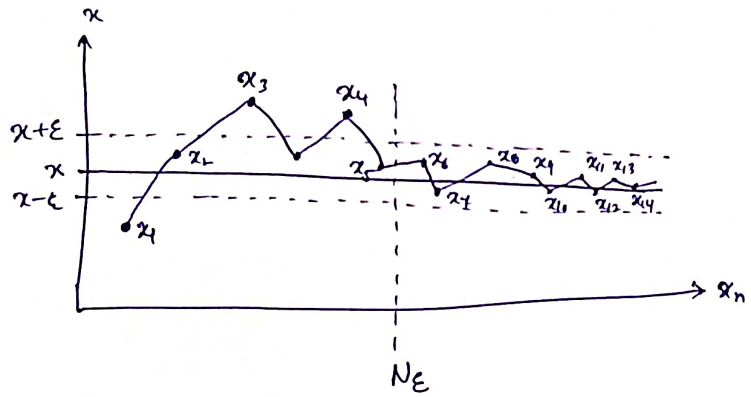
Definition: A sequence (x_n) in $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \quad \forall \; n \geq N_\varepsilon \qquad \text{——} (1)$$

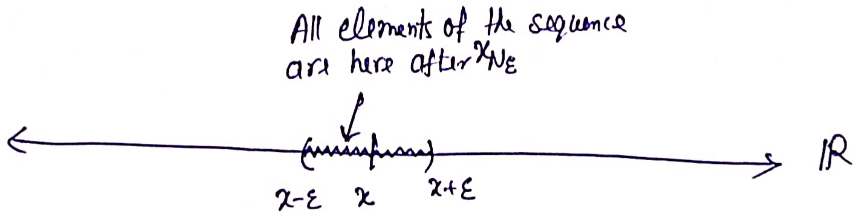
* In this case, x is called limit of (x_n).
* A sequence is called divergent if it is not convergent.
* (1) is equivalent to

$$x_n \in (x-\varepsilon, x+\varepsilon) \quad \forall \; n \geq N_\varepsilon .$$

* We can replace ε by $c\varepsilon$ or ε/c for some $c > 0$.
* **Notation:** $x_n \to x$ as $n \to \infty$ or $\lim (x_n) = x$.



or



* For $a \in \mathbb{R}$, if $0 \leq a < r$ for every $r > 0$, then $a = 0$. $\to$ One property of real numbers.

Proof: This can be proved by contradiction. Suppose $a > 0$, then $0 \leq a < r$ is not satisfied for $r = \frac{a}{2} > 0$, a contradiction.

Theorem: The limit of a sequence is always unique.

Proof: Suppose $x_n \to x$ and $x_n \to y$ as $n \to \infty$. Then, for every $\varepsilon > 0$, there exist N_1 and N_2 such that

$$|x_n - x| < \varepsilon/2 \qquad \forall\, n \geq N_1$$

and

$$|x_n - y| < \varepsilon/2 \qquad \forall\, n \geq N_2$$

Let $N = \max\{N_1, N_2\}$. Consider

$$|x-y| = |x - x_n + x_n - y| \leq |x_n - x| + |x_n - y|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \qquad \forall\, n \geq N.$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude

$$x = y.$$

$*$ $x_n \to x \Leftrightarrow |x_n - x| \to 0$

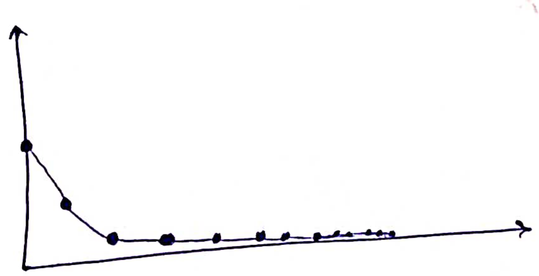
Example: (1) $x_n = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then

$$\left|\frac{1}{n} - 0\right| < \varepsilon \iff n > \frac{1}{\varepsilon}$$

Choose $N_\varepsilon = \underbrace{\left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1}_{\text{(by Archimedean property)}}$, we have

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \qquad \forall\, n \geq N_\varepsilon.$$

Hence, $\frac{1}{n} \to 0$ as $n \to \infty$.



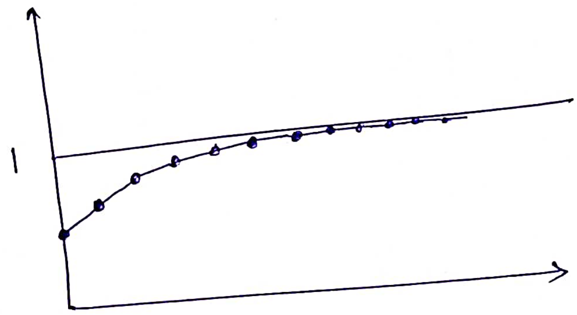
② $x_n = \frac{n}{n+1}$, $\forall\, n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \iff n > \frac{1}{\varepsilon} - 1$$

Choose $N_\varepsilon = \lfloor \frac{1}{\varepsilon} \rfloor$, we have

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon \qquad \forall\, n \geq N_\varepsilon$$

Hence, $x_n = \frac{n}{n+1} \to 1$ as $n \to \infty$.



Exercises: Test the convergence for the following sequences.

① $a_n = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n}$.

② $a_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}$.

③ $a_n = \frac{1}{n^2}$.

④ $a_n = \frac{1}{2^n}$.

