INDIAN INSTITUTE OF TECHNOLOGY (BHU) VARANASI
Even Semester 2020-2021
Mathematics I (MA 101)
Mid-semester Examination

Full Marks: 25 Date of Exam: 19-May-2021

Please read the following instructions carefully.

- Your submission must be neatly typed or hand written. The front page of your solution copy must include your roll number, email-id and discipline.
- Submit only ONE PDF file. *Do sign at the top and the bottom of every page of the PDF file.* A page without your signature will be discarded for evaluation.
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- The marks for a question is indicated by the numbers in the parentheses against the question.
- Start answering each question on a fresh page.
- All the notations and terminologies are usual and as per the class note.

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1. By the precise definition (ϵ-N definition) of limit, prove that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for any given $a > 0$. (3)

2. Examine convergence of the following series:

   (a) $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \cdots$

   (b) $1 - \frac{2^2}{2!} + \frac{3^3}{3!} - \frac{4^4}{4!} + \cdots$

   (2+2)

3. Show that the series
$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} + \cdots, \ a > 0,$$
   is

   (a) absolutely convergent if $p > 1$, and

   (b) conditionally convergent if $0 < p \leq 1$.

   (3)

4. Let $\sum u_n$ be a series of arbitrary terms. For all $n \in \mathbb{N}$, let $p_n = \frac{1}{2}(u_n + |u_n|)$ and $q_n = \frac{1}{2}(u_n - |u_n|)$. Show that

   (a) If $\sum u_n$ is absolutely convergent, then both $\sum p_n$ and $\sum q_n$ are convergent.

   (b) If $\sum u_n$ is conditionally convergent, then both $\sum p_n$ and $\sum q_n$ are divergent.

   (2+2)

5. Consider the sequence $\{x_n\}$ that is recursively defined by

$$x_1 = 1, x_2 = 2, x_{n+1} = x_n + x_{n-1}, n \geq 2.$$

(a) Show that $\{\frac{x_{n+1}}{x_n}\}$ is a Cauchy sequence.

(b) Find $\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n}$.

(3+1)

6. Let $\phi(x) = \frac{1}{4} + x - x^2$. For a given $r \in \mathbb{R}$, consider the sequence $\{x_n\}$, which is defined by

$$x_1 = r \;\text{ and }\; x_{n+1} = \phi(x_n),\; n \in \mathbb{N}.$$

If the sequence converges, let r_∞ be the limit.

(a) For $r = 0$, prove that $\{x_n\}$ is bounded and monotonic. Find $r_\infty = \beta$.

(b) Find all $r \neq 0$ such that $r_\infty = \beta$.

(3 + 2)

7. Consider the sequence $\{x_n\}$, which is defined by

$$x_1 = 1,\; x_{n+1} = x_n + \frac{1}{x_1 + x_2 + \cdots + x_n},\; n \in \mathbb{N}.$$

Does $\{x_n\}$ converge? Justify your answer.

(2)

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INDIAN INSTITUTE OF TECHNOLOGY (BHU) VARANASI

Odd Semester 2017-2018

Engineering Mathematics I (MA 101)

Mid-semester Examination (Closed Book)

Maximum Marks: 30 Date: 16-09-2017 Time: 10:45 AM-12:45 PM

Please read the following instructions carefully.

- This question paper is printed on both the sides of the sheet.
- Answer all the questions. Each question carries *three* marks.
- Start answering each question on a fresh page.
- All the notations and terminologies are usual ones and as per the textbook.
- Calculator and any other electronic gadget are not allowed in the examination hall.
- No query will be entertained. If any question is found with discrepancy, full marks will be awarded against the question.
- Any portion of your answer that is written with pencil will not be considered valid.
- Violation of any rule will lead to penalization.

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1. Prove that the sequence $\{a_n\}$ defined by

$$a_1 = \sqrt{5}, \ \ a_{n+1} = \sqrt{5 + a_n}, \ n = 1, 2, 3, \cdots$$

is convergent and converges to the positive root of $x^2 - x - 5 = 0$.

