Numerical Aspects of a Phase Field Model for Fracture Mechanics

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Outline

1. Fracture Mechanics
2. Phase Field Model
3. Numerical Examples
4. Conclusion
5. Future Scope

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Fracture Mechanics

- Griffith employed an energy-balance approach to find the total strain energy (U) in a linear material:

$$U = -\frac{\boldsymbol{\sigma}^2}{2E}.\pi a^2 \tag{1}$$

$\boldsymbol{\sigma}$: Cauchy′s stress; E : Young′s modulus; a : length of crack; $\beta = \pi$ (plane stress loading)

- Surface energy (S) absorbed to create new surfaces (γ : surface energy):

$$S = 2\gamma a \tag{2}$$

- Value of critical crack length:

$$\frac{\partial(S+U)}{\partial a} = 2\gamma - \frac{\boldsymbol{\sigma}^2}{E}\pi a = 0 \tag{3}$$

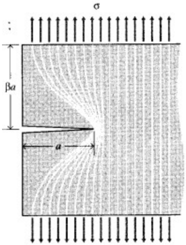

Figure 1: Idealization of unloaded region

- Solving:

$$\sigma = \sqrt{\frac{2E\gamma}{\pi a}} \tag{4}$$

- Including sinks like energy dissipation due to plastic flow near the crack tip:

$$\sigma = \sqrt{\frac{EG_c}{\pi a}} \tag{5}$$

G_c : critical strain energy release rate

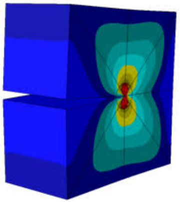

Figure 2: A fractured module

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- Most of the approaches in fracture models represent cracks as discrete discontinuities, except phase field model.

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Phase Field Model

- In phase field model, defect e.g. crack is approximated by a phase-field, which smoothens the boundary of a crack over a small region.
- Major advantage is that evolution of the crack follows from solution of a coupled system of Partial Differential Equations.

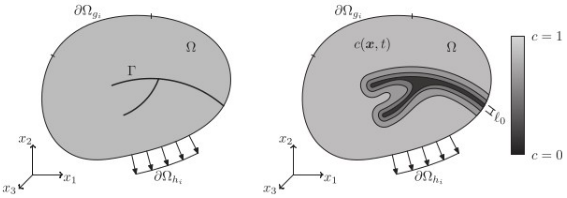

Figure 3: A phase field model

Phase Field Model

- Elastic energy (ψ) is defined as:

$$\psi(\epsilon) = [(1-\phi)^2 + k]\psi^+ + \psi^- \tag{6}$$

$$\psi^{\pm}(\boldsymbol{\epsilon}) = \frac{1}{2}\lambda\langle \boldsymbol{tr(\epsilon)}\rangle_{\pm}^2 + \mu \boldsymbol{tr(\epsilon_{\pm}^2)} \tag{7}$$

with

$$\langle x\rangle_+ = \begin{cases} x & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

ϕ : phase-field parameter; k : viscous parameter; λ,μ : Lame parameter; $\boldsymbol{\epsilon}_{\pm} = \sum_{I=1}^{3}\langle \epsilon_I\rangle_{\pm}\boldsymbol{n_I} \otimes \boldsymbol{n_I}$, ϵ_I : principle strains, $\boldsymbol{n_I}$: principle strain directions.

- Phase field variable (ϕ) is controlled by the tensile elastic energy ψ^+, contributing to crack propagation.
- Compressive elastic energy ψ^- doesn't contribute to crack propagation.

Phase Field Model

- The boundary value problem in the absence of body force in a elastic body Ω is: find $(\boldsymbol{u},\phi):\Omega \rightarrow \mathrm{R}^d$.

$$[(1-\phi)^2 + k]\nabla \cdot \boldsymbol{\sigma} = 0 \tag{8}$$

$$-G_c l_0 \nabla^2 \phi + [\frac{G_c}{l_0} + 2H^+]\phi = 2H^+ \tag{9}$$

l_0 : width of failure zone; $\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\epsilon})}{\partial \boldsymbol{\epsilon}}$;

$$H^+ = \begin{cases} \psi^+(\boldsymbol{\epsilon}) & \text{if } \psi^+(\boldsymbol{\epsilon}) < H_n^+ \\ H_n^+ & \text{otherwise, } (H_n^+ \text{ :tensile strain energy at load step n)} \end{cases}$$

- Boundary conditions: $[(1-\phi)^2 + k]\nabla \cdot \boldsymbol{\sigma} = \bar{\boldsymbol{t}}$ on Γ_N, $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on Γ_D, $\nabla\phi \cdot \boldsymbol{n} = 0$ on Γ_N.

