The derivative of the sine function

This document explains why the derivative of the sine function is the cosine function.

It starts by proving the two facts in the box below, and then uses them to differentiate the sine function from first principles.

Two useful limits

$$\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right) = 1 \qquad \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \right) = 0.$$

To prove the first fact above, let θ be an angle such that $0 < \theta < \pi/2$, and consider a sector of angle θ radians that's part of a circle of radius 1, as shown in Figure 1. By the formula for the area of a sector, this sector has area $\frac{1}{2}\theta$.

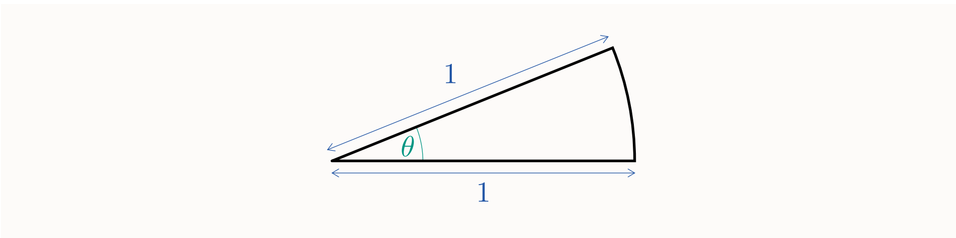

Figure 1 A sector of angle θ and radius 1

In Figure 2, two lines have been drawn perpendicular to one of the straight sides of the sector (and the line that's the other side of the sector has been extended a little), to form two right-angled triangles. Each of the two new lines passes through a vertex of the sector.

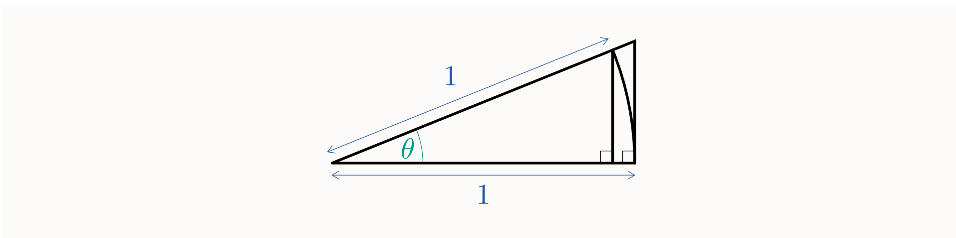

Figure 2 Two right-angled triangles drawn on the sector in Figure 1

The smaller of the two right-angled triangles has a hypotenuse of length 1, so by basic trigonometry it has base $\cos \theta$ and height $\sin \theta$. Hence it has area

$$\tfrac{1}{2} \sin \theta \cos \theta.$$

The larger of the two right-angled triangles has a base of length 1, so by basic trigonometry it has height $\tan\theta$. Hence it has area

$$\tfrac{1}{2}\tan\theta.$$

The area of the sector is larger than the area of the smaller triangle but smaller than the area of the larger triangle. Hence

$$\tfrac{1}{2}\sin\theta\cos\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta.$$

Multiplying these inequalities through by 2, and dividing them through by the positive number $\sin\theta$, gives

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Now consider what happens as θ gets closer and closer to zero. Both $\cos\theta$ and $1/\cos\theta$ get closer and closer to 1, so, since $\theta/\sin\theta$ is sandwiched between them, it must also get closer and closer to 1. It follows that the reciprocal of $\theta/\sin\theta$ also gets closer and closer to 1. That is, $(\sin\theta)/\theta$ gets closer and closer to 1.

This has nearly proved the first fact in the box; the only problem is that so far we've allowed θ only to take positive values. However,

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta},$$

so, as θ gets closer and closer to zero, $(\sin\theta)/\theta$ gets closer and closer to 1, no matter whether θ is positive or negative. This completes the proof of the first fact in the box.

To prove the second fact in the box, we can use the first fact, together with the half-angle identity for sine, which is

$$\sin^2\theta = \tfrac{1}{2}(1-\cos(2\theta)).$$

Replacing θ by $\tfrac{1}{2}\theta$ in this identity gives

$$\sin^2\left(\tfrac{1}{2}\theta\right) = \tfrac{1}{2}(1-\cos\theta),$$

and dividing both sides by $\tfrac{1}{2}\theta$ gives

$$\frac{\sin^2\left(\tfrac{1}{2}\theta\right)}{\tfrac{1}{2}\theta} = \frac{1-\cos\theta}{\theta}.$$

Swapping the sides and then slightly rearranging the right-hand side gives

$$\frac{1-\cos\theta}{\theta} = \left(\frac{\sin\left(\tfrac{1}{2}\theta\right)}{\tfrac{1}{2}\theta}\right)^2 \times \left(\tfrac{1}{2}\theta\right).$$

Now consider what happens as θ gets closer and closer to zero. The value of $\tfrac{1}{2}\theta$ also gets closer and closer to zero, and hence, by the first fact in the box, $\left(\sin\left(\tfrac{1}{2}\theta\right)\right)/\left(\tfrac{1}{2}\theta\right)$ gets closer and closer to 1. So the square of $\left(\sin\left(\tfrac{1}{2}\theta\right)\right)/\left(\tfrac{1}{2}\theta\right)$ gets closer and closer to 1. So the value of the expression on right-hand side of the equation above gets closer and closer to 1×0; that is, to 0. Hence the left-hand side $(1-\cos\theta)/\theta$ also gets closer and closer to zero, and it follows that so does its negative, $(\cos\theta - 1)/\theta$. This proves the second fact in the box.

We're now ready to differentiate the function $f(x) = \sin x$ from first principles.

The difference quotient for $f(x) = \sin x$ is

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}.$$

As usual, to find a formula for $f'(x)$, we have to work out what happens to the value of the difference quotient as h gets closer and closer to zero.

By the angle sum indentity for sine, we have

$$\sin(x+h) = \sin x \cos h + \cos x \sin h.$$

Hence

$$\frac{f(x+h) - f(x)}{h} = \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}.$$

Rearranging the right-hand side gives

$$\frac{f(x+h) - f(x)}{h} = \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right).$$

By the facts in the box near the beginning of this document, as h gets closer and closer to 0, the value of the expression on the right-hand side of this expression will get closer and closer to $\sin x \times 0 + \cos x \times 1$, which is equal to $\cos x$.

Hence the formula for the derivative of the function $f(x) = \sin x$ is $f'(x) = \cos x$.

As explained in Unit 7, you can also use this fact to deduce that the formula for the derivative of the function $f(x) = \cos x$ is $f'(x) = -\sin x$.