2. (a) Let $\{a_n\}$ be a sequence that satisfies $|a_{n+1} - a_n| < \frac{1}{3^n}$. Is the sequence $\{a_n\}$ convergent? Justify your answer.

   (b) Is the answer affirmative in Part (a) if we only assume that $|a_{n+1} - a_n| < \frac{1}{n}$? Justify your answer.

3. Find the radius of convergence of the power series

$$x + \frac{2^2x^3}{2!} + \frac{3^3x^5}{3!} + \frac{4^4x^7}{4!} + \cdots .$$

4. Test the convergence of the series $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^4+4}}$.

5. Test the convergence of the series $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)$.

6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^3+5}{2^n+5}$.

7. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt{n} + \sin^2 n}$.

8. With the help of Sandwich Theorem, prove that $\lim\limits_{n\to\infty} \dfrac{2017^n}{n!} = 0$.

9. Find the Taylor polynomial of order *five* generated by the function $\sin x$ at $x = \frac{\pi}{2}$.

10. Let f be a positive valued continuous function on $[-1, 1]$. Prove that there exists a $\beta > 0$ such that $f(x) \geq \beta$ for all x in $[-1, 1]$.

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Department of Mathematical Sciences, IIT(BHU)
Odd Semester 2019-2020

Mid Semester Examination

MA101 - Engineering Mathematics- I

Time: 2 Hours | Maximum Marks: 30

Instructions to all Candidates

i All questions are compulsory.

ii Usage of calculators or other electronic gadgets are NOT permitted.

iii Write each and every step clearly and show all your workings for proper evaluation.

iv Ineligible writing and incomplete working will result in penalty.

v Each question should begin on a new page.

1. Answer True or False. Justify your answer. $5 \times 1 = 5$

   (a) If the sequence (a_n) is a monotonic increasing then the sequence $(\frac{1}{a_n})$ is monotonic decreasing.

   (b) A real convergent sequence is always bounded.

   (c) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ is always convergent.

   (d) If $\sum a_n$ is convergent then $\sum |a_n|$ is also convergent.

   (e) If $f(x)$ has derivatives of all orders about a point x_0, then the Taylor series about x_0 of $f(x)$ will always converge to $f(x)$.

2. Find a positive integer N so that for all $n \geq N$, $\left|\frac{3n+1}{4n+5} - \frac{3}{4}\right| < \frac{1}{1000}$. (2)

3. Use $\epsilon - \delta$ definition of limit, to show that $\lim_{x\to 1} \frac{x^2 - x + 1}{x+1} = \frac{1}{2}$ (3)

$a_n =$ 4. Show that the sequence $(1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \dots,)$ is a Cauchy sequence. $a_{n-1} + a_{n-2} =$ (3)

tailed

5. Discuss the convergence of the sequence (a_n) and find the limit if it exists where (4)

$$a_1 = 1 \text{ and } a_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right), \text{ for } n \geq 2$$

6. If $\sum a_n$ is an absolutely convergent series and $\{b_n\}$ is a bounded sequence, then show that $\sum a_n b_n$ is absolutely convergent. (2)

7. Using the Taylor's formula with $x_0 = 0$ and $n = 3$, find the series expansion of $f(x) = \sqrt[3]{(1+x)}$, for $x > -1$. (3)

P.T.O...

8. Show that the series $\sum(-1)^n a_n$ converges, where a_n is given by

$$a_{2n-1} = \frac{1}{n} \text{ and } a_{2n} = \int_n^{n+1} \frac{dx}{x}, \text{ for } n = 1, 2, 3, \ldots.$$

Hence deduce that $\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)\right)$ exists.

9. Consider the series $\sum_n \frac{x^n}{(2n+1)^p}$, for $x, p \in \mathbb{R}$.
For what values of x and p, the above power series
(i) converges absolutely
(ii) converges
(iii) diverges.

