Suite Ex 3

* $z^2 + z + 1 = 0$

$\Delta = -3 = (i\sqrt{3})^2 \neq 0$

$z_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \boxed{j}$

$z_2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = \boxed{\bar{j}}$

* $z^3 = j \Rightarrow \boxed{z_k = e^{i\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)}}$

$z^3 = \bar{j} \Rightarrow z_k = e^{-i\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)}$

3) $(j+z)^6 + (j^2 - z^2)^3 + (j-z)^6 = 0$

$\Rightarrow z \neq j \quad (j - z \neq 0)$ $\quad (2j^6 = 0$ Impossible$)$

$\left(\frac{j+z}{j-z}\right)^6 + \boxed{\left(\frac{j+z}{j-z}\right)^3} + 1 = 0$

on pose $\boxed{t} = \left(\frac{j+z}{j-z}\right)^3$

$$t^2 + t + 1 = 0 \quad (\mathcal{E})$$

t est sol de $(\mathcal{E})$.

$t = j$ ou $t = \bar{j}$

$\left(\boxed{\dfrac{j+z}{j-z}}\right)^3 = j$ ou $\left(\dfrac{j+z}{j-z}\right)^3 = \bar{j}$

$\Rightarrow$ ① $\dfrac{j+z}{j-z}$ est une racine cubique de j

ou

② $\dfrac{j+z}{j-z}$ est une racine cubique de $\bar{j}$

$$\Rightarrow \begin{cases} ① \quad \dfrac{j+z}{j-z} = e^{i\left(\boxed{\frac{2\pi}{3} + \frac{2k\pi}{3}}\right)} & k \in \{0,1,2\} \\ ② \quad \dfrac{j+z}{j-z} = e^{-i\left(\frac{2\pi}{3} + \frac{2k\pi}{3}\right)} & k \in \{0,1,2\} \end{cases}$$

$$\Rightarrow \begin{cases} ① \quad Z = i\, j \tan\left(\frac{\pi}{9} + \frac{k\pi}{3}\right) \quad k \in \{0,1,2\} \\ ② \quad Z = i\, j \tan\left(-\frac{\pi}{9} - \frac{k\pi}{3}\right) \quad k \in \{0,1,2\}. \end{cases}$$

Exercice 4 — 30 min — 5 pts

A/ Soit la fonction f définie sur $]-\infty, \pi[$ par : $\begin{cases} f(x) = \sqrt{x^2+4} - x - 2 & si \ \ x < 0 \\ f(x) = \tan\left(\frac{x}{2}\right) & si \ \ 0 \le x < \pi \end{cases}$

1) Montrer que f est continue en 0.

- $f(0) = \tan(0) = 0$
- $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{0^+} \tan\left(\frac{x}{2}\right) = 0 = f(0)$
- $\lim\limits_{0^-} f(x) = \lim\limits_{0^-} \sqrt{x^2+4} - x - 2 = 0 = f(0)$

$\underline{\underline{d'}}$ $\boxed{f \text{ Cont en } 0}$.

2) $\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \text{finies}$

$\Rightarrow$ f d^lble en 0.

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \ell = f'_d(a)$$

$$\lim_{a^-} \frac{f(x) - f(a)}{x - a} = \ell' = f'_g(a) =$$

f Cont à dte et Cont à gche en a $\Longleftrightarrow$ f cont en a

Si f est d^lble à droite en a et f d^lble à gche en a, + $f'_d(a) = f'_g(a)$ $\Longrightarrow$ f d^lble en a

$$\lim_{0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\sqrt{x^2+4} - x - 2}{x}$$

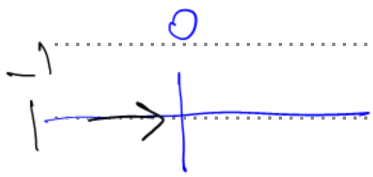
$$= \lim_{x \to 0^-} \frac{\sqrt{x^2 \cdot \left(1 + \frac{4}{x^2}\right)} - x - 2}{x}$$

