Exercice

M invariant par $h \iff h(M) = M' = M$

$\iff z' = z$

$z = x + iy$ F.alg

$z + \bar{z} = 2x$

$z - \bar{z} = 2iy$

$z\bar{z} = x^2 + y^2$

$\iff \dfrac{z + i z\bar{z}}{1 + z\bar{z}} = z$

$\iff \cancel{z} + i z\bar{z} = \cancel{z} + z^2\bar{z}$

$\iff i z\bar{z} - z^2\bar{z} = 0$

$\iff z\bar{z}(i - z) = 0$

$\iff z\bar{z} = 0$ ou $z = i$

$z\bar{z} = 0 \iff z = 0 \iff M = O$

$z = i \iff M = A$

les pts invariants par f sont O et A seulement

Soit f dep $\quad f(A) = A$, $f(B) = B \Rightarrow f = Id$.

$f = $ Anti $\quad f(A) = A$, $f(B) = B \Rightarrow f = S_{(AB)}$

- f dep fixe uniquement $A \Rightarrow R_{(A, \alpha \neq 0)}$
- f Antid fixe " " A $\cancel{\Rightarrow}$ il n'y a p.
- f déplacement qui fixe A $(f(A) = A)$

$\xrightarrow{\text{uniqu}} f = R_{(A, \alpha \neq 0)}$

$\rightarrow f = Id$.

Suite 2 :

$\vec{AM}$ et $\vec{AM'}$ Colinéaires

$\Downarrow$

A, M, M' alignés

$$\overrightarrow{AM'} = \alpha\ \overrightarrow{AM}$$

$$\Updownarrow$$

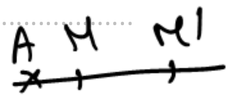
$$z' - z_A = \alpha\ (z - z_A) \qquad (z \neq A)$$

$$\frac{z' - z_A}{z - z_A} = \alpha \in \mathbb{R}^*$$

A≠M

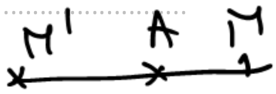
* M'≠A

$$(\overrightarrow{AM}, \overrightarrow{AM'}) \equiv 0 \quad (2\pi)$$



ou

$$\equiv \pi \quad (2\pi)$$



$$z_{\overrightarrow{AM}} = z - i$$

$$z_{\overrightarrow{AM'}} = z' - i \overset{?}{=} \alpha\ (z - i) \ (*)$$

$$\text{or } z' - i = \frac{z + i z\bar{z}}{1 + z\bar{z}} - i = \frac{z + i\cancel{z\bar{z}} - i - i\cancel{z\bar{z}}}{1 + z\bar{z}}$$

$$\Rightarrow z' - i = \frac{z - i}{1 + z \cdot \bar{z}}$$

$$\Rightarrow z' - i = \underbrace{\frac{1}{1+|z|^2}}_{\in \mathbb{R}^*_+} \cdot (z - i)$$

$\Rightarrow \overrightarrow{AM'}$ et $\overrightarrow{AM}$ Sont Co.in.sens.

$\Rightarrow A, M, M'$ alignes

3) $(z \neq 0$ et $z \neq -i) \Leftrightarrow M \neq O$ et $M \neq B$

$$(\vec{u}, \widehat{\overrightarrow{OM'}}) \overset{?}{\equiv} \frac{\pi}{2} - (\underset{\neq 0}{\overrightarrow{MO}}, \underset{\neq 0}{\overrightarrow{MB}}) \ (2\pi)$$

$M \neq O$ Comme $f(O) = O$ $M' \neq O$.

