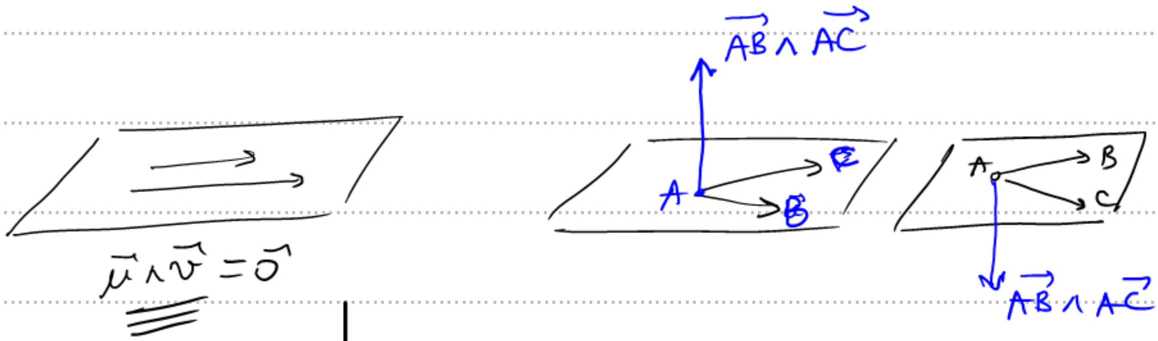
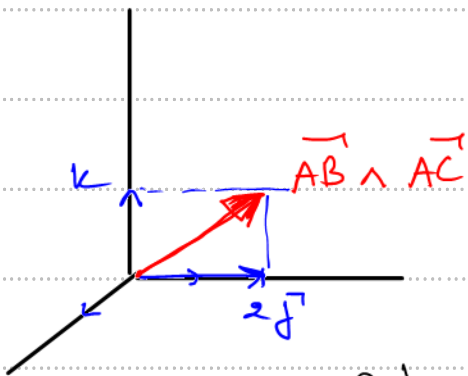
Exercice 1

1) $\overrightarrow{AB}\begin{pmatrix}1\\0\\0\end{pmatrix} \wedge \overrightarrow{AC}\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix} \begin{vmatrix}0&1\\0&-2\end{vmatrix} \\ -\begin{vmatrix}1&2\\0&-2\end{vmatrix} \\ \begin{vmatrix}1&2\\0&1\end{vmatrix}\end{pmatrix}$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix} = 0\vec{i} + 2\vec{j} + \vec{k}$$





b) $\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{pmatrix}0\\2\\1\end{pmatrix} \neq \vec{0}\begin{pmatrix}0\\0\\0\end{pmatrix}$

A, B et C ne sont pas alignés d'où ils déterminent un plan de vecteur normal $\begin{pmatrix}0\\2\\1\end{pmatrix}$

$(ABC) : 0x + 2y + z + d = 0$

$P = (ABC) : 2y + z + d = 0$

$A(0,0,1)$ $\quad 2\times 0 + 1 + d = 0$

$d = -1$

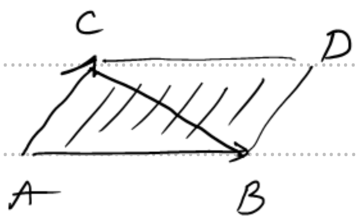
$$\boxed{P : 2y + z - 1 = 0}$$

c) $\text{Aire}(ABC) = \dfrac{\|\vec{AB} \wedge \vec{AC}\|}{2}$

$\vec{AB} \wedge \vec{AC} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \|\vec{AB} \wedge \vec{AC}\| = \sqrt{0^2 + 2^2 + 1^2}$