⑤ $a_n = (-1)^n$

Definition: A sequence (x_n) is said to be diverges to infinity if for every $M > 0$, there exists $N \in \mathbb{N}$ such that

(However large)

$$x_n > M \qquad \forall\, n \geq N.$$

In this case, we write $x_n \to \infty$.

Definition: A sequence (x_n) is said to be diverges to minus infinity if for every $M < 0$, there exists $N \in \mathbb{N}$ such that
(however small)

$$x_n < M \quad \forall\, n \geq N.$$

In this case, we write $x_n \to -\infty$.

Example: ① $x_n = n \to \infty$ as $n \to \infty$.

② $x_n = -n \to -\infty$ as $n \to \infty$.

Theorem: Suppose $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then the following results hold:

① $x_n \pm y_n \to x \pm y$ as $n \to \infty$.

② $x_n y_n \to xy$ as $n \to \infty$.

③ $c x_n \to cx$ as $n \to \infty$, for any $c \in \mathbb{R}$.

④ If $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

⑤ If $x_n \leq z_n \leq y_n$ for all $n \in \mathbb{N}$, and if $x = y$, then $z_n \to x$ as $n \to \infty$ (Sandwich Theorem)

Example: ① $x_n = \dfrac{\sin(n)}{n} \quad \forall\, n \in \mathbb{N}$.

Note that $-1 \leq \sin(n) \leq 1 \Rightarrow -\dfrac{1}{n} \leq \dfrac{\sin(n)}{n} \leq \dfrac{1}{n}$

$$\underbrace{-\frac{1}{n} \to 0 \qquad \frac{1}{n} \to 0}$$

$$\Downarrow$$

$$\frac{\sin(n)}{n} \to 0 \text{ as } n \to \infty.$$

② $x_n = \dfrac{1}{(n+1)^2} \quad \forall\, n \in \mathbb{N}$.

Note that $0 \leq \left| \dfrac{1}{(n+1)^2} - 0 \right| = \dfrac{1}{(n+1)^2} \leq \dfrac{1}{n}$

$$\underbrace{0 \to 0 \qquad \frac{1}{n} \to 0} \Rightarrow \frac{1}{(n+1)^2} \to 0 \text{ as } n \to \infty.$$

Ratio Test: Suppose $x_n > 0$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L, \quad \text{for some } L \geq 0.$$

Then

① If $L < 1$, then $x_n \to 0$

② If $L > 1$, then $x_n \to \infty$.

Example: ① $x_n = \frac{n}{2^n} \quad \forall\, n \in \mathbb{N}$.

Consider

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} < 1.$$

Hence, $x_n = \frac{n}{2^n} \to 0$ as $n \to \infty$.

Exercises: Estiblish either the convergence or divergence of the following sequences:

① $x_n = \frac{n}{n+1}$

② $x_n = \frac{n^2}{n+1}$

③ $x_n = \frac{(-1)^n n}{n+1}$

④ $x_n = \frac{2n^2+3}{n^2+1}$.

Remark: Ratio test fails when $L = 1$. For example,

① $x_n = \frac{1}{n} \quad \forall\, n \in \mathbb{N}$

Here, $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$

We also know $x_n = \frac{1}{n} \to 0$ as $n \to \infty$.

② $x_n = n \quad \forall\, n \in \mathbb{N}$

Here, $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

We also know $x_n = n \to \infty$ as $n \to \infty$.

In these examples, we get $L=1$ but one is convergent and other is divergent.

Definition: A sequence (x_n) is said to be bounded above if there exists a real number M such that

$$x_n \le M \quad \forall\, n \in \mathbb{N}.$$

In this case, M is called the upper bound for (x_n).

Definition: A sequence (x_n) is said to be bounded below if there exist a real number m such that

$$x_n \ge m \quad \forall\, n \in \mathbb{N}.$$

In this case, m is called the lower bound for (x_n).

Definition: A sequence (x_n) is said to be bounded if it is both bounded above and bounded below.

or

A sequence (x_n) is said to be bounded if there exists $M>0$ such that

$$|x_n| \le M \quad \forall\, n \in \mathbb{N}$$

$$\Downarrow$$

$$-M \le x_n \le M \quad \forall\, n \in \mathbb{N}.$$

Definition: A sequence that is not bounded is called an unbounded sequence.

* Every convergent sequence is bounded. but the converse is not true.

Example: $x_n = (-1)^n$ is bounded but not convergent.

* NOT bounded $\Rightarrow$ not convergent. [This may be used to prove some sequence is not convergent].

Example: $x_n = n \ \forall\, n \in \mathbb{N}$ is not bounded and so, not convergent.

Definition: A sequence (x_n) is said to be monotonically increasing sequence if

$$x_n \leq x_{n+1} \quad \forall n \in \mathbb{N}$$

or

$$x_1 \leq x_2 \leq x_3 \leq x_4 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$$

Definition: A sequence (x_n) is said to be monotonically decreasing sequence if

$$x_n \geq x_{n+1} \quad \forall n \in \mathbb{N}$$

or

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$$

Definition: A sequence (x_n) is said to be monotonic sequence if it is either monotonically increasing or monotonically decreasing.

