Exercice 1 : 30 min 6 pts

Soit n un entier ≥ 2 . On pose : $U_n = \int_0^{\frac{\pi}{4}} \tan^n(x)$.

1) Calculer : U_2

$$U_1 = \int_0^{\frac{\pi}{4}} \tan(x)\, dx$$

	$f(x)$	$F(x)$	
$n \neq 0$, $n \neq 1$	$\frac{1}{x^n}$	$\frac{-1}{(n-1)x^{n-1}} + cte$	
	$\frac{1}{x}$	$\ln(x)$	$]0, +\infty[$.
$n \neq 1$	$\frac{u'}{u^n}$	$\frac{-1}{(n-1)u^{n-1}}$	
	$\frac{u'}{u}$	$\ln(u)$	$u(x) > 0$ I

$$I_1 = - \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x}\, dx = - \Big[\ln(\cos x)\Big]_0^{\frac{\pi}{4}}$$

$$= \ln(\cos 0) - \ln\left(\cos \frac{\pi}{4}\right)$$

$$= 0 - \ln\left(\frac{\sqrt{2}}{2}\right).$$

$$* \; U_2 = \int_0^{\frac{\pi}{4}} \tan^2(x)\, dx$$

$$1 + \tan^2(x) \xrightarrow{\text{Prim}} \tan x + c$$

$$U_2 = \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) - 1 \, dx$$

$$= \Big[\underbrace{\tan x - x} \Big]_0^{\frac{\pi}{4}}$$

$$= \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - (\tan 0 - 0)$$

$$\boxed{U_2 = 1 - \frac{\pi}{4}.}$$

2) a) $U_n = \int_0^{\frac{\pi}{4}} \boxed{\tan^n x}\, dx \overset{?}{\geq} 0$

$+$

$$\begin{cases} 0 \leq x \leq \frac{\pi}{4} \\ \tan \nearrow \end{cases} \Rightarrow \tan(0) \leq \tan x \leq \tan \frac{\pi}{4}$$

$$\bullet \; \boxed{0 \leq \tan x} \leq 1$$

• $a = 0 < b = \frac{\pi}{4} \Rightarrow U_n \geq 0 \quad \forall n \geq 0$

b) Montrer que la suite (U_n) est décroissante.

$$U_{n+1} - U_n = \int_0^{\frac{\pi}{4}} \tan^{n+1}(x)\,dx - \int_0^{\pi/4} \tan^n(x)\,dx$$

$$= \int_0^{\frac{\pi}{4}} \underbrace{\tan^{n+1}(x) - \tan^n(x)}_{\text{signe}}\,dx$$

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

$$\int_0^{\frac{\pi}{4}} \underbrace{\tan^n(x)}_{+} \underbrace{(\tan x - 1)}_{-}\,dx$$

on a pour $0 < x \leq \frac{\pi}{4} \Rightarrow 0 \leq \boxed{\tan x \leq 1}$

$\tan^n(x)(\tan x - 1) \leq 0$ et $0 < \frac{\pi}{4}$

$\Rightarrow U_{n+1} - U_n \leq 0$ d'où (U_n) est $\downarrow$

3) a) Montrer que, pour tout entier $n \geq 2$, $U_{n+2} + U_n = \frac{1}{n+1}$.

$$U_{n+2} + U_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n+2}\,dx + \int_0^{\frac{\pi}{4}} (\tan x)^n\,dx$$

$$= \int_0^{\pi/4} (\tan x)^{n+2} + (\tan x)^n\,dx$$

$$= \int_0^{\pi/4} \underbrace{(\tan x)^n}_{U^n} \times \underbrace{(\tan^2(x) + 1)}_{U'} \, dx \qquad U(x) = \tan x$$

$$\boxed{f = U' U^n \longrightarrow F(x) = \frac{U^{n+1}}{n+1} + c^{te}}$$

$$= \left[\frac{(\tan x)^{n+1}}{n+1} \right]_0^{\pi/4}$$

$$= \frac{\left(\tan \frac{\pi}{4}\right)^{n+1}}{n+1} - (0)$$

$$\boxed{U_{n+2} + U_n = \frac{1}{n+1}} \quad n \geq 2.$$

b) Rappel

Si (U_n) Converge Vers ℓ

$\Rightarrow (U_{n+1})$ (U_{n+2}) (U_{2n}) . U_N $(N \geq n)$

Convergent Vers ℓ.