$\sqrt{x^2} = |x|$

$$= \lim_{0^-} \frac{-x\sqrt{1+\frac{4}{x^2}} - x - 2}{x}$$

$$= \lim_{0^-} \frac{\cancel{x}\left(-\sqrt{1+\frac{4}{x^2}} - 1 - \frac{2}{x}\right)}{\cancel{x}} \qquad \frac{4}{x^2} \to +\infty \qquad \frac{2}{x} \to -\infty$$

$$=$$



$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{0^-} \frac{\left(\boxed{\sqrt{x^2+4}} - \boxed{(x+2)}\right)\left(\sqrt{x^2+4}+x+2\right)}{x\left(\sqrt{x^2+4}+x+2\right)}$$

$$= \lim_{0^-} \frac{\cancel{x^2}+\cancel{4} - \cancel{x^2} - \cancel{4} - 4x}{x\left(\sqrt{x^2+4}+x+2\right)}$$

$$= \lim_{0^-} \frac{-4}{\sqrt{x^2+4}+x+2} = \frac{-4}{4} = -1$$

f est d^{ble} à gche en 0 et $f'_g(0) = -1$

$$* \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{\tan\left(\frac{x}{2}\right)}{x} \to 0$$

$$= \lim_{x \to 0^+} \frac{\tan\left(\frac{1}{2}x\right)}{x} = \frac{1}{2}$$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x \to 0} \frac{\sin(ax)}{x} = a$$

$$\boxed{\lim_{x \to 0} \frac{\tan(ax)}{x} = a} = \lim_{0} \frac{\frac{\sin(ax)}{\cos ax}}{x}$$

$$= \lim_{0} \frac{\sin(ax)}{x \cdot \cos(ax)}$$

$$= \lim_{0} \underbrace{\frac{\sin(ax)}{x}}_{a} \cdot \frac{1}{\cos(ax)}$$

$$a \times 1 = a.$$

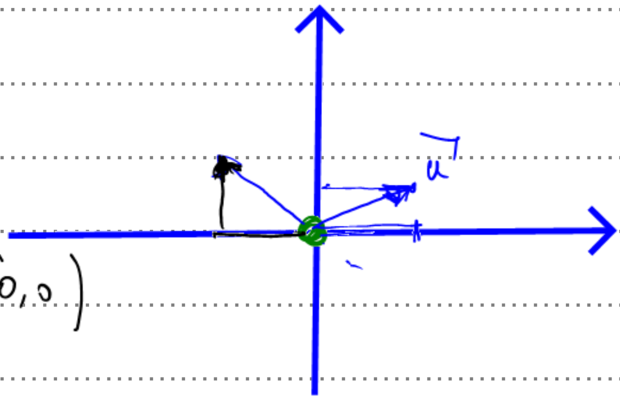
f dérivable à droite en 0 $\quad f'_d(0) = \frac{1}{2}$ = pente.

f " gauche en 0 $\quad f'_G(0) = -1$ =

f n'est pas dér en 0.

Interprétation graphique:

- $f'_d(0) = \frac{1}{2}$



$\mathcal{C}_f$ admet une demi-tgte à dte du pt $(0,0)$ de vect dir $\vec{u}\begin{pmatrix}1\\ \frac{1}{2}\end{pmatrix}$

$f'_G(0) = -1 \Rightarrow \mathcal{C}_f$ admet une demi-tgte à gauche du pt $(0,0)$ de vect directeur $\begin{pmatrix}1\\ -1\end{pmatrix} \xrightarrow{\text{colin}} \begin{pmatrix}-1\\ 1\end{pmatrix}\vec{v}$

$T_d : y = f'_d(0)(x-0) + f(0) \qquad x \geq 0$

$y = \frac{1}{2}(x-0) + 0$

$$\boxed{T_d : y = \frac{1}{2}x \qquad x \geq 0}$$

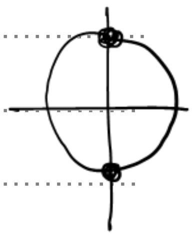
$T_g : y = f'_g(0)(x-0) + f(0) \qquad x \leq 0$

$y = -x \qquad x \leq 0$

3) Sur $[0,\pi[$ $\quad f(x)=\tan\left(\frac{x}{2}\right)$

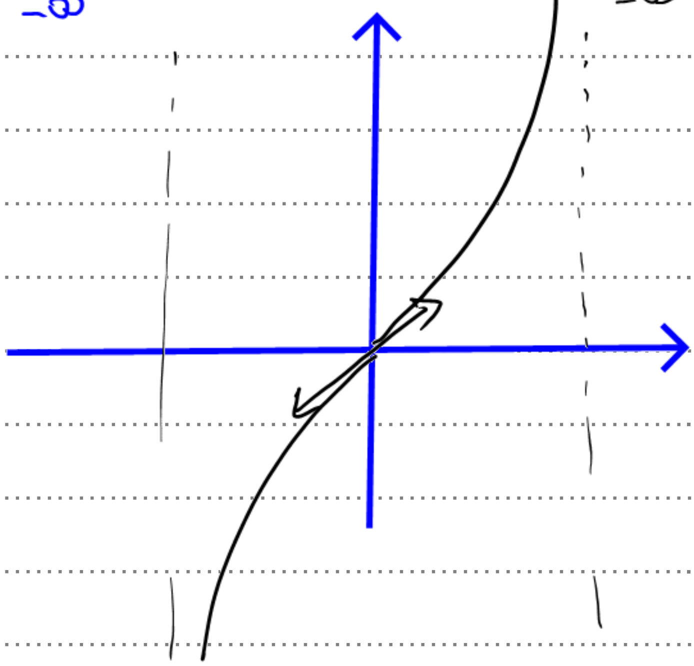
I)