Definition: A sequence (x_n) is said to be strictly increasing sequence if

$$x_n < x_{n+1} \quad \forall n \in \mathbb{N}$$

or

$$x_1 < x_2 < x_3 < x_4 < \cdots < x_n < x_{n+1} < \cdots$$

Definition: A sequence (x_n) is said to be strictly decreasing sequence if

$$x_n > x_{n+1} \quad \forall n \in \mathbb{N}$$

or

$$x_1 > x_2 > x_3 > x_4 > \cdots > x_n > x_{n+1} > \cdots$$

Examples:

1. $(1, 2, 2, 3, 3, 3, \ldots) \rightarrow$ monotinically increasing
2. $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots) \rightarrow$ monotinically decreasing
3. $(1, 2, 3, 4, \ldots) \rightarrow$ strictly increasing
4. $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \rightarrow$ strictly decreasing.
5. $x_n = (-1)^n \rightarrow$ NOT monoto

Monotone Convergence Theorem: A monotone sequence is convergent if and only if it is bounded. (8)

* Monotonically increasing + bounded above $\Rightarrow$ Convergent
* Monotonically decreasing + bounded below $\Rightarrow$ Convergent.

Examples: (1) $x_n = \frac{1}{n} \quad \forall\, n \in \mathbb{N}$.

Clearly, $x_n \geq 0 \quad \forall\, n \in \mathbb{N}$

Also, $\underbrace{x_{n+1} - x_n}_{\text{(way to check monotone sequence)}} = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{n(n+1)} \leq 0 \Rightarrow x_{n+1} \leq x_n \quad \forall\, n \in \mathbb{N}$

So, (x_n) is monotonically decreasing and bounded below and hence, it is convergent.

(2) $x_1 = 1, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \forall\, n \in \mathbb{N}$. ——— (A)

Soln: Claim: $x_n > \sqrt{2} \quad \forall\, n > 1$.

Note that $x_2 = \frac{1}{2} + 1 = \frac{3}{2} = 1.5 > \sqrt{2} \Rightarrow$ the claim is true for $n = 2$.

Assume the claim is true for $n = k$, that is, $x_k > \sqrt{2}$

or

$x_k = \sqrt{2} + \delta$, for some $\delta > 0$.

Consider

$$x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k} = \frac{\sqrt{2}+\delta}{2} + \frac{1}{\sqrt{2}+\delta}$$

$$= \frac{(\sqrt{2}+\delta)^2 + 2}{2(\sqrt{2}+\delta)} = \frac{4 + \delta^2 + 2\sqrt{2}\delta}{2(\sqrt{2}+\delta)}$$

$$= \sqrt{2}\left(\frac{4 + \delta^2 + 2\sqrt{2}\delta}{4 + 2\sqrt{2}\delta}\right) = \sqrt{2}\left(1 + \frac{\delta^2}{4 + 2\sqrt{2}\delta}\right)$$

$$> \sqrt{2}$$

Hence, by induction on n, $x_n > \sqrt{2} \quad \forall\, n > 1$

Now, consider

$$x_{n+1} - x_n = \left(\frac{x_n}{2} - \frac{1}{x_n}\right) - \left(\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}\right) \quad \left(\rightarrow \text{Sometime this may help not always}\right)$$

$$= \frac{x_n}{2} + \frac{1}{x_n} - x_n = \frac{x_n^2 + 2 - 2x_n^2}{2x_n}$$

$$= \frac{2 - x_n^2}{2x_n} \leq 0$$

$\Rightarrow \quad x_{n+1} \leq x_n \quad \forall\, n \in \mathbb{N} \setminus \{1\}.$

$\Rightarrow \quad (x_n)$ is monotonically decreasing.

Hence, (x_n) is convergent by monotone convergence theorem.

<u>What is the limit?</u>

If $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} x_{n+1} = x$ and $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

So, take limit in Ⓐ, we get

$$x = \frac{x}{2} + \frac{1}{x}$$

$$\Rightarrow 2x^2 = x^2 + 2$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow \boxed{x = \sqrt{2}}$$

So, $x_n \to \sqrt{2}$ as $n \to \infty$.

Existence of limit does not imply that the sequence is convergent

* Do not use the above concept if the sequence is not convergent.

For example, let $x_1 = 1$ and

$$x_{n+1} = -x_n \quad \forall\, n \in \mathbb{N}.$$

$$\Downarrow$$

$$x_n = (-1)^{n+1} \quad \forall\, n \in \mathbb{N}$$

Not convergent

n.t $\quad x_{n+1} = -x_n \Rightarrow x = -x \Rightarrow x = 0$ if $\lim_{n\to\infty} x_n = x$.

3. Let $x_n = x^n$ then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x|<1 \\ 1 & \text{if } x=1 \\ \infty & \text{if } |x|>1 \end{cases}$$

Soln: First, consider $x>0$.

Case I: If $0<x<1$, then $x^n > 0 \ \forall\, n\in\mathbb{N} \Rightarrow (x_n)$ is bounded below.

Also, $x_{n+1} = x^{n+1} = x\cdot x^n < x^n = x_n \Rightarrow (x_n)$ is strictly decreasing.

By monotone convergence theorem, (x_n) is convergent.

Let $\lim_{n\to\infty} x_n = a$. Then

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x^{n+1} = x \lim_{n\to\infty} x^n = ax, \quad 0<x<1.$$

This is true only when $a=0$, that is

$$\lim_{n\to\infty} x^n = 0 \quad \text{if } 0<x<1.$$

Case II: If $x>1$, then $x = 1+\delta$ for some $\delta>0$. [Also, $x_{n+1} = x^{n+1} > x^n = x_n$ (x_n) is strictly increasing]

Note that

$$x_n = x^n = (1+\delta)^n = \sum_{k=0}^{n} \binom{n}{k} \delta^k$$

$$= \binom{n}{0}\delta^0 + \binom{n}{1}\delta^1 + \cdots + \binom{n}{n}\delta^n$$

$$= 1 + n\delta + \frac{1}{2} n(n-1)\delta^2 + \cdots + \delta^n$$

$$> 1 + n\delta.$$

(however large)

Given $M \geq 0$, choose $N\in\mathbb{N}$ such that $N > M/\delta$, we have

$$x_n > M \quad \forall\, n \geq N.$$

$$\Rightarrow x_n \to \infty \text{ as } n\to\infty.$$

Similarly, the proof can be done for $x<0$. }

Consequences! ① $\frac{1}{2^n} \to 0$ as $n\to\infty$

② $\left(\frac{3}{2}\right)^n \to \infty$ as $n\to\infty$.