**** ALL THE BEST ****

INDIAN INSTITUTE OF TECHNOLOGY (BHU) VARANASI
Even Semester 2020-2021
Mathematics I (MA 101)
Mid-semester Examination

Full Marks: 25 Date of Exam: 19-May-2021

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   (2+2)

3. Show that the series
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   (3)

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7. Consider the sequence $\{x_n\}$, which is defined by

$$x_1 = 1,\ x_{n+1} = x_n + \frac{1}{x_1 + x_2 + \cdots + x_n},\ n \in \mathbb{N}.$$

Does $\{x_n\}$ converge? Justify your answer.

(2)

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DEPARTMENT OF MATHEMATICS

Indian Institute of Technology (BHU)

27 May 2022

Engineering Mathematics-I (Code: MA 101)

Time: 24 hour | *Mid-Term* | Maximum Marks: 25

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1. Let $1 \leq x_1 \leq x_2 \leq 2$ and $x_{n+2} = \sqrt{x_{n+1}x_n}$, $n \in \mathbb{N}$. Show that x_n converges. [5]
2. Prove or disprove the following statements:
   "Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum_{n=1}^{\infty} c_n$ is convergent." Justify your answer. [5]
3. Determine the values of x for which the power series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}\left(\frac{2x+1}{x}\right)^n$ converges? [5]
4. Let $x_n = \left(\frac{a^n+b^n+c^n}{3}\right)^{1/n}$ where $1 < a < b < c$. Show that the sequence converges and find its limit. [5]
5. Determine the following limit using $\varepsilon - \delta$ definition of limit:

$$\lim_{x \to 4} \frac{2x-5}{6-3x}$$

[5]

Indian Institute of Technology (BHU), Varanasi

Department of Mathematical Sciences

End Semester Examination

Course: Mathematics I (MA 101) **Maximum Marks: 35**

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---

1. A function f is defined on $[1, \infty)$ by (5)

$$f(x) = \frac{(-1)^{n-1}}{n}, \quad \text{for } n \leq x < n+1 \ (n = 1, 2, 3, \dots).$$

Examine convergence of the integrals (i) $\int_1^\infty f(x)dx$, (ii) $\int_1^\infty |f|(x)dx$.

2. Find the values of $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy\, dx$ and $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx\, dy$. Noting the values are different can we conclude that Fubini's theorem is violated here? (4)

3. Let $f : [0, 1] \to (0, \infty)$ be a continuous function, then show that (5)

$$\lim_{n\to\infty} \left(f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n}{n}\right) \right)^{\frac{1}{n}} = e^{\int_0^1 \ln f(x)\ dx}.$$

4. Let f be a continuous function, then prove that (2)

$$\int_0^y \left(\int_0^u f(x)\ dx \right) du = \int_0^y f(t)(y-t)\ dt.$$

5. For what values of p and q the following integral is convergent (5)

$$\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx.$$

6. Describe the two regions in (a, b) space for which the function $f_{a,b}(x, y) = ay^2 + bx$ restricted to the circle $x^2 + y^2 = 1$ has exactly two and exactly four critical points. (4)

7. Let $a < b$ and (5)

$$f(x) = \begin{cases} 0, & \text{if } x \in [a, b] \cap \mathbb{Q} \\ x, & \text{if } x \in [a, b] \text{ is irrational,} \end{cases}$$

where $\mathbb{Q}$ denotes the set of rational numbers. Find the upper and lower Riemann integrals of $f(x)$ over $[a, b]$, and conclude whether $f(x)$ is Riemann integrable.

8. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be Riemann integrable over $[x, 1]$ for all x with $0 < x \leq 1$. Prove or disprove that f is Riemann integrable over $[0, 1]$. (2)

9. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by (3)

$$f(x, y) = \begin{cases} \frac{\cos y \sin x}{x}, & \text{if } x \neq 0, \\ \cos y, & \text{if } x = 0. \end{cases}$$

Discuss the continuity of f.

