Birth and Death Process — M/G/1 Queue – Pollaczek - Khintchine Formula, Series Queues – Open and Closed Networks

BIRTH AND DEATH PROCESS

U.Q

Let $x(t)$ be the number of the customers at time t [population at time t] in which two types of events occur namely **births** and **deaths**. Then the continuous time discrete random process $\{X(t);$ With state space $\{0,1,2...\}$ is called a **birth and death process** if the following postulates are satisfied.

If $X(t) = n$, that is the size of the population at time t is n or the system is in state n. Let λ_n, $n = 0,1,2,...$ be the rate at which births occur in the state n and μ_n, $n = 1,2,....$ be the rate at which deaths occur in state n, such that

i. P [1 birth in $(t, t+h)$] $= \lambda_n h + O(h)$
ii. P [0 birth in $(t, t+h)$] $= 1 - \lambda_n h + O(h)$
iii. P [2 or more births in $(t, t+h)$] $= O(h)$
iv. Births in $(t, t+h)$ are independent of time since the last birth
v. P [1 death in $(t, t+h)$] $= \mu_n h + O(h)$
vi. P [0 death in $(t, t+h)$] $= 1-\mu_n h + O(h)$
vii. P [2 or more deaths in $(t, t+h)$] $= O(h)$
viii. Deaths occuring in $(t, t+h)$ are independent of time since the last death
ix. Births and deaths occur indepedently of each other at any time.

Probability Distribution of X(t)

U.Q

Let $X(t)$ be a birth and death process.

Let $X(t) = n$ be the event with corresponding probability $P_n(t) = P[X(t) = n]$

Then $P_n(t+h) = P[X(t+h) = n]$ be the probability that the size of the population is n at time $t+h$.

The event $X(t+h) = n$ means that the process will be in state n at time $t+h$ if one of the following mutually exclusive and exhaustive events occurs.

i. $X(t) = n$ and no birth or death in $(t, t+h)$
ii. $X(t) = n-1$ and 1 birth and no death in $(t, t+h)$
iii. $X(t) = n+1$ and no birth and one death in $(t, t+h)$
iv. $X(t) = n$ and 1 birth and 1 death in $(t, t+h)$

Then, $P[X(t) = n] = P_n(t+h)$

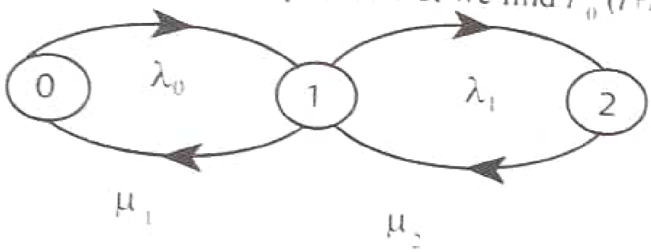
$$= P_n(t)[1 - \lambda_n h + O(h)][1 - \lambda_n h + O(h)] +$$
$$P_{n-1}(t)[\lambda_{n-1} h + O(h)][1 - \mu_{n-1} h + O(h)] +$$
$$P_{n+1}(t)[1 - \lambda_{n-1} h + O(h)][\mu_{n+1} h + O(h)] +$$
$$P_n(t)[\lambda_n h + O(h)][\mu_n h + O(h)] \qquad [\because P_n(t+h) = P[(i)+(ii)+(iii)+(iv)]$$
$$= P_n(t)[1 - \lambda_n h - \mu_n h] + O(h) + P_{n-1}(t)[\lambda_{n-1} h] + O(h) +$$
$$P_{n+1}(t)[\mu_{n+1}(h)] + O(h) + P_n(t)$$

Since $O(h)[1 - \lambda_n h + O(h)] = O(h)$ & $\lambda_n \mu_n h^2 - \lambda_n O(h) + O(h) = O(h)$

$$\Rightarrow P_n(t+h) = P_n(t)[1 - (\lambda_n + \mu_n)h] + P_{n-1}(t)\lambda_{n-1} h + P_{n+1}(t)\mu_{n+1} h + O(h)$$
$$\therefore P_n(t+h) - P_n(t) = -P_n(t)(\lambda_n + \mu_n)h + P_{n-1}(t)\lambda_{n-1} h + P_{n+1}(t)\mu_{n+1} h + O(h)$$
$$\Rightarrow \frac{P_n(t+h) - P_n(t)}{h} = -P_n(t)(\lambda_n + \mu_n) + P_{n-1}(t)\lambda_{n-1} + P_{n+1}(t)\, P_{n-1}(t)\mu_{n+1} + \frac{O(h)}{h}$$
$$\lim_{h\to 0}\left[\frac{P_n(t+h) - P_n(t)}{h}\right] = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1} P_{n-1}(t) + P_{n+1}(t)\mu_{n+1}$$

i.e. $P_n^{\,|}(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1} P_{n-1}(t) + P_{n+1}(t)\mu_{n+1}$ ....(1)

This is true of all n≥1, Since the state space has 0, we find $P_0(t+h)$ by a similar reasoning.



(State diagram)

$$P_0(t+h) = P_0(t)[1 - \lambda_0 h + O(h)] + P_1(t)[1 - \lambda_0 h + O(h)][\mu_1 h + O(h)]$$
$$= P_0(t) - \lambda_0 h P_0(t) + P_1(t)\mu_1 h + O(h)$$
$$\Rightarrow P_0(t+h) - P_0(t) = -\lambda_0 h P_0(t) + \mu_1 h P_1(t) + O(h)$$
$$\therefore \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \mu_1 P_1(t) + \frac{O(h)}{h}$$
$$\therefore \lim_{h\to 0} \frac{P_0(t+h) - P_0(t)}{h} = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

i.e. $P_0^{\,|}(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$ ....(2)

Solving (1) and (2) we get $P_n(t)$ for $n \geq 0$, which gives the probability distribution of $X(t)$.