At $x=0$, it is clear that $\lim_{n\to\infty} x_n = 1$.

* If $x=-1$, then $x_n = (-1)^n$ which is not convergent. [It is called oscillating sequence]

[In general, a sequence which does not converge or diverge]

Exercises: ① Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

② Test for the convergence for the following sequences

ⓐ $x_1 = 2, \quad x_{n+1} = 2 + \frac{1}{x_n}, \quad n\in\mathbb{N}$.

ⓑ $x_1 = 1, \quad x_{n+1} = \frac{1}{4}(2x_n+3), \quad n\in\mathbb{N}$.

ⓒ $x_1 = 1, \quad x_{n+1} = \sqrt{2x_n}, \quad n\in\mathbb{N}$.

* A montonically increasing sequence either converges or diverges to $+\infty$.

* A monotonically decreasing sequence either converges or diverges to $-\infty$.

Subsequences: Let (n_k) be a strictly increasing sequence of natural numbers. Then (x_{n_k}) is called a subsequence of the sequence (x_n).

$(x_{n_k}) = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$

Example! ① $x_n = (-1)^n \quad \forall\, n\in\mathbb{N}$.

$x_{2n} = (1,1,1,\ldots) \to$ a subsequence of (x_n)

$x_{2n+1} = (-1,-1,\ldots) \to$ a subsequence of (x_n).

② $x_n = \frac{1}{n} \quad \forall\, n\in\mathbb{N}$

$x_{n^2} = \frac{1}{n^2} \to$ a subsequence of x_n.

Theorem: If a sequence (x_n) converges to x then all subsequences are also converges to x.

* A sequence (x_n) converges to $x \Leftrightarrow$ every subsequence of (x_n) converges to x.

* If (x_n) is such that $x_{2n}\to x$ and $x_{2n+1}\to x$, for some $x\in\mathbb{R}$ then $x_n \to x$.

* (x_{2n}) and (x_{2n+1}) are called complementary pair of sequences.

Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has at least one convergent subsequence.

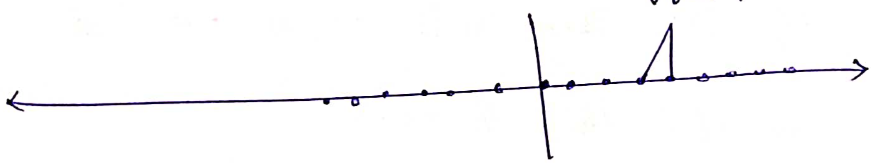
Example! $x_n = (-1)^n \quad \forall\, n \in \mathbb{N}$

$$x_{2n} = (1, 1, 1, \dots) \rightarrow \text{convergent sequence.}$$

Cauchy Sequence: A sequence (x_n) is said to be cauchy if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall\, n, m \in N_\varepsilon.$$

(difference between any two points is less than ε)



N_ε

Example! $x_n = \frac{1}{n} \quad \forall\, n \in \mathbb{N}$

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, n, m > \left[\frac{2}{\varepsilon}\right] + 1 = N_\varepsilon.$$

Hence, $\left(\frac{1}{n}\right)$ is a cauchy sequence.

Lemma: A cauchy sequence of real numbers is bounded.

Proof: Let (x_n) be a cauchy sequence. Let $\varepsilon = 1$. Then $\exists\, N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < 1 \qquad \forall\, n, m \geq N_1$$

In particular, $|x_n - x_{N_1}| < 1 \quad \forall\, n \geq N_1$

$$\Rightarrow |x_n| \leq |x_{N_1}| + 1 \quad \forall\, n \geq N_1$$

or

$$|x_n| = |x_n - x_{N_1} + x_{N_1}| \leq |x_n - x_{N_1}| + |x_{N_1}| \leq |x_{N_1}| + 1.$$

Take

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N_1 - 1}|, |x_{N_1}|+1\}$$

then it follows that

$$|x_n| \leq M \quad \forall\, n \in \mathbb{N}.$$

Hence, (x_n) is bounded.

<u>Cauchy Convergence Criterion</u>: A sequence of real numbers is convergent if and only if it is Cauchy.

<u>Proof</u>: Let (x_n) be a convergent sequence and $\lim_{n\to\infty} x_n = x$.

By definition, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2 \quad \forall\, n \geq N.$$

Consider

$$\begin{aligned} |x_n - x_m| &= |x_n - x + x - x_m| \\ &= |x_n - x| + |x_m - x| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, n \geq N. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a cauchy sequence.

Conversely, let (x_n) is a Cauchy sequence. By previous lemma, (x_n) is bounded. By the Bolzano-Weierstrass Theorem, there is a subsequence (x_{n_k}) which converge to some real number, say, x^*.