$x \xrightarrow{f} \tan x$ est dérivable en tout réel $\neq \frac{\pi}{2}+k\pi \quad k\in\mathbb{Z}$



$$f'(x) = 1+\tan^2 x = \frac{1}{\cos^2 x} > 0$$

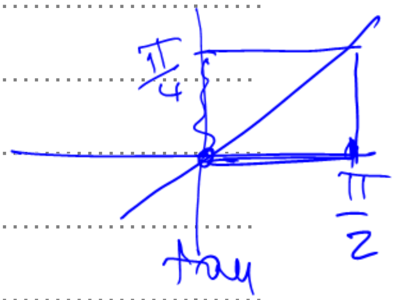
x	$-\pi/2$		$\pi/2$		$3\pi/2$
$f'(x)$		$+$	||		
f	$-\infty$	$\nearrow$	$+\infty$ || $-\infty$	$\nearrow$	$+\infty$



$$f(x) = \tan\left(\frac{1}{2}x\right) = \tan\big(U(x)\big)$$

$$\begin{tikzcd} {[0,\pi[} \arrow[r, "U"] \arrow[rr, bend right, "f"'] & U([0,\pi[) \arrow[r, "\tan"] & {} \end{tikzcd}$$

- U est linéaire donc $d^{\underline{ble}}$ sur $[0,\pi[$
- $U\big([0,\pi[\big) = \left[0, \frac{\pi}{2}\right[$



$$\begin{tikzcd} {[0,\pi[} \arrow[r, bend left, "U"] & U\big([0,\pi[\big) = \left[0,\frac{\pi}{2}\right[\arrow[r, bend left] & {} \end{tikzcd}$$

$$\begin{cases} ① & U \ d^{\underline{ble}} \text{ sur } [0,\pi[\\ ② & \tan \ d^{\underline{ble}} \text{ sur } U\big([0,\pi[\big) = \left[0,\frac{\pi}{2}\right[\end{cases}$$

$\Rightarrow$ $f = \tan \circ U$ est $d^{\underline{ble}}$ sur $\left[0,\frac{\pi}{2}\right[$

$$f'(x) = U'(x) \times \tan'\big(U(x)\big)$$

$$= \frac{1}{2}\left(1 + \tan^2\left(\frac{1}{2}x\right)\right) > 0.$$

$$\big(\tan(ax)\big)' = a\left(1+\tan^2(ax)\right)$$

x	0		π
$f'(x)$		$+$	‖
$f(x)$	0	$\nearrow$	$+\infty$ ‖

$$\lim_{x\to\pi^-} \tan\left(\frac{x}{2}\right) = +\infty \qquad \frac{x}{2} \to \frac{\pi}{2}^-$$

B/ h d^{ble} sur $[0,+\infty[$

$$h'(x) = \frac{2}{1+x^2} \qquad h(1) = \frac{\pi}{2} \qquad \lim_{+\infty} h = \pi$$

$$\varphi(x) = h(\sqrt{x}) + h\left(\frac{1}{\sqrt{x}}\right)$$

$$= \underbrace{\boxed{h\big(V(x)\big)}}_{g_1(x)} + \underbrace{\boxed{h\big(W(x)\big)}}_{g_2(x)}$$

$$]0,+\infty[\xrightarrow{V} \underbrace{V(]0,+\infty[)}_{]0,+\infty[} \xrightarrow{h}$$

g d^{ble} sur I
$g(x) > 0$ sur I $\Longrightarrow$ $\sqrt{g(x)}$ d^{ble} sur I.

