn;
  if n {
     A := <u> [q, tl n];
     Res := <mult> [n, A]
  }
  else {
   Res := 1
  }
}
write Res
```

Self-interpreter

Recursion Theorem gives us (existence of) `facRefl` such that

$$[\![\texttt{facRefl}]\!]^{\texttt{WHILE}}(d) = [\![\texttt{fac2}]\!]^{\texttt{WHILE}}[\texttt{facRefl}, d]$$

1st Impact of Theorem

- Any acceptable programming language is already "closed under recursion". [Neil Jones]
- Usage in interpreted languages: needs a new self-interpreter called for each recursive call, leading to stacks of self-interpreters running each other,
- leading to exponential runtime,
- thus built-in "reflection" mechanisms to avoid this problem.

2nd Impact of Theorem

- Reflection in compiled languages more difficult as there is no source code or AST.
- So no reflection in C or C++.
- But in Java as there is a ByteCode interpreter. The **Reflection** library allows to introspect classes/methods.
- This allows one to circumvent static type checking to work with "unknown" classes
- Frameworks like *Spring* heavily use reflection.

"Plug-in" technology needs reflection!

Kleene's Recursion Theorem semantically validates the principle of reflection in programming

Proof of Recursion Theorem

- Main technique: again self-application!
- We can apply a program to itself and still get a program (S-1-1 Theorem i.e. specialiser)
- and can plug that into given program p:

$$f(r,d) = [\![p]\!]^{\mathtt{L}}\left([\![\mathtt{spec}]\!]^{\mathtt{L}}(r,r)\ ,\ d\right) \tag{10.1}$$

- f is $\mathtt{L}$-computable by a program $\mathtt{p}^{\mathtt{self}}$

$$[\![p^{\mathtt{self}}]\!]^{\mathtt{L}} = f \tag{10.2}$$

Proof of Recursion Theorem

- Now we consider another self-application:

$$q = [\![\texttt{spec}]\!]^{\texttt{L}}(\texttt{p}^{\texttt{self}}, \texttt{p}^{\texttt{self}}) \tag{10.3}$$

- which returns an L-program which is our desired program q ;
- it remains to show that for this program q:

$$[\![\texttt{q}]\!]^{\texttt{L}}(\texttt{d}) = [\![\texttt{p}]\!]^{\texttt{L}}(\texttt{q}, \texttt{d})$$

Proof of Recursion Theorem

We compute the required equation just by using the definitions we made:

$$
\begin{aligned}
[\![q]\!]^{\texttt{L}}(d) &= [\![[\![\texttt{spec}]\!]^{\texttt{L}}(p^{\texttt{self}}, p^{\texttt{self}})]\!]^{\texttt{L}}(d) && \textit{use Eq. 10.3, the definition of } q \\
&= [\![p^{\texttt{self}}]\!]^{\texttt{L}}(p^{\texttt{self}}, d) && \textit{use S-1-1 Thm. 10.1 with } p \textit{ and } s = p^{\texttt{self}} \\
&= f(p^{\texttt{self}}, d) && \textit{use Eq 10.2, definition } p^{\texttt{self}} \\
&= [\![p]\!]^{\texttt{L}}([\![\texttt{spec}]\!]^{\texttt{L}}(p^{\texttt{self}}, p^{\texttt{self}}), d) && \textit{use Eq. 10.1, definition of } f \\
&= [\![p]\!]^{\texttt{L}}(q, d) && \textit{use Eq. 10.3, definition } q \textit{ backwards}
\end{aligned}
$$

END

Next time:
Why non-computable problems remain non-computable when using different notions of "effective procedure".