Now, since (x_n) is cauchy. For every $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon/2 \quad \forall\, n, m \geq N_1$$

In particular,

$$|x_n - x_{n_k}| < \varepsilon/2 \quad \forall \; n, n_k \geq N_1$$

Next, since (x_{n_k}) converges to x^*. For every $\varepsilon > 0$, $\exists$ $N_2 \in \mathbb{N}$ such that

$$|x_{n_k} - x^*| < \varepsilon/2 \quad \forall \; n_k \geq N_2.$$

Let $N := \max\{N_1, N_2\}$ then

$$\begin{aligned}
|x_n - x^*| &= |x_n - x_{n_k} + x_{n_k} - x^*| \\
&\leq |x_n - x_{n_k}| + |x_{n_k} - x^*| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall \; n \geq N.
\end{aligned}$$

Hence, $x_n \to x^*$ as $n \to \infty$.

This proves the result.

Example: $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \forall\, n \in \mathbb{N}.$

Note that

$$|S_{2n} - S_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geq \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

This implies (S_n) is not Cauchy and hence, not convergent.

Definition: A sequence (x_n) is said to be contractive if there exist a constant r, $0 < r < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq r\, |x_{n+1} - x_n| \quad \forall\, n \in \mathbb{N}.$$

The number r is called the constant of contractive sequence.

Theorem: Every contractive sequence is a Cauchy sequence, and therefore is convergent.

Proof: Note that

$$|x_{n+2} - x_{n+1}| \le \gamma\,|x_{n+1} - x_n| \le \gamma^2\,|x_n - x_{n-1}|$$

$$\vdots$$

$$\le \gamma^n\,|x_2 - x_1|.$$

For $m > n$, consider

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \cdots + x_{n+1} - x_n|$$

$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$\le \left(\gamma^{m-2} + \gamma^{m-1} + \cdots + \gamma^{n-1}\right)|x_2 - x_1|$$

$$= \frac{\gamma^{n-1}(1 - \gamma^{m-n})}{1-\gamma}\,|x_2 - x_1|$$

$$\le \frac{\gamma^{n-1}}{1-\gamma}\,|x_2 - x_1|.$$

Since $0 < \gamma < 1$, we know that $\gamma^n \to 0$ as $n \to \infty$. Therefore,

$$|x_m - x_n| \to 0 \quad \text{as } n \to \infty.$$

$\Rightarrow$ (x_n) is a Cauchy sequence.

Example: $x_{n+1} = \frac{1}{7}(x_n^3 + 2)$ $\forall\, n \in \mathbb{N}$. Here $0 < x_1 < 1$.

Soln: Since $0 < x_1 < 1$, we have.

$0 < x_n < 1$ $\rightarrow$ Can be easily proved by induction on n.

Consider

$$|x_{n+2} - x_{n+1}| = \frac{1}{7}\left|x_{n+1}^3 + 2 - x_n^3 - 2\right| = \frac{1}{7}\left|x_{n+1}^3 - x_n^3\right|$$

$$= \frac{1}{7}\left|x_{n+1}^2 + x_n^2 + 2x_{n+1}x_n\right|\,|x_{n+1} - x_n| \le \frac{3}{7}\,|x_{n+1} - x_n|.$$

Therefore, (x_n) is contractive and hence convergent.

Let $\lim_{n\to\infty} x_n = x$. Then

$$x = \frac{1}{7}(x^3+2) \Rightarrow x^3 - 7x + 2 = 0 \rightarrow$$ solution of this equation lies between 0 and 1 is the limit of the sequence (x_n).

Some more Definitions and Results:

① A sequence (x_n) is said to be constant if

$$x_{n+1} = x_n \quad \forall\, n \in \mathbb{N}.$$

② A sequence (x_n) is said to be eventually constant if there exists $K \in \mathbb{N}$ such that

$$x_n = x_K \quad \forall\, n \geq K.$$

③ A sequence (x_n) is said to be alternating sequence if x_n changes sign alternately, that is, $x_n x_{n+1} < 0 \quad \forall\, n \in \mathbb{N}$.

Example: $(-1)^n$ and $\frac{(-1)^n}{n}$ are alternating sequence.

Theorem: Let (x_n) be a sequence such that

$$|x_{n+1} - x| \leq r\,|x_n - x| \quad \forall\, n \in \mathbb{N}, \text{ for some } r,\ 0 < r < 1,$$

Then $x_n \to x$ as $n \to \infty$.

Theorem: Suppose $x_n \to x$ and $y_n \to y$ as $n \to \infty$, $y_n \neq 0$ $\forall n \in \mathbb{N}$, $y \neq 0$. Then

$$\frac{1}{y_n} \to \frac{1}{y} \quad \text{and} \quad \frac{x_n}{y_n} \to \frac{x}{y}$$

Theorem: Every sequence of real numbers has a monotone subsequence.

Theorem: Let (x_n) be a sequence and $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim_{n\to\infty} a_n = 0$ and if for some constant $r > 0$ and $N \in \mathbb{N}$, we have

$$|x_n - x| < r a_n \quad \forall n \geq N$$

then it follows that $\lim_{n\to\infty} x_n = x$.

Theorem: If (x_n) is a convergent sequence and if $x_n \geq 0$ $\forall n \in \mathbb{N}$, then

$$x = \lim_{n\to\infty} x_n \geq 0.$$

Theorem: ① If (x_n) is unbounded and monotonically increasing, then $\lim(x_n) = +\infty$
② If (x_n) is unbounded and monotonically decreasing, then $\lim(x_n) = -\infty$.