$$\cdot (\vec{u}, \widehat{\overrightarrow{OM'}}) \equiv \arg(z') \ (2\pi)$$

$$\cdot (\overrightarrow{MO}, \widehat{\overrightarrow{MB}}) \equiv \arg\left(\frac{z_B - z_M}{z_O - z_M}\right) (2\pi)$$

$$\equiv \arg\left(\frac{-i - z}{-z}\right) (2\pi)$$

$$\equiv \arg\left(\frac{i + z}{z}\right) (2\pi)$$

on a $z' = \dfrac{z + i z\bar{z}}{1 + z\bar{z}}$

stop

$z \neq 0 \quad z \neq -i$

$z' = 0 \Rightarrow z + i z\bar{z} = 0$

$\Rightarrow z(1 + i\bar{z}) = 0$

$z = 0$ ou $1 + i\bar{z} = 0$

$i\bar{z} = -1$

$\bar{z} = \dfrac{-1}{i} = i$

$\Rightarrow z = -i$

or $z \neq 0$ et $z \neq -i \Rightarrow z' \neq 0$

$\Rightarrow \arg(z') \equiv \arg\left(\dfrac{z + i z\bar{z}}{1 + z\bar{z}}\right) (2\pi)$

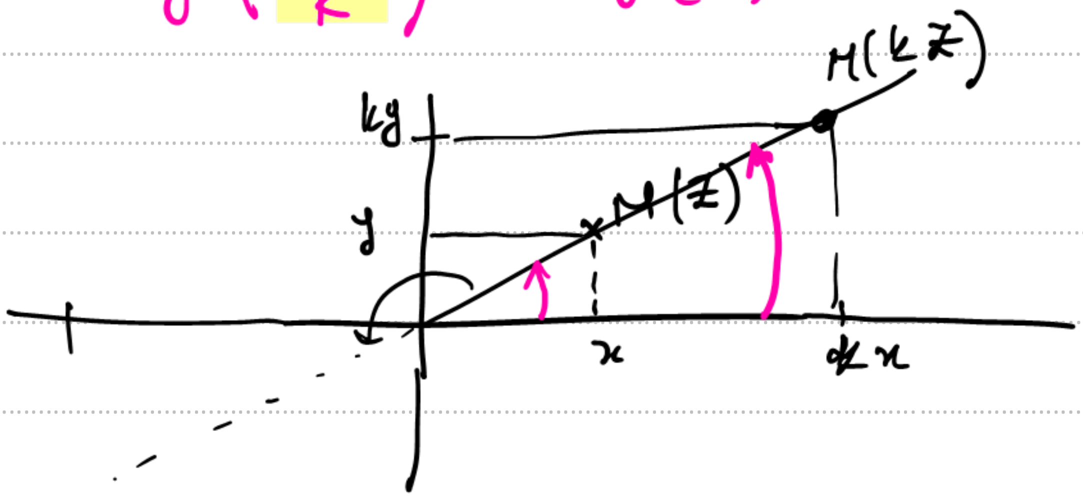
$\arg(z') \equiv \arg(z + i z\bar{z}) (2\pi)$ $\quad 1 + z\bar{z} \in \mathbb{R}^*_+$

$\equiv \arg z(1 + i\bar{z}) \quad (2\pi)$

$\equiv \arg(z) + \arg(1 + i\bar{z}) (2\pi)$

$\arg(kZ) \equiv \arg(Z)\ (2\pi) \quad k>0$

$\arg\left(\frac{Z}{k}\right) \equiv \arg(Z)\ (2\pi) \quad k>0$



$\arg(kZ) \equiv \pi + \arg(Z)\ (2\pi) \quad k<0.$

$\arg(\text{reel strict } \oplus) \equiv 0\ (2\pi)$

$\arg(\text{réel strict } \ominus) \equiv \pi\ (2\pi).$

Suite Ex1 3) a)