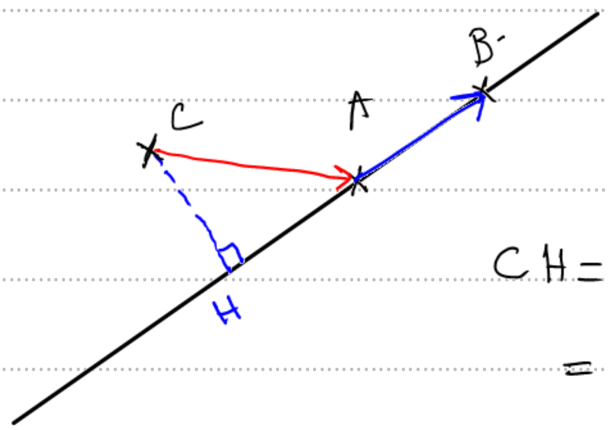
$= \sqrt{5}$

$\text{Aire}(ABC) = \dfrac{\sqrt{5}}{2}$



$\|\vec{AB} \wedge \vec{AC}\| = \text{Aire } ABDC$

d)



$CH = d(C, (AB))$

$= \dfrac{\|\vec{AC} \wedge \vec{AB}\|}{\|\vec{AB}\|}$

$$\overrightarrow{CA} \wedge \overrightarrow{AB} = -\left(\overrightarrow{AC} \wedge \overrightarrow{AB}\right)$$

$$(\alpha \vec{u}) \wedge \vec{v} = \alpha \left(\vec{u} \wedge \vec{v}\right)$$

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = -\overrightarrow{AC} \wedge \overrightarrow{AB}$$

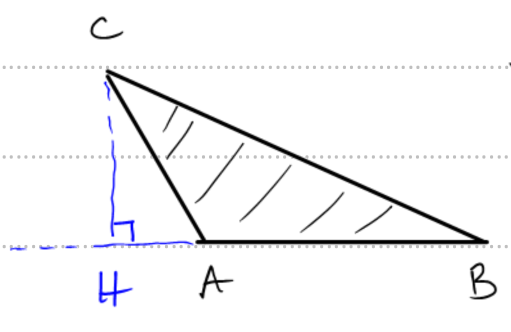
$$\|\overrightarrow{AB} \wedge \overrightarrow{AC}\| = \|\overrightarrow{AC} \wedge \overrightarrow{AB}\| = \|\overrightarrow{CA} \wedge \overrightarrow{AB}\|$$

App

$$d\left(C, (AB)\right) = \frac{\|\overrightarrow{AC} \wedge \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

$$\overrightarrow{AB}\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow AB = \|\overrightarrow{AB}\| = \sqrt{1^2+0+0} = 1$$

2ème methode:

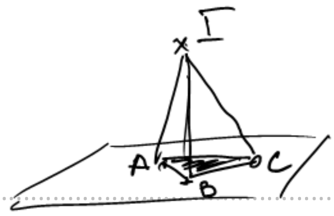


$$Aire(ABC) = \frac{AB \times CH}{2}$$

$$\frac{2\,Aire(ABC)}{AB} = CH$$

$$CH = d\left(C, (AB)\right)$$

2)



$\left(\text{I, A, B et C non coplanaire}\right) \Longleftrightarrow$ IABC est un tetraedre

$\det\left(\overrightarrow{IA}, \overrightarrow{IB}, \overrightarrow{IC}\right) \neq 0 \Longleftrightarrow$

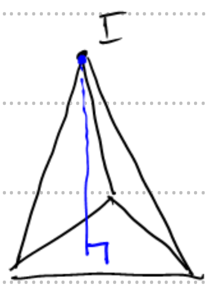
$I \notin (ABC) \Longrightarrow$ IABC est un tetraedre

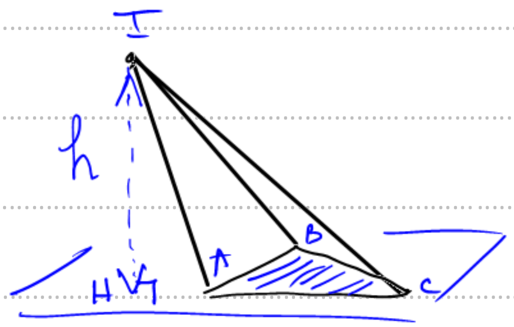
$\boxed{\text{ou}}$ $(ABC) : 2y + z - 1 = 0$

$I(-2, \underset{y}{1}, \underset{z}{2}) \notin (ABC)$ car

$2 + 2 - 1 = 3 \neq 0$

d'où IABC est un tetraedre.