Theorem: If (x_n) and (y_n) are convergent sequences and if $x_n \leq y_n$ $\forall n \in \mathbb{N}$ then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Theorem: If (x_n) is a convergent sequence and if $a \leq x_n \leq b$ $\forall n \in \mathbb{N}$ then

$$a \leq \lim_{n\to\infty} x_n \leq b.$$

Theorem: If the sequence (x_n) converges to x then the sequence $|x_n|$ converges to $|x|$.

Theorem: If the sequence (x_n) converges to x and $x_n \geq 0$, then $\sqrt{x_n}$ of positive square roots converges to $\sqrt{x}$.

Theorem: ① If (x_n) is bounded and increasing then $\lim_{n\to\infty}(x_n) = \sup\{x_n : n \in \mathbb{N}\}$
② If (x_n) is bounded and decreasing then $\lim_{n\to\infty}(x_n) = \inf\{x_n : n \in \mathbb{N}\}$.

Exercises: ① For $n \in \mathbb{N}$, let $x_n = \sqrt{n+1} - \sqrt{n}$. Show that (x_n) and $(\sqrt{n}\, x_n)$ are convergent sequences. Find their limits.

② Check the converges of the following sequences. Find the limit, if convergence hold:

ⓐ $x_n = (n!)^{\frac{1}{n}} \quad \forall\, n \in \mathbb{N}$

ⓑ $x_n = (n!)^{\frac{1}{n^2}} \quad \forall\, n \in \mathbb{N}$.

ⓒ $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}$.

ⓓ $x_1 = 1, \quad x_{n+1} = 1 + \frac{1}{x_n} \quad \forall\, n \in \mathbb{N}$.

ⓔ $x_1 = 1, \quad x_{n+1} = \frac{2x_n + 3}{4}, \quad \forall\, n \in \mathbb{N}$

ⓕ $x_1 = 1, \quad x_{n+1} = \frac{x_n}{1 + x_n} \quad \forall\, n \in \mathbb{N}$.

ⓖ $x_1 = 0, \quad x_{n+1} = x_n + \frac{1}{4}(1 - 2x_n^2) \quad \forall\, n \in \mathbb{N}$.

Definition: The limit inferior of a sequence (x_n) is defined by

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \left(\inf_{k \geq n} x_k \right).$$

or

$$\alpha_1 = \inf\{x_1, x_2, x_3, \ldots\}$$
$$\alpha_2 = \inf\{x_2, x_3, x_4, \ldots\}$$
$$\alpha_3 = \inf\{x_3, x_4, x_5, \ldots\}$$
$$\vdots$$
$$\alpha_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$
$$\vdots$$

Then $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n \leq \cdots$ and

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right).$$

Definition: The limit superior of (x_n) is defined by

$$\limsup_{n\to\infty}(x_n) = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right).$$

or

$$\beta_1 = \sup\{x_1, x_2, x_3, \ldots\}$$
$$\beta_2 = \sup\{x_2, x_3, x_4, \ldots\}$$
$$\beta_3 = \sup\{x_3, x_4, x_5, \ldots\}$$
$$\vdots$$
$$\beta_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$
$$\vdots$$

Then $\beta_1 \geq \beta_2 \geq \beta_3 \geq \ldots \geq \beta_n \geq \ldots$ and

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right).$$

Theorem: A bounded sequence (x_n) is convergent if and only if
$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Theorem! If $x_n \to x$ then $\liminf\limits_{n\to\infty} x_n = \limsup\limits_{n\to\infty} x_n = x.$

Example: Find $\liminf\limits_{n\to\infty} x_n$ and $\limsup\limits_{n\to\infty} x_n$, where $x_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$.

Soln: Note that

$$\alpha_1 = \inf\{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} = 0$$
$$\alpha_2 = \inf\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} = 0$$
$$\alpha_3 = \inf\{\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots\} = 0$$
$$\vdots$$
$$\alpha_n = \inf\{\tfrac{1}{n}, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \ldots\} = 0$$
$$\vdots$$
$$\Downarrow$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \alpha_n = 0$$

Also, note that

$$\beta_1 = \sup\{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} = 1$$
$$\beta_2 = \sup\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} = \tfrac{1}{2}$$
$$\beta_3 = \sup\{\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots\} = \tfrac{1}{3}$$
$$\vdots$$
$$\beta_n = \sup\{\tfrac{1}{n}, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \ldots\} = \tfrac{1}{n}$$
$$\vdots$$
$$\Downarrow$$
$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty}\left(\tfrac{1}{n}\right) = 0.$$

Series: Let (x_n) be a sequence of real numbers. A series of real numbers is an expression of the form

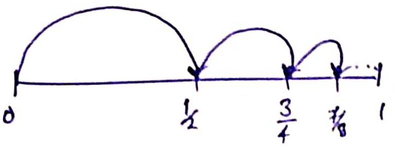
$$x_1 + x_2 + x_3 + \cdots,$$

or more compactly,

$$\sum_{n=1}^{\infty} x_n.$$

* x_n is called n^{th} term of the series.

Example: Moving on a road of length 1 k.m.



→ moving half distance of the remaining distance at a time.

Then,

$$\boxed{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1}$$

Definition: The sum of first n terms of (x_n) is called the n^{th} partial sum of the series. and its sequence is called sequence of partial sums.

That is,

$$S_n = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x_i \rightarrow n^{th} \text{ partial sums.}$$

* $(S_n) \rightarrow$ sequence of partial sums.
* Note that

$$\text{Sequence of partial sums.} \left\{ \begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ S_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ S_n &= x_1 + x_2 + x_3 + \cdots + x_n \\ &\vdots \end{aligned} \right.$$

* Also, observe that $S_n = S_{n-1} + x_n$.