$$\arg(z') \equiv \arg(z) + \arg(1+i\bar{z}) \ (2\pi)$$

$$\Rightarrow \arg(z') \equiv \arg(z) + \arg i.(-i+\bar{z}) \ (2\pi)$$

$$\equiv \arg(z) + \arg(i) + \arg(-i+\bar{z}) \ (2\pi)$$

$$\equiv \frac{\pi}{2} + \arg(z) - \arg(\overline{-i+\bar{z}}) \ (2\pi)$$

$$\equiv \frac{\pi}{2} + \arg(z) - \arg(i+z) \ (2\pi)$$

$$\arg(z') \equiv \frac{\pi}{2} + \arg\left(\frac{z}{i+z}\right) \ (2\pi)$$

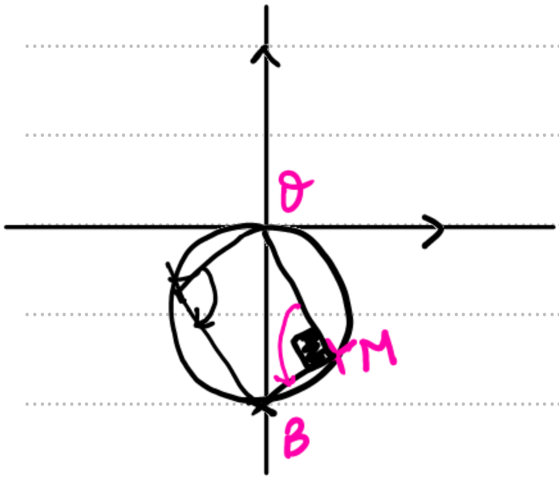
$$(\vec{u}, \overrightarrow{OM'}) \equiv \frac{\pi}{2} + \arg\left(\frac{z-0}{z-(-i)}\right) \ (2\pi)$$

$$\equiv \frac{\pi}{2} + (\overrightarrow{MB}, \overrightarrow{MO}) \ (2\pi)$$

$$(\vec{u}, \overrightarrow{OM'}) = \frac{\pi}{2} - (\overrightarrow{MO}, \overrightarrow{MB}) \ (2\pi)$$

b) $M \in \mathcal{C}$ de diametre $[OB] \Rightarrow$ .....

$\Rightarrow$ .... $\Rightarrow M' \in$ Ens^{ble}



I) $M \in \mathcal{C} \setminus \{O, B\}$.

$$\underbrace{(\vec{MO}, \vec{MB}) \equiv \frac{\pi}{2} (2\pi)}_{1^{er} \text{ cas}} \quad \text{ou} \quad \underbrace{(\vec{MO}, \vec{MB}) \equiv -\frac{\pi}{2} (2\pi)}_{2^{eme} \text{ cas.}}$$

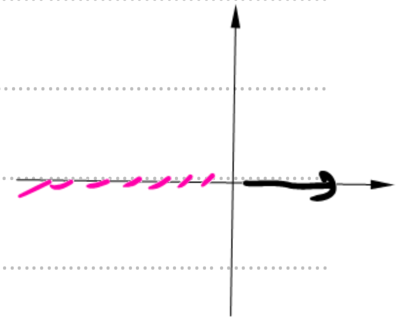
1er cas

$$(\widehat{\vec{u}, \vec{OM'}}) \equiv \frac{\pi}{2} - \frac{\pi}{2} (2\pi) \quad \boxed{\text{ou}} \quad (\widehat{\vec{u}, \vec{OM'}}) \equiv \pi (2\pi)$$

$$\equiv 0 \ (2\pi$$

$\Downarrow$

$\vec{u}, \overrightarrow{OM'}$ Col. $\hat{m}$ sens. $\boxed{\text{ou}}$ $\vec{u}, \overrightarrow{OM'}$. C. sens.



Si $M \in \mathcal{C} \setminus \{O, B\} \Longrightarrow M' \in (O, \vec{u}) \setminus \{O\}$.