H = le proj $\perp$ de I sur (ABC)

$$\mathcal{V} = \frac{\text{Aire}(ABC) \times IH}{3}$$

$$IH = d(I, (ABC))$$

ou $\mathcal{V} = \left| \frac{\det(\overrightarrow{IA}, \overrightarrow{IB}, \overrightarrow{IC})}{6} \right|$

$$= \boxed{\frac{\left|(\overrightarrow{AB} \wedge \overrightarrow{AC}) \cdot \overrightarrow{AI}\right|}{6}}$$

$$(ABC) : 2y + z - 1 = 0$$

$$d(I, (ABC)) = \frac{|2 \times 1 + 2 - 1|}{\sqrt{0^2 + 2^2 + 1^2}} = \frac{3}{\sqrt{5}}$$

$I(-2, 1, 2)$

norme

$\overrightarrow{n}_{(ABC)} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

d'où $\mathcal{V} = \frac{\text{Aire}(ABC) \times d(I, (ABC))}{3}$

$$= \frac{\frac{\cancel{\sqrt{5}}}{2} \times \frac{3}{\cancel{\sqrt{5}}}}{3} \quad \frac{3}{2} \times \frac{1}{3}$$

$$= \frac{1}{2} \text{ U.V.}$$

ou $\quad \mathcal{V} = \dfrac{\left|\left(\overrightarrow{AB} \wedge \overrightarrow{AC}\right) \cdot \overrightarrow{AI}\right|}{6} = \dfrac{|3|}{6} = \dfrac{1}{2}$

$\overrightarrow{AI}\begin{pmatrix}-2\\1\\1\end{pmatrix} \cdot \begin{pmatrix}0\\2\\1\end{pmatrix} = 2+1$

c) $\quad \mathcal{V} = \dfrac{1}{2} = \dfrac{Aire(ABC) \times d(I,P)}{3}$

$\dfrac{3}{2} = \dfrac{\sqrt{5}}{2} \cdot d(I,P)$

$\dfrac{2}{\sqrt{5}} \times \dfrac{3}{2} = d(I,P)$

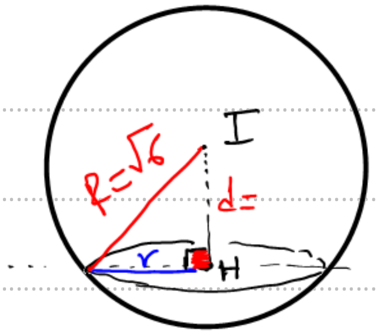
$\left\{ \dfrac{3}{\sqrt{5}} = d(I,P) \right.$

3) $\mathcal{S}$ = Sphere de centre I et passant A

$\Leftrightarrow$ Rayon $IA = \sqrt{(-2)^2+1+1} = \sqrt{6}$

$\overrightarrow{AI}\begin{pmatrix}-2\\1\\1\end{pmatrix} \qquad I\left(\boxed{-2}, \boxed{1}, \boxed{2}\right)$

$M(x,y,z) \in \mathcal{S} : \left(x+\boxed{2}\right)^2 + \left(y-\boxed{1}\right)^2 + \left(z-\boxed{2}\right)^2 = \sqrt{6}^2$