$\exists n_0 \quad > n \geq n_0$

$n_0 \quad U_n \quad \ell$

$|U_n - \ell| < 0{,}1$

(U_n) est ↓ et minorée par 0

donc U_n Converge vers $\ell \geq 0$

$\lim_{+\infty} U_n = \ell$

et aussi $\lim_{+\infty} U_{n+2} = \ell$.

or $\left(U_{n+2} + U_n\right) = \frac{1}{n+1}$

donc $\lim_{n\to+\infty} \left(U_{n+2} + U_n\right) = \lim_{+\infty} \frac{1}{n+1} = 0$

$\ell + \ell = 0$

$2\ell = 0$

$\Rightarrow \boxed{\ell = 0}$

$\lim_{n \to +\infty} U_n = 0$

4) On pose, pour tout entier $n \geq 2$, $V_n = U_{n+4} - U_n$ et $S_n = \sum_{k=1}^{n} V_{4k-2}$.

a) Montrer que, pour tout entier $n \geq 2$, $V_n = \frac{1}{n+3} - \frac{1}{n+1}$.

$$v_n = U_{n+4} - U_n \overset{?}{=} \frac{1}{n+3} - \frac{1}{n+1}$$

$$v_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n+4}\, dx - \int_0^{\frac{\pi}{4}} (\tan x)^n\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n+4} - (\tan x)^n\, dx \qquad (\boxed{\tan^2 x})^2 - \boxed{1}^2$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^n \left((\tan x)^4 - 1\right) dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^n \left(\boxed{\tan^2 x} - \boxed{1}\right)\left(\boxed{\tan^2} x + \boxed{1}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} (1 + \tan^2 x)\left(\tan^{n+2} x - \tan^n x\right) dx$$

$$= \int_0^{\pi/4} \underbrace{(1+\tan^2 x)\tan^{n+2} x}_{U' \; U^{n+2}} - \underbrace{(1+\tan^2 x)\tan^n(x)}_{U' \; U^n}\, dx$$

$$= \left[\frac{U^{n+3}}{n+3} - \frac{U^{n+1}}{n+1}\right]_0^{\pi/4}$$

$$= \left[\frac{\tan x^{n+3}}{n+3} - \frac{\tan x^{n+1}}{n+1} \right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{n+3} - \frac{1}{n+1} \right) - \left(0 \quad 0 \right)$$

$$\boxed{v_n = \frac{1}{n+3} - \frac{1}{n+1}}$$

b)

$$S_n = \sum_{k=1}^{n} V_{4k-2} = \underset{\substack{k=1\\ \downarrow}}{V_2} + \underset{\substack{k=2\\ \downarrow}}{V_6} + \underset{k=3}{V_{10}} + \underset{k=4}{V_{14}} + \dots + V_{4n-2}$$

$$v_n = U_{n+4} - U_n \quad \forall n \geq 2$$

$n=2$ $\quad v_2 = \cancel{U_6} - U_2$

$n=6$ $\quad v_6 = \cancel{U_{10}} - \cancel{U_6}$

$\quad v_{10} = \cancel{U_{14}} - \cancel{U_{10}}$

$\quad v_{14} = \cancel{U_{18}} - \cancel{U_{14}}$

$\quad \vdots$

$\quad v_{4n-2} = U_{4n+2} - \cancel{U_{4n-2}}$

$$\boxed{S_n = U_{4n+2} - U_2 .}$$

$$V_{n} = U_{n+4} - U_{n} \quad \forall n$$

$$V_{2} = \cancel{U_6} - U_2$$

$$V_{6} = \cancel{U_{10}} - \cancel{U_6}$$

$$U_{n+2} + U_{n} = \frac{1}{n+1}$$

$$\left(U_{4} + U_{2} = \frac{1}{2+1} = \frac{1}{3} \right.$$

$$U_{6} + U_{4} = \frac{1}{4+1} = \frac{1}{5}$$

$$\left(U_{8} + U_{6} = \frac{1}{7} \right.$$

$$U_{10} + U_{8} = \frac{1}{9}$$

$$\vdots \qquad \vdots$$

$$U_{4n} + U_{4n-2} = \frac{1}{4n-1}$$

$$U_{4n+2} + U_{4n} = \frac{1}{4n+1}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots-\frac{1}{4n-1}+\frac{1}{4n+1}$$

$$1-(U_4+U_2)+(\cancel{U_6}+U_4)-(\cancel{U_8}+\cancel{U_6})\dots+(U_{4n+2}+\cancel{U_{4n}})$$

$$1-U_2+U_{4n+2}$$

$$=\cancel{1}-\cancel{1}+\frac{\pi}{4}+U_{4n+2}$$

donc $\lim\limits_{n\to+\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots-\frac{1}{4n-1}+\frac{1}{4n+1}\right)$.

$$=\lim_{+\infty}\left(\frac{\pi}{4}+\underbrace{U_{4n+2}}_{\to 0}\right)=\frac{\pi}{4}$$