Definition: A series $\sum_{n=1}^{\infty} x_n$ is said to be a convergent series if the corresponding sequence (S_n) of partial sums converges.

* If $S_n \to s$ then we say that $\sum_{n=1}^{\infty} x_n$ converges to s, and s is called the sum of the series, and we write

$$s = \sum_{n=1}^{\infty} x_n.$$

* A series which does not converge is called divergent series.

Example: ① Consider the geometric series

$$1 + x + x^2 + \cdots$$

The above series converges if and only if $|x| < 1$.

Soln: The sequence of partial sums is

$$S_n = 1 + x + x^2 + \cdots + x^{n-1} = \begin{cases} n & \text{if } x = 1 \\ \dfrac{1-x^n}{1-x} & \text{if } x \neq 1. \end{cases}$$

If $x = 1$, then (S_n) is monotonically increasing and unbounded, and hence not convergent.

Also, if $x = -1$, then

$$\frac{1-(-1)^n}{2} = S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

which is clearly not convergent.

Next, let $|x| \neq 1$. Consider

$$\left| S_n - \frac{1}{1-x} \right| = \left| \frac{1-x^n}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^n}{|1-x|} \to 0 \text{ as } n \to \infty, \text{ if } |x| < 1.$$

[redacted]

$$\sum_{n=1}^{\infty} x^{n-1} = \begin{cases} \dfrac{1}{1-x} & \text{if } |x| < 1 \\ \infty & \text{if } x > 1 \\ \infty & \text{if } x = 1 \\ \text{does not converge} & \text{if } x \leq -1. \end{cases}$$

(sequence of partial sums is monotonically increasing but not bounded above).

[redacted]

② Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Soln: We know

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not Cauchy, and therefore not convergent. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

In fact, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ $\rightarrow$ show that (S_n) is monotonically increasing and not bounded above.

Necessary Condition: If $\sum_{n=1}^{\infty} x_n$ converges then $x_n \to 0$ as $n \to \infty$. The converse is not true.

Proof: Let $S_n = \sum_{i=1}^{n} x_i$ and $S_n \to S$ as $n \to \infty$. Then

$$x_n = S_n - S_{n-1} \to S - S = 0 \quad \text{as } n \to \infty.$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, how-ever,

$$\frac{1}{n^2} \to 0 \text{ and } \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Example: $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges $\left(\because \frac{n}{n+1} \to 1 \neq 0 \text{ as } n \to \infty\right)$.

Exercises: ① Check the convergence of the following series:

ⓐ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ ⓑ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Euler's Series) ⓒ $\sum_{n=1}^{\infty} \frac{3}{10^n}$ ⓓ $\sum_{n=1}^{\infty} 2^n$

② Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $0 < p \leq 1$.

Comparison Test: Let (x_n) and (y_n) be real sequences such that

$$0 \le x_n \le y_n \quad \text{for } n \geq N, \text{ for some } N \in \mathbb{N}.$$

Then

① $\sum_{n=1}^{\infty} y_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ converges

② $\sum_{n=1}^{\infty} x_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} y_n$ diverges.

Example: Show that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Soln: Note that

$$n! = n(n-1)(n-2)\cdots 3.2.1$$
$$\geq 2.2.2.\cdots 2.2.1$$
$$= 2^{n-1}$$

$$\Rightarrow \frac{1}{n!} \le \frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}.$$

We know that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ converges $\left[\because \sum_{n=1}^{\infty} x^n \text{ converges for } |x|<1\right]$

Hence, by Comparison test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Example: Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

Soln: Note that

$$\frac{1}{\sqrt{n+1}} \geq \frac{1}{n} \quad \forall n \geq 2.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and therefore, by comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

Cauchy Criterion for Series: The series $\sum_{n=1}^{\infty} x_n$ converges if and only if for every ε there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon \quad \forall m > n \geq N_\varepsilon.$$

Limit Comparison Test I: Suppose that (x_n) and (y_n) are strictly positive sequences and

$$L = \lim_{n\to\infty} \frac{x_n}{y_n} \quad \text{exists.}$$

① If $L > 0$ then $\sum_{n=1}^{\infty} x_n$ Converges $\Leftrightarrow$ $\sum_{n=1}^{\infty} y_n$ Converges

② If $L = 0$ and if $\sum_{n=1}^{\infty} y_n$ Converges $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ Converges.

* If $L = \infty$ then $\sum_{n=1}^{\infty} x_n$ Converges $\Rightarrow$ $\sum_{n=1}^{\infty} y_n$ Converges.

Examples: ① Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ Converges.

Soln: Let $x_n = \frac{1}{n^2+n}$ and $y_n = \frac{1}{n^2}$ then

$$L = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1/n^2+n}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ Converges.

② Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

Soln: Let $x_n = \frac{1}{\sqrt{n+1}}$ and $y_n = \frac{1}{\sqrt{n}}$ then

$$L = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1. > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, by limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

Exercises: Check the convergence of the following series:

① $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ ② $\sum_{n=1}^{\infty} \cos(n)$ ③ $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

④ $\sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$ ⑤ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)}$ ⑥ $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n$.

D' Alembert's ratio Test: Suppose (x_n) is a sequence of positive terms such that

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \quad \text{exists.}$$

Then,

① If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges

② If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

Cauchy's Root Test: Suppose (x_n) is a sequence of positive terms such that

$$L = \lim_{n\to\infty} (x_n)^{\frac{1}{n}} \quad \text{exists.}$$

Then,

① If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges

② If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

* Both the above tests are silent for the case $L = 1$. In this case, we can not assert either way. For example, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, however, $\frac{x_{n+1}}{x_n} \to 1$ and $(x_n)^{1/n} \to 1$, for both cases.