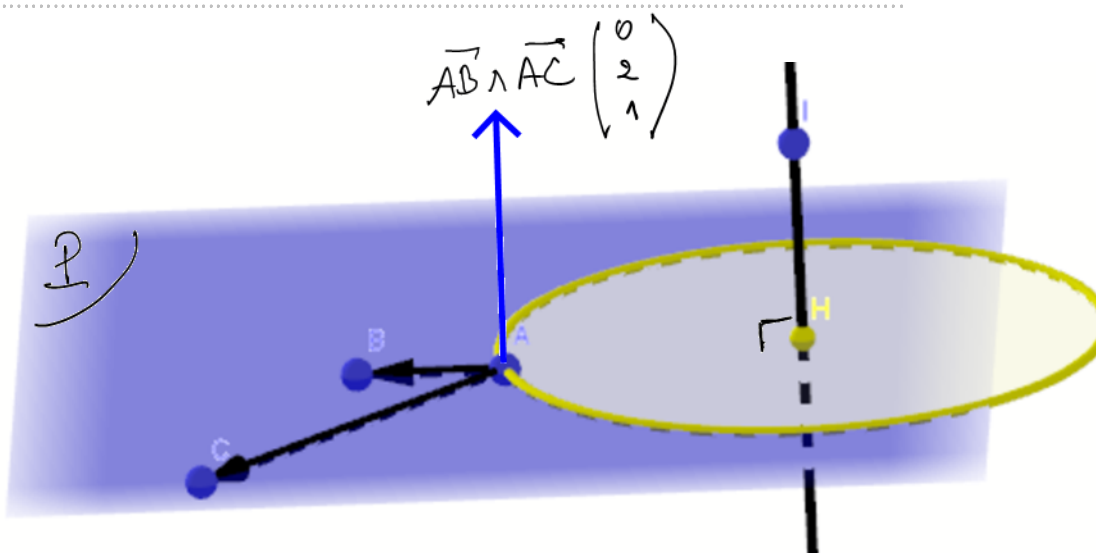


$d(I,P)=\frac{3}{\sqrt{5}} < R=\sqrt{6}$

donc $S\cap P$ est un cercle de rayon

$r=\sqrt{\sqrt{6}^2-d^2}=\sqrt{6-\frac{9}{5}}=\sqrt{\frac{21}{5}}$

- le centre de $\mathscr{C}=S\cap P$ est le projeté orthog de I sur (ABC)



$H=P\cap(IH)$

(IH) c'est la droite qui passe par $\boxed{I}$ $(-2,1,2)$ et de vect dir $\vec{AB}\wedge\vec{AC}\boxed{\begin{pmatrix}0\\2\\1\end{pmatrix}}$

$H(x, y, z) \in (IH)$ $\begin{cases} x = -2 + 0\alpha = -2 \\ y = 1 + 2\alpha \\ z = 2 + \alpha \end{cases}$

③ $H \in P$: $2y + z - 1 = 0$

③ $2(1 + 2\alpha) + 2 + \alpha - 1 = 0$

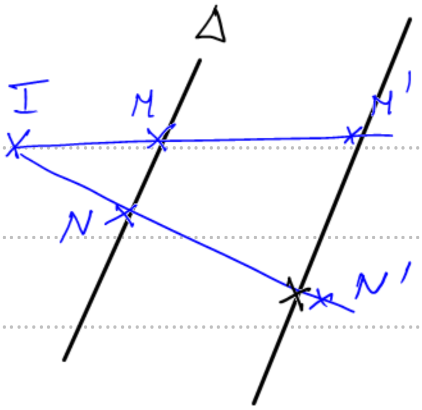
$5\alpha + 3 = 0 \Rightarrow \alpha = -\frac{3}{5}$

$\begin{cases} x = -2 \\ y = 1 - \frac{6}{5} = -\frac{1}{5} \\ z = 2 - \frac{3}{5} = \frac{7}{5} \end{cases}$