II Si $M = O \Longrightarrow M' = O$

Si $M = B \Longrightarrow M' = ?$

$$z' = \frac{-i + i\,(-i)\,\overline{(-i)}}{1 + (-i)\,\overline{(-i)}}$$

$$z' = \frac{-i + i \cdot (-i)\, i}{2}$$

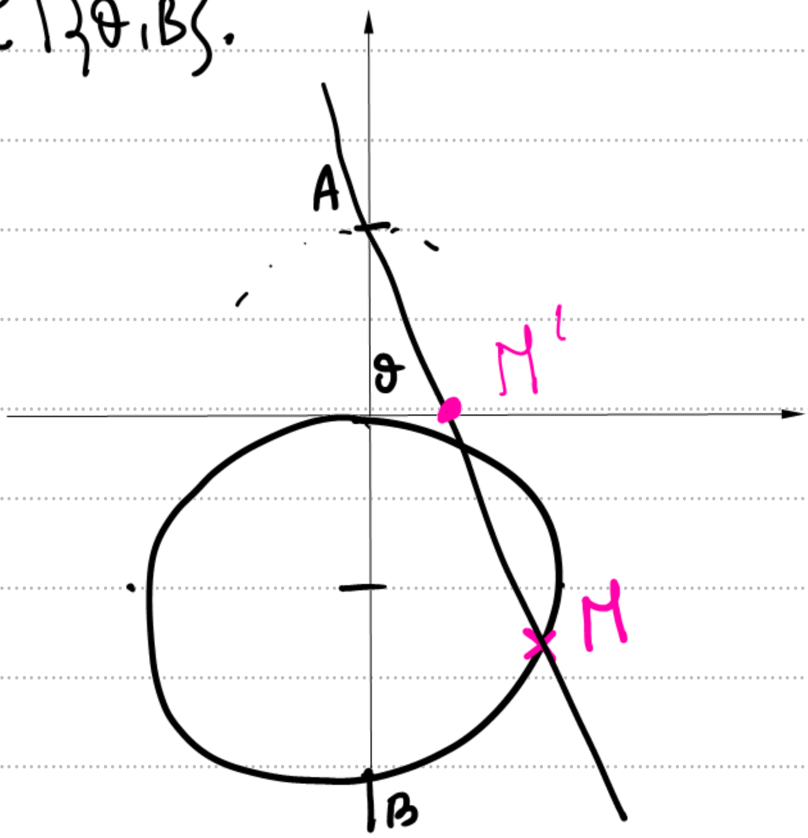
$$= \frac{-i + i}{2} = 0$$

Si $M = B \longrightarrow M' = B' = O$

Conclusion

Si $M \in \mathcal{C} \Longrightarrow M' \in (O, \vec{u})$.

c) $M \in \mathcal{C} \setminus \{O, B\}$.



d'apres 2) $M' \in (AM)$

3)b) $M \in \mathcal{C} \Rightarrow M' \in (O, \vec{u})$

4) a) $z \neq i \iff M \neq A$

$|z'-z| = |z'-i| \iff M \in \mathcal{C}_{(O,1)} \setminus \{A\}$??

$|z'-z| = |z'-i| \iff \left|\frac{z+iz\bar{z}}{1+z\bar{z}} - z\right| = \left|\frac{1}{1+|z|^2}(z-i)\right|$

$\iff \left|\frac{\cancel{z} + iz\bar{z} - \cancel{z} - z^2\bar{z}}{1+z\bar{z}}\right| = \frac{|z-i|}{|1+z\bar{z}|}$

$\iff \frac{|iz\bar{z} - z^2\bar{z}|}{|1+z\bar{z}|} = \frac{|z-i|}{|1+z\bar{z}|}$

$\iff |iz\bar{z} - z^2\bar{z}| = |z-i|$

$\iff |z\bar{z}.(i-z)| = |z-i|$

$\iff |z\bar{z}|\,|i-z| = |z-i|$ $\left(z \neq i \Updownarrow M \neq A\right)$

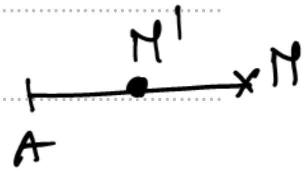
$\iff |z\bar{z}| = 1$

$$\Leftrightarrow \left||z|^2\right| = 1$$

$$\Leftrightarrow |z|^2 = 1 \Leftrightarrow OM = 1$$

$$\Leftrightarrow M \in \mathcal{C} \setminus \{A\}.$$

b) $M \in \mathcal{C} \setminus \{A\}$.