① $x \xrightarrow{V} \sqrt{x}$ est d^{ble} sur $]0, +\infty[$

② $V(]0, +\infty[) =]0, +\infty[$

Cont et st $\nearrow$

③ h est d^{ble} sur $[0, +\infty[$ en particulier sur $]0, +\infty[$

① + ② + ③ $\Longrightarrow$ $x \xrightarrow{g_1} h(V(x))$ est d^{ble} sur $]0, +\infty[$

$$g_1(x) = h(V(x))$$

$$g_1'(x) = \boxed{V'}(x) \cdot \boxed{h'}(V(x))$$

$$\boxed{x} \xrightarrow{\boxed{V}} \boxed{V(x)} \xrightarrow{\boxed{h}}$$

$$g_1'(x) = \frac{1}{2\sqrt{x}} \times h'(\boxed{\sqrt{x}})$$

$$= \frac{1}{\cancel{2}\sqrt{x}} \; \frac{\cancel{2}}{1+\sqrt{x}^2}$$

$$\boxed{g_1'(x) = \frac{1}{\sqrt{x}\,(1+x)}}$$

$$* \quad g_2(x) = h\left(\boxed{\frac{1}{\sqrt{x}}}\right)$$

$$]0,+\infty[\xrightarrow{w} w(]0,+\infty[) \xrightarrow{h}$$

① $x \longrightarrow 1$ est d^{ble} sur $]0,+\infty[$

② $x \longrightarrow \sqrt{x}$ est d^{ble} sur $]0,+\infty[$

③ $\sqrt{x} \neq 0$ sur $]0,+\infty[$

$\longrightarrow$ $\boxed{w}$ est d^{ble} sur $]0,+\infty[$

∘ $w(]0,+\infty[) = ?$

$$w(x) = \frac{1}{\sqrt{x}}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$w'(x) = \frac{-\left(\sqrt{x}\right)'}{\left(\sqrt{x}\right)^2}$$

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$$

$$= \frac{-\frac{1}{2\sqrt{x}}}{x}$$

$$\left(\sqrt{g}\right)' = \frac{g'}{2\sqrt{g}}$$

$$w'(x) = \frac{-1}{2x\sqrt{x}} < 0$$

w Cont et Strict $\searrow$ sur $]0,+\infty[$

$$w\left(]0,+\infty[\right) = \left]\lim_{+\infty} w \,;\, \lim_{0^+} w\right[$$

$$=]0 , +\infty[$$

$$]0,+\infty[\xrightarrow{w}]0,+\infty[\xrightarrow{h}$$

$$\begin{cases} \cdot\ w \text{ est } d^{ble} \text{ sur }]0,+\infty[\\ \cdot\ h \text{ est } d^{ble} \text{ sur } w\left(]0,+\infty[\right) =]0,+\infty[\end{cases}$$

$\Longrightarrow g_e = h \circ w$ est d^{ble} sur $]0,+\infty[$

$$x \xrightarrow{w} w(x) \xrightarrow{h}$$

$$g_2'(x) = w'(x) \cdot h'\left(w(x)\right)$$

$$= \frac{-1}{2x\sqrt{x}} \cdot h'\left(\boxed{\frac{1}{\sqrt{x}}}\right)$$

$$= \frac{-1}{\cancel{2}x\sqrt{x}} \quad \frac{\cancel{2}}{1+\left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \frac{-1}{x\sqrt{x}} \quad \frac{1}{1+\frac{1}{x}}$$