$H\left(-2, -\frac{1}{5}; \frac{7}{5}\right)$

$\mathcal{C}$ – Centre H rayon $r = \sqrt{\frac{21}{5}}$

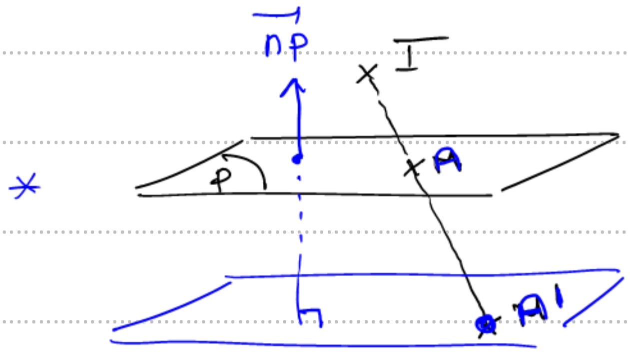
1)



$\overrightarrow{IM'} = k\,\overrightarrow{IM}$

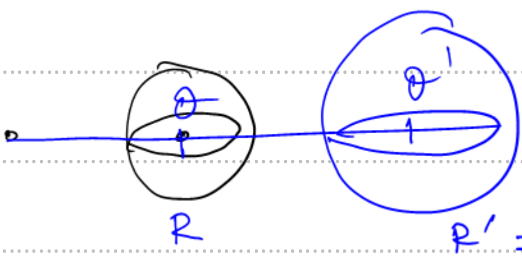
Si $I \in \Delta$ $\quad h(\Delta) = \Delta$.

2)



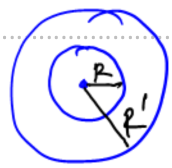
* Si $I \in P$ $\quad h(P) = P$

3)



$R' = |k|\,R$.

Si I = centre de h = centre de S.



$R' = |k| \cdot R$.

4] a) $M(x,y,z) \xrightarrow{h_{(I,k)}} M'(x',y',z')$

Expression Analytique.

$$\begin{cases} x' = \boxed{k}\,x + \boxed{(1-k)}\,\boxed{x_I} \\ y' = k\,y + (1-k)\,y_I \\ z' = k\,z + (1-k)\,z_I \end{cases}$$

Si $k=1$ $\begin{cases} x'=x \\ y'=y \\ z'=z \end{cases} \Longleftrightarrow M'=M.$

App $h = h_{\left(I,\frac{1}{5}\right)}$ $\boxed{I(-2,1,2)}$

$1-\frac{1}{5}=$

$$\begin{cases} x' = \frac{1}{5}x + \frac{4}{5}\times -2 \\ y' = \frac{1}{5}y + \frac{4}{5}\times 1 \\ z' = \frac{1}{5}z + \frac{4}{5}\times 2 \end{cases}$$

$$\begin{cases} x' = \frac{1}{5}x - \frac{8}{5} \\ y' = \frac{1}{5}y + \frac{4}{5} \\ z' = \frac{1}{5}z + \frac{8}{5} \end{cases}$$

S est une sphere de centre I et passant par A de rayon $IA = \sqrt{6}$

$\Rightarrow h(S)$ la sphere de centre $h(I) = I$ et de rayon $R' = \left|\frac{1}{5}\right| \cdot R = \frac{\sqrt{6}}{5}$

c)

soit $A' = h(A)$ $\quad A(0,0,1)$

$$\begin{cases} x_{A'} = \frac{1}{5} \cdot 0 - \frac{8}{5} = -\frac{8}{5} \\ y_{A'} = \frac{1}{5} \cdot 0 + \frac{4}{5} = \frac{4}{5} \\ z_{A'} = \frac{1}{5} \cdot 1 + \frac{8}{5} = \frac{9}{5} \end{cases}$$

S' la sphere de centre I et passant $A'\left(-\frac{8}{5} ; \frac{4}{5} , \frac{9}{5}\right)$