*Hirshikesh, A FEniCS implementation of the phase field method for quasi-static brittle fracture,2018

Phase Field Model

- Staggered algorithm is used to solve weak forms:(spatial discretization of coupled equations)
  1. Initialization: Displacement ($\boldsymbol{u}$), phase field (ϕ) and history (H) are known at initial time step.
  2. Compute History: It is obtained from it's definition at that particular time step.
  3. Compute Phase field: Phase field (ϕ) is solved using given displacement field ($\boldsymbol{u}$).
  4. Compute Displacement field: Updated phase field (ϕ) is used to solve for displacement field ($\boldsymbol{u}$).
- The numerical implementation of this algorithm has been done in FEniCS.

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Numerical Examples

Problem Statement (Shear Loading):

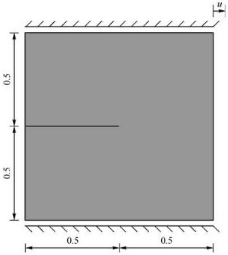


Figure 4: Plate with edge crack

- Given : Quasi-static shear loading in 2D plate with plane strain and existing crack.
- Unknowns : Displacement field ($\boldsymbol{u}$) and phase field (ϕ).
- Output : Crack propagation inside the plane.

Numerical Examples

Problem Statement (Shear Loading): Other given conditions-

- Griffith critical energy $(G_c) = 2700$ J$/m^2$
- Poisson ratio $(\nu) = 0.3$
- Young's modulus $(E) = 210$ GPa
- Plane strain and existing crack.
- Width of failure zone $(l_0) = 0.011$ mm
- Viscous damping parameter $(k) = 10^{-6}$.
- Staggered algorithm for displacement and phase field.

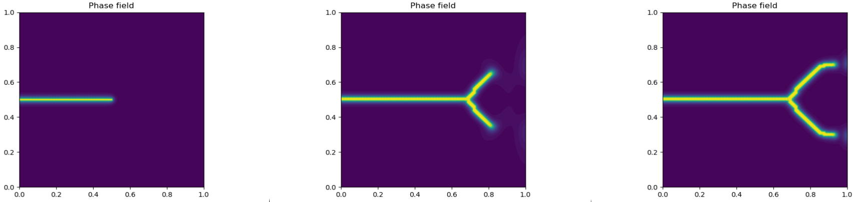
Numerical Examples

Case 1:

- Mesh is refined.
- History function used :

$$H(\boldsymbol{u}) = \frac{1}{2}k_n[tr(\epsilon)]^2 + \mu tr((\epsilon^{\boldsymbol{dev}})^2)$$

$k_n = \lambda + \mu$; $\epsilon^{\boldsymbol{dev}}$ = Deviatoric strain tensor.



u = 8.3 x $10^{-3}mm$. u = 15.5 x $10^{-3}mm$. u = 30 x $10^{-3}mm$.

Figure 5: Crack progression at various loads increments

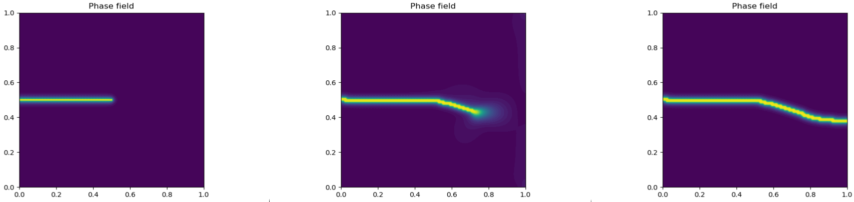
Numerical Examples

Case 2:

- Mesh is refined.
- History function used :

$$H(\boldsymbol{u}) = \frac{1}{2} k_n \left[\frac{1}{2} \left[tr(\epsilon) + abs(tr(\epsilon)) \right] \right]^2 + \mu tr((\epsilon^{\boldsymbol{dev}})^2)$$

$k_n = \lambda + \mu$; $\epsilon^{\boldsymbol{dev}}$ = Deviatoric strain tensor.