$$= \frac{-1}{\cancel{x}\sqrt{x} \quad \frac{(x+1)}{\cancel{x}}}$$

$$g_2'(x) = \frac{-1}{\sqrt{x}\ (x+1)}$$

$$\varphi(x) = g_1(x) + g_2(x)$$

φ est d^{ble} sur $]0;+\infty[$

et $\varphi'(x) = g_1'(x) + g_2'(x)$

$$= \frac{1}{\sqrt{x}(x+1)} - \frac{1}{\sqrt{x}(x+1)}$$

$$\boxed{\varphi'(x) = 0 \quad \forall x>0.}$$

2) $\varphi'(x) = 0$ donc $\varphi(x) = cte$ sur $]0,+\infty[$

$\varphi(x) = \varphi(x_0) \qquad x_0 > 0$

$$\boxed{\varphi(x) = h(\sqrt{x}) + h\left(\frac{1}{\sqrt{x}}\right)}$$

$$\varphi(1) = h(\sqrt{1}) + h\left(\frac{1}{\sqrt{1}}\right)$$

$$= h(1) + h(1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

$$\varphi(x) = \pi \quad \forall x > 0$$

$$h(\sqrt{x}) + \boxed{h\left(\frac{1}{\sqrt{x}}\right)} = \pi \quad x>0$$

d'où $\boxed{h\left(\frac{1}{\sqrt{x}}\right) = \pi - h(\sqrt{x}) \quad x>0}$

D/ $$U_n = \frac{1}{n+1} \sum_{k=n}^{2n} h(\sqrt{k})$$

1/ * $n \leq k \leq 2n$

* $0 < \sqrt{n} \leq \sqrt{k} \leq \sqrt{2n}$

$$h'(x) = \frac{2}{1+x^2} > 0 \quad \forall x \geq 0$$

h est strict $\nearrow$ sur $[0, +\infty[$

$$h(\sqrt{n}) \leq h(\sqrt{k}) \leq h(\sqrt{2n})$$

$$\sum_{k=n}^{2n} h(\sqrt{n}) \leq \sum_{k=n}^{2n} h(\sqrt{k}) \leq \sum_{k=n}^{2n} h(\sqrt{2n})$$

$$\frac{1}{n+1}\sum_{k=n}^{2n} h(\sqrt{n}) \leq \underbrace{\frac{1}{n+1}\sum_{k=n}^{2n} h(\sqrt{k})}_{} \leq \frac{1}{n+1}\sum_{k=n}^{2n} h(\sqrt{2n})$$

cte % k

$$\cancel{\frac{1}{n+1}} h(\sqrt{n})\,(\cancel{2n-n+1}) \leq U_n \leq \cancel{\frac{1}{n+1}} h(\sqrt{2n})\,(\cancel{2n-n+1})$$

$$\boxed{h(\sqrt{n}) \leq U_n \leq h(\sqrt{2n})}$$

$$\sum_{k=p}^{q} cte = cte\,(q-p+1).$$

2°) $h(\sqrt{n}) \leq U_n \leq h(\sqrt{2n})$

- $\lim\limits_{n\to+\infty} h(\sqrt{n}) = \pi$; $\sqrt{n} \to +\infty$

- $\lim\limits_{n\to+\infty} h(\sqrt{2n}) = \pi$; $\sqrt{2n} \to +\infty$

$$d'où \quad \lim_{n \to +\infty} U_n = \pi .$$

$$v_n = \frac{1}{n+1} \sum_{k=n}^{2n} \boxed{\ln\left(\frac{1}{\sqrt{k}}\right)}$$

$$= \frac{1}{n+1} \sum_{k=n}^{2n} \pi - \ln(\sqrt{k})$$

$$= \frac{1}{n+1} \left(\sum_{k=n}^{2n} \pi - \sum_{k=n}^{2n} \ln\sqrt{k} \right)$$

$$v_n = \frac{1}{n+1} \sum_{k=n}^{2n} \pi - U_n$$

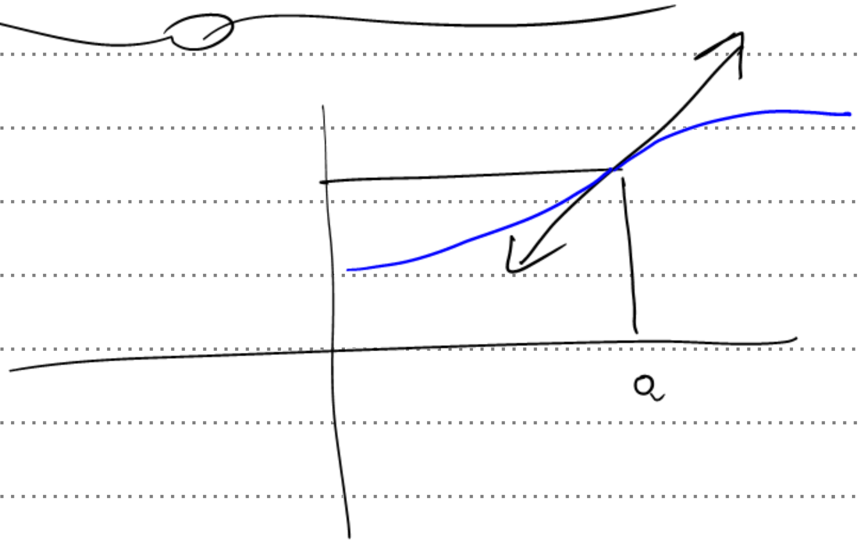
$$= \frac{1}{\cancel{n+1}} \pi \left(\cancel{2n - n+1}\right) - U_n$$

$$v_n = \pi - U_n$$

$$\lim_{n \to +\infty} v_n = \pi - \pi = 0$$

$$\sum (U_k + V_k) = \sum U_k + \sum V_k$$

$$\sum \alpha U_k = \underset{\substack{\downarrow \\ \underline{cte}}}{\alpha} \sum U_k$$



x		a	
f''	$+$	0	$-$