P est le plan passant par A et de vect normal $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow h(P)$ " " " $h(A) = A'$ et de $\vec{n}$

Vecteur normal ($P' \parallel P$)

$P' : 2y + z + d = 0$

$A'\left(-\frac{8}{5}, \frac{4}{5}, \frac{9}{5}\right) \in P'$

$\frac{8}{5} + \frac{9}{5} + d = 0 \Rightarrow d = -\frac{17}{5}$

d'où $P' : 2y + z - \frac{17}{5} = 0$

$: 10y + 5z - 17 = 0$

d) $S \cap P = \mathcal{C}_{(H, r)}$

$\Rightarrow \underbrace{h(S)}_{S'} \underset{\cap}{\cap} \underbrace{h(P)}_{P'} = \mathcal{C}_{(H', r')}$.

$H' = h(H) \qquad r' = \left|\frac{1}{5}\right| . r.$

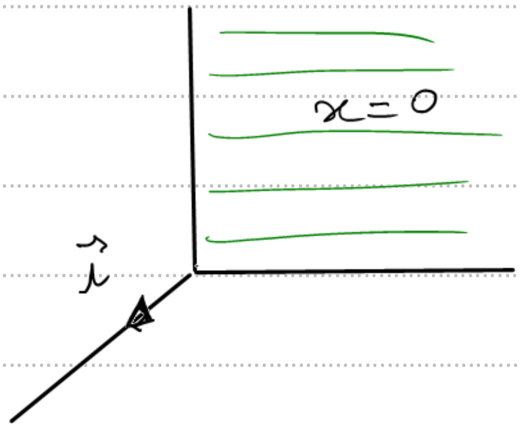
$= \frac{1}{5} . \sqrt{\frac{21}{5}}$

$S' \cap P'$ est le cercle de

centre H' rayon $r' = \frac{1}{5}\sqrt{\frac{21}{5}}$

$$\begin{cases} x_{H'} = \frac{1}{5} \times -2 - \frac{8}{5} = \dots \\ y_{H'} = \frac{1}{5} \times -\frac{1}{5} + \frac{4}{5} = - \\ z_{H'} = \frac{1}{5} \times \frac{7}{5} + \frac{8}{5} = - \end{cases}$$

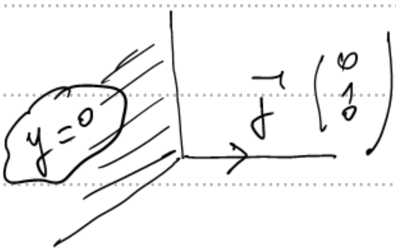
$H\left(-2, -\frac{1}{5}; \frac{7}{5}\right.$



$\vec{i}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\vec{j}\begin{pmatrix}0\\1\\0\end{pmatrix}$

$k\begin{pmatrix}0\\0\\1\end{pmatrix}$



Ex2

$$\begin{cases} a-1 \equiv 0 \ (2^4) \\ a-1 \equiv 0 \ (5^4) \\ 2^4 \wedge 5^4 = 1 \end{cases} \Rightarrow a-1 \equiv 0 \ (2^4 \times 5^4)$$

d'où $a \equiv 1 \ (10^4)$

2) $b = (9217)^4$

$9217 \equiv 2 \ (5) \Rightarrow 9217^4 \equiv 2^4 \ (5)$

or $2^4 \equiv 1 \ (5)$

donc $\boxed{b \equiv 1 \ (5)}$

$2^4 = 16$

* $9217 \equiv 1 \ (16)$

donc $9217^4 \equiv 1 \ (16)$

$\boxed{b \equiv 1 \ (2^4)}$

3) $b_n = \boxed{b^{5^n} - 1}$

a) Montrer que pour tout entier, naturel n, $b_{n+1} = (b_n + 1)^5 - 1$.