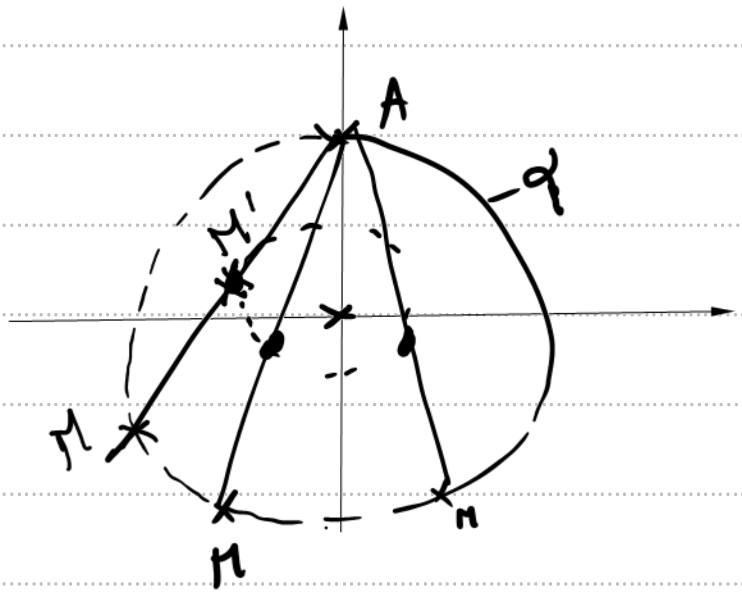


$\Downarrow$

$$|z' - z| = |z' - i|$$

$\Rightarrow MM' = AM'$ Comme $\underline{M' \in (AM)}$
d'après 1°

$\Rightarrow M' = M * A.$



$M' = A * M \iff \overrightarrow{AM'} = \frac{1}{2}\overrightarrow{AM}$

$M' = H_{(A,\frac{1}{2})}(M)$

Comme M varie sur $\mathcal{C} \setminus \{A\}$.

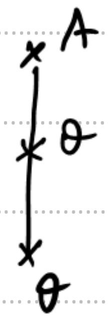
$\Rightarrow \underbrace{H(M)}$ " " $H(\mathcal{C}) \setminus H(A)$

$\Rightarrow M'$ " sur le cercle de

centre $H(O) = O'$ et de rayon $\frac{1}{2} \times 1 = \frac{1}{2}$

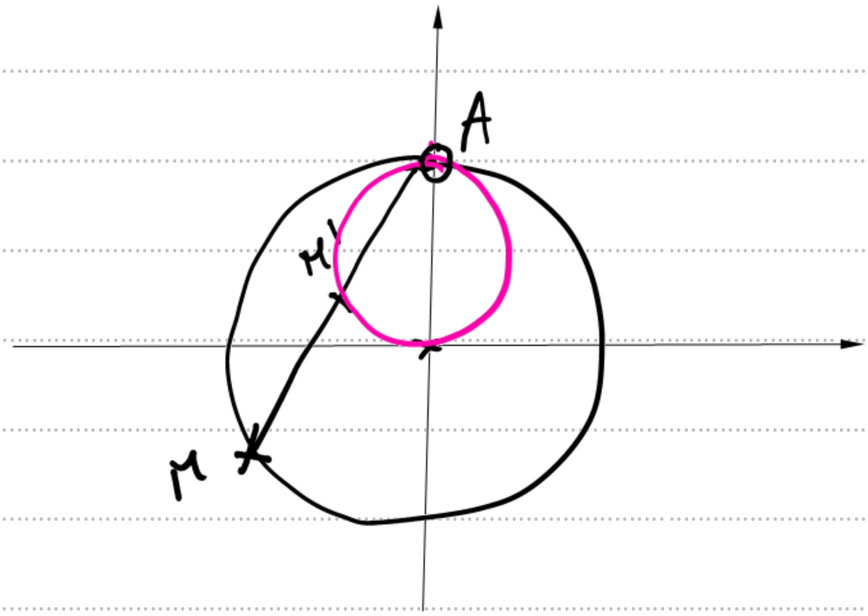
$\overrightarrow{AO'} = \frac{1}{2}\overrightarrow{AO}$

M' varie $\mathcal{C}_{(O',\frac{1}{2})} \setminus \{A\}$



$|\Gamma \quad \mathcal{C} = \mathcal{C}_{(O,R)}$

$h_{(A,k)}(\mathcal{C}) = \mathcal{C}_{(h(O);\,|k|\cdot R)}$



B] $Z = e^{i\theta}$ $\qquad$ $Z\bar{Z} = |Z| = |e^{i\theta}| = 1$

$$Z' = \frac{Z + iZ\bar{Z}}{1 + Z\bar{Z}} = \frac{e^{i\theta} + i}{1 + 1}$$

$$\rightarrow \boxed{Z' = \frac{1}{2}\left(e^{i\theta} + i\right)}$$

- $e^{i\theta} \times e^{-i\theta} = 1$ $\qquad$ $e^{i\theta} = \cos\theta + i\sin\theta$
- $e^{i\theta} + e^{-i\theta} = 2\cos\theta$
- $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$1+e^{i\alpha}=e^{i\frac{\alpha}{2}}\left(e^{-i\frac{\alpha}{2}}+e^{i\frac{\alpha}{2}}\right)$$