$2^{\text{ème}}$ Methodes

$$S_n=\sum_{k=1}^{n} v_{\boxed{4k-2}}.$$

$$=\sum_{k=1}^{n}\left(\frac{1}{4k+1}-\frac{1}{4k-1}\right)$$

$$=\frac{1}{5}-\frac{1}{3}+\frac{1}{9}-\frac{1}{7}+\dots+\frac{1}{4n+1}-\frac{1}{4n-1}$$

donc $\underbrace{1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{4n+1}}$

$= 1 + S_n$

$= 1 + U_{4n+2} - U_2 \xrightarrow{n \to 0} 1 - U_2 = \frac{\pi}{4}$

$U_{4n+2} \downarrow 0$

$F(x) = \int_a^x f(t)\,dt$ I

① f cont I

② $\boxed{a \in I}$

$F'(x) = f(x)$

$F(x) = \int_a^{u(x)} f(t)\,dt$

① u d^{ble} I

② $u(I) = J$

③ f cont sur J.

④ $\boxed{a \in J}$

$\Rightarrow F'(x) = u'(x)\, f(u(x))$.

Exercice 2 25 min 4 pts

On pose : $g(x) = \int_x^{2x} \frac{dt}{\sqrt{1+t^3}}$ $= \left[F(t)\right]_x^{2x} = F(2x) - F(x)$ $\int_a^b f(t)\,dt = \left[F(t)\right]_a^b = F(b) - F(a)$

1) Montrer que g est définie sur $\mathbb{R}^+$.

2) Montrer que g est dérivable sur $\mathbb{R}^+$ et calculer $g'(x)$ pour tout $x \in$.

on pose $f(x) = \frac{1}{\sqrt{1+x^3}}$

$x \longrightarrow 1+x^3$ poly str positif sur $\mathbb{R}^+$

$\Longrightarrow$ f cont sur $\mathbb{R}^+$

Soit F une primitive de f sur $\mathbb{R}^+$

$g(x) = \underbrace{F(2x)} - \underbrace{F(x)}$

$g(x) = F(v(x)) - \underbrace{F(x)}$

$\mathbb{R}^+ \xrightarrow{v} v(\mathbb{R}^+) = [0,+\infty[\xrightarrow{F}$

$F \circ v$

. v def sur $\mathbb{R}^+$

. $v([0,+\infty[) = [0,+\infty[$

. F def sur $v([0,+\infty[)$

$\Longrightarrow F \circ v$ def sur $\mathbb{R}^+$

$g = F \circ v - F$ est dérivable sur $\mathbb{R}^+$

2) $g(x) = F(v(x)) - F(x) \quad x \geq 0$

$x \longrightarrow \boxed{F(x) \text{ est dérivable sur } \mathbb{R}^+} \qquad F'(x) = f(x)$

$[0,+\infty[\xrightarrow{v} v([0,+\infty[) = [0,+\infty[\xrightarrow{F}$

• v est linéaire $\Rightarrow$ • v dérivable sur $[0,+\infty[$

• F est dérivable sur $[0,+\infty[= v([0,+\infty[)$

$\Longrightarrow \boxed{F \circ v \text{ est}}$ dérivable sur $\mathbb{R}^+$

par suite $g = F \circ v - F$ est dérivable

sur $\mathbb{R}^+$

$g(x) = F(\boxed{2x}) - F(x)$

$g'(x) = 2 \times F'(2x) - F'(x)$

$= 2 f(2x) - f(x)$

$= 2 \dfrac{1}{\sqrt{1+8x^3}} - \dfrac{1}{\sqrt{1+x^3}}$

$$g'(x) = \frac{\boxed{2\sqrt{1+x^3}}}{\sqrt{1+8x^3}} - \frac{\boxed{\sqrt{1+8x^3}}}{\sqrt{1+x^3}}$$

$$= \frac{4(1+x^3) - (1+8x^3)}{\boxed{\sqrt{1+8x^3}\sqrt{1+x^3}\left(2\sqrt{1+x^3} + \sqrt{1+8x^3}\right)} \to D}$$

$$= \frac{3 - 4x^3}{D}$$

Signe $g'(x)$ = Signe $3 - 4x^3$

- $3 - 4x^3 = 0 \iff x^3 = \frac{3}{4}$

$$x = \sqrt[3]{\frac{3}{4}}$$

- $3 - 4x^3 > 0$ $(+)$

$$3 > 4x^3$$

$$\iff \frac{3}{4} > x^3$$

$$\sqrt[3]{\frac{3}{4}} > \sqrt[3]{x^3}$$

$$\sqrt[3]{\frac{3}{4}} > x$$

$$\sqrt[3]{x^3} = x$$

x	0		$\sqrt[3]{\frac{3}{4}}$		$+\infty$
$g'(x)$		$+$	0	$-$	
		↗		↘	

b) $$\frac{x}{\sqrt{1+8x^3}} \overset{?}{\le} \int_x^{2x} \frac{1}{\sqrt{1+t^3}}\,dt \overset{?}{\le} \frac{x}{\sqrt{1+x^3}}.$$