$$(\boxed{b_n} + 1)^5 - 1 = (\boxed{b^{5^n} - 1} + 1)^5 - 1$$

$$= \left(b^{5^n}\right)^5 - 1$$

$(a^n)^m = a^{n \cdot m}$

$$= b^{5 \times 5^n} - 1$$

$$= b^{5^{n+1}} - 1$$

$$= b_{n+1}$$

b) En déduire que pour tout entier naturel n, $b_{n+1} = b_n^5 + 5b_n^4 + 10b_n^3 + 10b_3^2 + 5b_n$.

$$(X + Y)^{\boxed{n}} = \sum_{k=0}^{\boxed{n}} C_{\boxed{n}}^k X^{n-k} Y^k$$

$$= C_n^0 X^n + C_n^1 X^{n-1} Y + \dots + C_n^{n-1} X Y^{n-1} + C_n^n Y^n$$

$$\boxed{(b_n + 1)^{\boxed{5}} = \sum_{k=0}^{\boxed{5}} C_5^k b_n^{5-k} \cdot 1^k}$$

Triangle Pascal.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
$\boxed{1 \ 5 \ 10 \ 10 \ 5 \ 1}$

$= b_n^{\boxed{5}} + 5 b_n^{\boxed{4}} \cdot 1 + 10 b_n^{\boxed{3}} 1^{\boxed{2}} + 10 b_n^{\boxed{2}} \cdot 1^{\boxed{3}} + 5 b_n 1^4 + 1^5$

$= b_n^5 + 5 b_n^4 + 10 b_n^3 + 10 b_n^2 + 5 b_n + 1$

$b_{n+1} = (b_n + 1)^5 - 1$

$= b_n^5 + 5 b_n^4 + 10 b_n^3 + 10 b_n^2 + 5 b_n$

4) a) Montrer que si 5^{n+1} divise b_n alors 5^{n+2} divise b_n^5.

Si 5^{n+1} divise b_n $\quad b_n = 5^{n+1} \times q \quad q \in \mathbb{Z}^+$

$b_n^5 = (5^{n+1} \cdot q)^5 = (5^{n+1})^5 \times q^5$

$= 5^{5n+5} \times q^5$

$= 5^{n+2} \times \underbrace{5^{4n+3} \times q^5}_{\in \mathbb{Z}}$

$\Rightarrow \underline{b_n^5 \equiv 0 \ (5^{n+2})}$

b) $b_n \equiv 0 \; (5^{n+1})$?

* $n=0$ $\quad b_0 = b^{5^0} - 1 = b - 1$

or $b \equiv 1 \; (5)$

$b_0 = b - 1 \equiv 0 \; (5^{0+1})$ Vraie

* Soit $n \in \mathbb{N}$ Supp

$\longrightarrow b_n \equiv 0 \; (5^{n+1}) \longleftarrow$

* Mque $b_{n+1} \equiv 0 \; (5^{n+2})$

on a $b_n \equiv 0\,(5^{n+1})$ $k \in \mathbb{Z}$

$\Rightarrow 5^{n+1}$ divise $b_n \rightarrow b_n = \boxed{5^{n+1}}\,k$

4°) a) $\Rightarrow 5^{n+2}$ divise $b_n^5 \Rightarrow b_n^5 = 5^{n+2}q'$

or $b_{n+1} = b_n^5 + 5\,\boxed{b_n}^4 + 10\,\boxed{b_n}^3 + 10\,\boxed{b_n}^2 + 5\,\boxed{b_n}$

$= \underbrace{b_n^5}_{5^{n+2}\cdot q'} + \underbrace{5\times(5^{n+1}k)^4 + 2\times 5\,(5^{n+1}k)^3 + 2\times 5\,(5^{n+1}k)^2 + 5\cdot 5^{n+1}k}_{5^{n+2}\times Q}$

$b_{n+1} = 5^{n+2}\,(Q'')$ $Q'' \in \mathbb{Z}$.

Conclusion :

$b_n \equiv 0\,(5^{n+1})$