$$=e^{i\frac{\alpha}{2}}\times 2\cos\left(\frac{\alpha}{2}\right)$$

$$1+e^{i\alpha}=\underbrace{2\cos\left(\frac{\alpha}{2}\right)}\, e^{i\frac{\alpha}{2}}$$

$$=r\,e^{i\alpha}$$

$$z'=\frac{1}{2}\left(e^{i\theta}+i\right)$$

$$=\frac{1}{2}i\left(-ie^{i\theta}+1\right)$$

$$-i=e^{-i\frac{\pi}{2}}$$

$$=\frac{1}{2}i\underbrace{\left(e^{i\left(\theta-\frac{\pi}{2}\right)}+1\right)}$$

$$=\frac{1}{2}\cdot\underbrace{i\quad 2\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)e}\,{}^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}$$

$$=\cancel{1}\ \cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\ e^{i\frac{\pi}{2}}\times e^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}$$

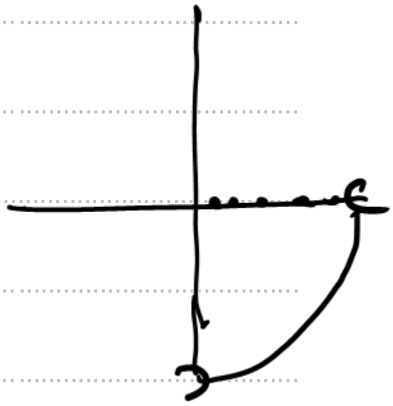
$$Z' = \cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) e^{i\left(\frac{\theta}{2} - \frac{\pi}{4} + \frac{\pi}{2}\right)}$$

$$= \cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$$

or $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < \frac{\theta}{2} - \frac{\pi}{4} < 0$$



$$\Rightarrow \cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) > 0$$

d'où une forme exponentielle de Z'

c'est $Z' = \cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$

$$e^{i\alpha} + e^{i\beta} = e^{i\left(\frac{\alpha}{2} + \frac{\beta}{2}\right)}\left(e^{i\left(\frac{\alpha}{2} - \frac{\beta}{2}\right)} + e^{i\left(\frac{\beta}{2} - \frac{\alpha}{2}\right)}\right)$$

$$e^{i\theta} + i = e^{i\theta} + e^{i\frac{\pi}{2}} = \ldots$$

Équa :

$$Z' = \cos(\psi)\, e^{i\alpha}$$

Mais $\cos(\psi) < 0$

$$Z' = \left(-\cos(\psi)\right)\left(-e^{i\alpha}\right)$$

$$= \left(-\cos(\psi)\right) e^{i(\alpha+\pi)}$$

F. exp.

2)

$$\underbrace{\cos^2\theta}_{} - 2(1+\sin\theta)$$

$$= 1 - \sin^2\theta - 2 - 2\sin\theta$$

$$= -1 - 2\sin\theta - \sin^2\theta$$

$$= -(1 + 2\sin\theta + \sin^2\theta)$$

$$= -(1+\sin\theta)^2 \checkmark$$

b) $Z^2 - \cos\theta\, Z + \frac{1+\sin\theta}{2} = 0$

$a = 1$

$b = -\cos\theta$

$c = \frac{1+\sin\theta}{2}$

$\Delta = b^2 - 4ac$

$= \cos^2(\theta) - 4\left(\frac{1+\sin\theta}{2}\right)$

$= \cos^2\theta - 2(1+\sin\theta)$

$\Delta = -(1+\sin\theta)^2$

$\Delta = (\delta?)^2$

$\Delta = i^2 (1+\sin\theta)^2$

$= \left(\underbrace{i(1+\sin\theta)}\right)^2$

$= (\delta)^2$ $\quad \delta = i(1+\sin\theta)$

$$\Delta$$

$\Delta = 0$ $(\sin\theta = -1)$

$\Downarrow$

$z = -\frac{b}{2a}$

$\Delta \neq 0$

$z_1 = \frac{-b+\delta}{2a}$ $z_2 = -b-$

$$\Delta = \left(\underbrace{i\,(1+\sin\theta)}\right)^2 \neq 0$$

car $\sin\theta \neq -1$

donc l'eq admet 2 sols.