$2x - x = x \ge 0$ $\qquad x \le 2x$

$$x \le t \le 2x$$

$$\sqrt{1+x^3} \le \sqrt{t^3+1} \le \sqrt{8x^3+1}$$

$$\frac{1}{\sqrt{1+8x^3}} \le \frac{1}{\sqrt{1+t^3}} \le \frac{1}{\sqrt{x^3+1}}$$

de , $t \longrightarrow \frac{1}{\sqrt{1+t^3}}$ Cont sur $[x, 2x] \subset \mathbb{R}^+$

$t \longrightarrow \frac{1}{\sqrt{1+8x^3}}$ (Constante % t)

$t \longrightarrow \frac{1}{\sqrt{1+x^3}}$ " "

$a = x \leq b = 2x$

$$\int_x^{2x} \boxed{\frac{1}{\sqrt{1+8x^3}}}\, dt \leq g(x) \leq \int_x^{2x} \frac{1}{\sqrt{1+x^3}}\, dt$$

cte

$$\frac{1}{\sqrt{1+8x^3}}(2x-x) \leq g(x) \leq \frac{1}{\sqrt{1+x^3}}(2x-x)$$

$$x \geq 0 \qquad \frac{x}{\sqrt{1+8x^3}} \leq g(x) \leq \frac{x}{\sqrt{1+x^3}}$$

4) $$\lim_{+\infty} \frac{x}{\sqrt{1+8x^3}} = \lim_{+\infty} \frac{x}{\sqrt{x^2\left(\frac{1}{x^2}+8x\right)}}$$

$$= \lim_{+\infty} \frac{\cancel{x}\ 1}{\cancel{x}\sqrt{\frac{1}{x^2}+\underbrace{8x}_{+\infty}}} = 0$$

$\frac{1}{x^2} \to 0$

$$\lim_{+\infty} \frac{x}{\sqrt{1+x^3}} = \lim_{+\infty} \frac{x}{\sqrt{x^2\left(\frac{1}{x^2}+x\right)}}$$

$$= \lim_{+\infty} \frac{\cancel{x}\ 1}{\cancel{x}\sqrt{\frac{1}{x^2}+x}} = 0$$

$\frac{1}{x^2} \to 0$, $x \to +\infty$

donc $\lim_{x \to +\infty} f(x) = 0$

Theo

$x \in [a, b]$ $\quad g(x) \leq f(x) \leq h(x)$

· g, f, h Cont sur $[a, b]$

$$\Rightarrow \int_a^b g(x)\,dx \leq \boxed{\int_a^b f(x)\,dx} \leq \int_a^b h(x)\,dx$$

$$\int_a^b 4\,dx = \left[4x\right]_a^b = 4(b-a)$$

$$\boxed{\int_a^b m\,dx = m\,(b-a)}$$

$$\int$$

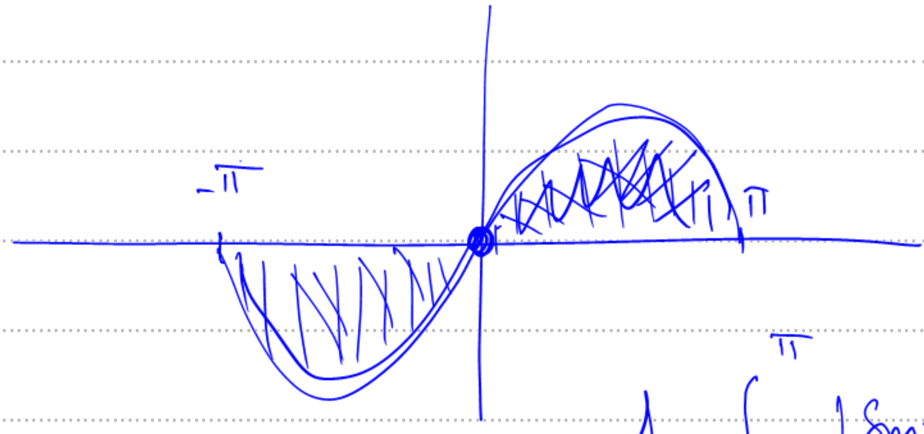
$$\int_0^1 \frac{1}{k+1}\,dx = \frac{1}{k+1}(1-0)$$

$$= \frac{1}{k+1}$$

$$\int_a^b f(x)\,dx = \pm\,\text{Aire}$$

$$\int_0^{\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi}$$

$$= 1 + 1 = 2$$



$$A = \int_{-\pi}^{\pi} |\sin x| \, dx$$

$$= \int_{-\pi}^{0} -\sin x \, dx + \int_{0}^{\pi} \sin x \, dx$$

$$=$$