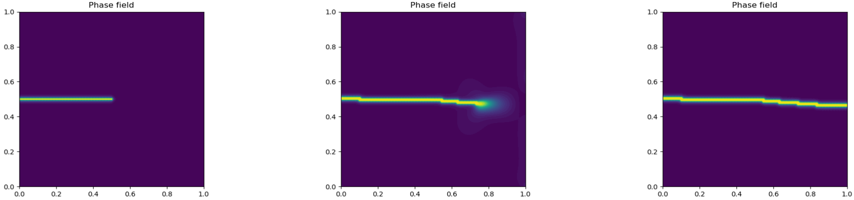
u = 8.3 x 10^{-3} *mm*.   u = 12.7 x 10^{-3} *mm*.   u = 15.5 x 10^{-3} *mm*.

Figure 6: Crack progression at various loads increments

Numerical Examples

Case 3:

- $l_0 = 7.5 \text{ x } 10^{-3} mm.$
- Mesh is refined.
- l_0 is a model parameter that controls the width of the smooth approximation of the crack.



$u = 8.3 \text{ x } 10^{-3} mm.$ $u = 15.5 \text{ x } 10^{-3} mm.$ $u = 30 \text{ x } 10^{-3} mm.$

Figure 7: Crack progression at various loads increments

Numerical Examples

Problem Statement (Tension Loading):

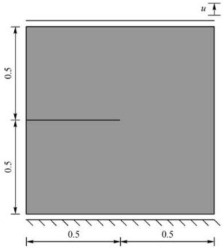


Figure 8: Plate with edge crack

- Given : Quasi-static Tension loading in 2D plate with plane strain and existing crack.
- Unknowns : Displacement field ($\boldsymbol{u}$) and phase field (ϕ).
- Output : Crack propagation inside the plane.

Numerical Examples

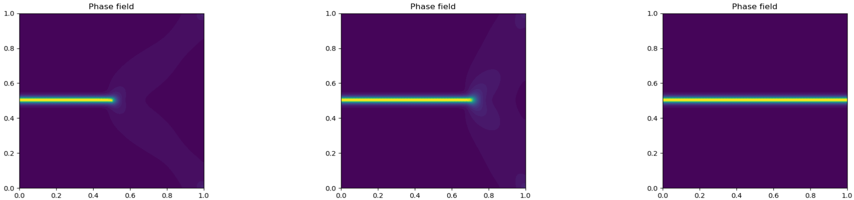
Problem Statement (Tension Loading): Other given conditions-

- Griffith critical energy (G_c) = 2700 J/m^2
- Poisson ratio (ν) = 0.3
- Young's modulus (E) = 210 GPa
- Width of failure zone (l_0) = 0.011 mm
- Plane strain and existing crack.
- Viscous damping parameter (k) = 10^{-6}.
- Staggered algorithm for displacement and phase field.

Numerical Examples

Case 1:

- Mesh is refined.
- History function used is same as earlier one.



u = 5.1 x $10^{-3} mm$. u = 6.5 x $10^{-3} mm$. u = 8.3 x $10^{-3} mm$.

Figure 9: Crack progression at various loads increments

Numerical Examples

Python Code:

```
#### Spaces #####
W = VectorFunctionSpace(mesh,'CG',1)
u , v = TrialFunction(W),TestFunction(W)

V = FunctionSpace(mesh,'CG',1)
p , q = TrialFunction(V), TestFunction(V)


####### classes #########
#Coordinate system with bottom as reference axis:
class top(SubDomain):
    def inside(self,x,on_boundary):
        tol = 1e-10
        return abs(x[1]-1.0) < tol and on_boundary

class bottom(SubDomain):
    def inside(self,x,on_boundary):
        tol = 1e-10
        return abs(x[1]) < tol and on_boundary

class middle(SubDomain):
    def inside(self,x,on_boundary):
        tol = 1e-3
        return abs(x[1]-0.5) < tol and x[0] <= 0.5

def epsilon(u):
    return 0.5*(grad(u) + grad(u).T)

def sigma(u):
    return lmbda*nabla_div(u)*Identity(2) + 2*mu*epsilon(u)

kn = lmbda + mu

def hist(u):
    return 0.5*kn*( 0.5*(tr(epsilon(u)) + abs(tr(epsilon(u)))) )**2 + mu*tr(dev(epsilon(u))*dev(epsilon(u)))
```

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Conclusion

- Phase-field model provides a smooth representation of a crack and removes requirement to numerically track discontinuities in the displacement field.
- Results obtained are in good agreement with the standard results from the literature.

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Future Scope

- Artificial viscous regularization can be used leading to more robust overall performance.
- An adaptive local refinement strategy that allows for efficient simulation of complex crack patterns can be used taking advantage of the character of the phase-field evolution to determine when refinement is needed.
- Phase field model can be extended and studied to the dynamic case as well.