Indian Institute of Technology (BHU), Varanasi

Department of Mathematical Sciences

End Semester Examination

Course: Mathematics I (MA 101) **Maximum Marks: 35**

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2. Find the values of $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dy\, dx$ and $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3} dx\, dy$. Noting the values are different can we conclude that Fubini's theorem is violated here? (4)

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4. Let f be a continuous function, then prove that (2)

$$\int_0^y \left(\int_0^u f(x)\ dx \right) du = \int_0^y f(t)(y-t)\ dt.$$

5. For what values of p and q the following integral is convergent (5)

$$\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}} dx.$$

6. Describe the two regions in (a, b) space for which the function $f_{a,b}(x, y) = ay^2 + bx$ restricted to the circle $x^2 + y^2 = 1$ has exactly two and exactly four critical points. (4)

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8. Let $f : [0,1] \longrightarrow \mathbb{R}$ be Riemann integrable over $[x, 1]$ for all x with $0 < x \leq 1$. Prove or disprove that f is Riemann integrable over $[0,1]$. (2)

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$$f(x,y) = \begin{cases} \frac{\cos y \sin x}{x}, & \text{if } x \neq 0, \\ \cos y, & \text{if } x = 0. \end{cases}$$

Discuss the continuity of f.

INDIAN INSTITUTE OF TECHNOLOGY (BHU) VARANASI
Even Semester 2020-2021
Mathematics I (MA 101)
End-semester Examination

Full Marks: 35 Date of Exam: 8 AM, 10-July-2021 Deadline of Submission: 8 AM, 12-July-2021

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---

1. Show that the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{x^n} dx$ exists if and only if $n < m + 1$. (3)

2. Show that the integral $\displaystyle\int_0^1 \frac{\sin \frac{1}{x}}{x^p} dx$ converges absolutely for $0 < p < 1$. (3)

3. Consider the function
$$f(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n},\ n = 1, 2, 3, \dots \\ 0 & \text{if } x = 0. \end{cases}$$
Prove that f is integrable on $[0, 1]$. Evaluate $\int_0^1 f(x)dx$.

(2 + 1)

4. Is the function $f(x, y) = |xy|$ differentiable at $(0, 0)$? Justify your answer.

(1)

5. Evaluate the integral $\iint_R x\ dxdy$, where R is the region $1 \le x(1 - y) \le 2$ and $1 \le xy \le 2$.

(3)

6. Let R be the bounded region in the first octant when the cylinder $x^2 + z^2 = 1$ is cut by the planes $y = 0$, $z = 0$ and $x = y$. Find the volume of the region R by double integration.

(3)

7. Consider the following series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}.$$

Find the radius and interval of convergence of the series. Also, find the sets x-values for which the series converges absolutely and converges conditionally.

(2+1)

8. If p and q are positive real numbers, then find the condition for the convergence of series

$$\frac{2^p}{1^q} + \frac{3^p}{2^q} + \frac{4^p}{3^q} + \cdots.$$

(3)

9. Let f be defined on $[-2, 2]$ by

$$f(x) = \begin{cases} 3x^2 \cos\left(\frac{\pi}{x^2}\right) + 2\pi \sin\left(\frac{\pi}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is Riemann integrable on $[-2, 2]$. Hence, evaluate $\int_{-2}^{2} f(x)dx$.

(2 + 1)

10. Use the transformation $x = \frac{u}{v}$, $y = uv$ where $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

as an integral over an appropriate region G in the uv-plane. Then, evaluate the uv-integral over G. Here R is the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 4$ and the lines $y = x$, $y = 9x$.

(3)

11. Find the extreme values of $x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and $z = x + y$.

(3)

12. Let $f(x) = \begin{cases} \frac{1}{4}(x^2 + y^2)\log(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

    (a) Show that $f_{xy}(x, y) = f_{yx}(x, y)$ at all points (x, y).

    (b) Discuss the continuity of f_{xy} at $(0, 0)$.

(2+2)

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