$$z_1 = \frac{\cos\theta + i\,(1+\sin\theta)}{2}$$

$$z_2 = \frac{\cos\theta - i\,(1+\sin\theta)}{2}$$

$$\Rightarrow \quad z_1 = \frac{1}{2}\left(\cos\theta + i\sin\theta + i\right)$$

$$= \frac{1}{2}\left(e^{i\theta} + i\right)$$

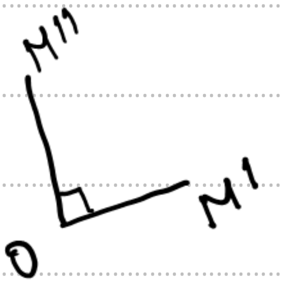
$$Z_2 = \frac{1}{2}\left(\underbrace{\cos\theta - i\sin\theta} - i\right)$$

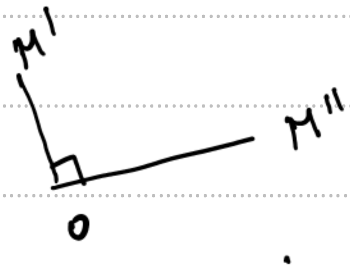
$$= \frac{1}{2}\left(e^{-i\theta} - i\right)$$

$$Z_2 = \overline{Z_1}$$

$$S_{\mathbb{C}} = \left\{Z_1, \overline{Z_1}\right\}.$$

c) $M(z)$ $M'(z')$ $M''(\overline{z'})$





$$\frac{Z_{\overrightarrow{OM''}}}{Z_{\overrightarrow{OM'}}} \; ? \neq 0$$

$$Z_{\overrightarrow{OM'}} = 0 \iff z' = 0$$

$$\Rightarrow \frac{1}{2}\left(e^{i\theta} + i\right) = 0$$

$$\Rightarrow e^{i\theta} = -i$$

$$\Rightarrow \theta = -\frac{\pi}{2} \text{ or } \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$\frac{z_{\overrightarrow{OM''}}}{z_{\overrightarrow{OM'}}} = \frac{\overline{z'}}{z'} = \frac{\overline{z'} \times \overline{z'}}{z' \times \overline{z'}}$$

$$= \frac{\overline{z'^2}}{|z'|^2}$$

$$z' = \underbrace{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)}_{\text{mod}} e^{i\left(\frac{\theta}{2} + \frac{\pi}{4}\right)}$$

donc $$\frac{z_{\overrightarrow{OM''}}}{z_{\overrightarrow{OM'}}} = \frac{\cancel{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)^2} e^{-i\left(\theta + \frac{\pi}{2}\right)}}{\cancel{\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)^2}}$$

$$= e^{-i\left(\theta + \frac{\pi}{2}\right)}$$

pour que $\overline{OM'} \perp \overline{OM''}$

$\Rightarrow e^{-i\left(\theta+\frac{\pi}{2}\right)}$ imaginaire.

$e^{i\alpha} = \cos\alpha + i\sin\alpha \in \mathbb{R}$

$\Updownarrow$

$\sin\alpha = 0 \iff \alpha = k\pi$

$e^{i\alpha} = \cos\alpha + i\sin\alpha \in i\mathbb{R}.$

$\Updownarrow$

$\cos\alpha = 0 \iff \alpha = \frac{\pi}{2} + k\pi$

$OM'M''$ rectangle en O $\Rightarrow \cos\left(-\left(\theta+\frac{\pi}{2}\right)\right) = 0$

$\cos\left(\theta+\frac{\pi}{2}\right) = 0 \Rightarrow \theta + \frac{\pi}{2} = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

or $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\iff -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow 0 < \theta + \frac{\pi}{2} < \pi$

$\theta + \frac{\pi}{2} = \frac{\pi}{2}$

$\theta = 0$.