Examples: ① Test for the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Soln: Here, $x_n = \frac{n^2}{2^{n+1}}$. Consider

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+2}} \cdot \frac{2^{n+1}}{n^2} = \frac{1}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1$$

By Ratio test, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

② Test for convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$.

Soln: Here, $x_n = \left(\frac{n}{2n+1}\right)^n$. Consider

$$L = \lim_{n\to\infty} (x_n)^{1/n} = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1.$$

By Root Test, $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ converges.

Exercises: Test for the convergence of the following series:

① $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ ② $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ ③ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$.

Definition: A series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$, where (x_n) is a sequence of positive terms, is called alternating series.

Leibnitz's Test: Suppose (x_n) is a sequence of positive terms such that

① $x_n \geq x_{n+1}$ $\forall\, n \in \mathbb{N}$.

② $x_n \to 0$ as $n \to \infty$.

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} x_n$ converges.

Example: Test for the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Soln: Let $x_n = \frac{1}{n}$ then

① $x_n \geq x_{n+1}$ $\forall\, n \in \mathbb{N}$

② $x_n \to 0$ as $n \to \infty$

By Leibnitz's test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges and hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Absolute Convergence: A series $\sum_{n=1}^{\infty} x_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |x_n|$ is convergent.

* Every absolutely convergent series is convergent. That is,

$$\sum_{n=1}^{\infty} |x_n| \text{ Converges} \Rightarrow \sum_{n=1}^{\infty} x_n \text{ Converges.}$$

Definition: A series $\sum_{n=1}^{\infty} x_n$ is said to be conditionally convergent if $\sum_{n=1}^{\infty} x_n$ convergent but $\sum_{n=1}^{\infty} |x_n|$ is not convergent.

Examples: ① $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \rightarrow$ absolutely convergent

② $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2} \rightarrow$ absolutely convergent

③ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow$ conditionally convergent.

Grouping of Series: Given a series $\sum_{n=1}^{\infty} x_n$, we can construct many other series $\sum_{k=1}^{\infty} y_k$ by leaving the order of the terms x_n fixed, but inserting parentheses that group together finite number of terms. For example.

$$\underbrace{1}_{1^{st}\text{ term}} - \underbrace{\tfrac{1}{2}}_{2^{nd}\text{ term}} + \underbrace{\left(\tfrac{1}{3}-\tfrac{1}{4}\right)}_{3^{rd}\text{ term}} + \underbrace{\left(\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}\right)}_{4^{th}\text{ term}} - \underbrace{\tfrac{1}{8}}_{5^{th}\text{ term}} + \underbrace{\left(\tfrac{1}{9}-\cdots+\tfrac{1}{13}\right)}_{6^{th}\text{ term}} - \cdots$$

Theorem: If a series $\sum_{n=1}^{\infty} x_n$ is convergent, then any series obtained from it by grouping the terms is also convergent and to the same value.

<u>Rearrangements of Series:</u> A rearrangement of the series is another series that is obtained from the given one by using all of the terms exactly once, but scrambling the order in which the terms are taken.

$$1 + \frac{1}{2} - \frac{1}{3} + \underbrace{\frac{1}{4} + \frac{1}{6} - \frac{1}{5}} + \underbrace{\frac{1}{8} + \frac{1}{9} - \frac{1}{7}} + \cdots$$

<u>Rearrangement Theorem:</u> Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series in $\mathbb{R}$. Then any rearrangement $\sum_{k=1}^{\infty} y_k$ of $\sum_{n=1}^{\infty} x_n$ converges to the same value.

* The above result may not be hold if we relax the condition of absolute convergence. For example.

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

ⓐ $$S = \underbrace{1 - \frac{1}{2} + \frac{1}{3}}_{=5/6} \underbrace{- \frac{1}{4} + \frac{1}{5}}_{(<0)} \underbrace{- \frac{1}{6} + \frac{1}{7}}_{(<0)} - \cdots \quad < \frac{5}{6}$$

ⓑ $$S = \underbrace{1 + \frac{1}{3} - \frac{1}{2}}_{=5/6} + \underbrace{\frac{1}{5} + \frac{1}{7} - \frac{1}{4}}_{(>0)} + \underbrace{\frac{1}{9} + \frac{1}{11} - \frac{1}{6}}_{(>0)} + \cdots \quad > 5/6$$

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0.$$

<u>Comparison Test II:</u> Suppose (x_n) and (y_n) are non-zero real sequences and

$$L = \lim_{n \to \infty} \left| \frac{x_n}{y_n} \right| \text{ exists. in } \mathbb{R}.$$

Then,

① If $L \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} y_n$ is absolutely convergent

② If $L = 0$, then $\sum_{n=1}^{\infty} y_n$ is absolutely convergent $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

Ratio Test: Let (x_n) be a non-zero sequence in $\mathbb{R}$ and
$$L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| \text{ exists in } \mathbb{R}.$$

Then,

① If $L<1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

② If $L>1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Root Test: Let (x_n) be a sequence in $\mathbb{R}$ and
$$L = \lim_{n\to\infty} |x_n|^{\frac{1}{n}} \quad \text{exists in } \mathbb{R}.$$

Then

① If $L<1$, then $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

② If $L>1$, then $\sum_{n=1}^{\infty} x_n$ is divergent.

Abel's Test: If (x_n) is a convergent monotone sequence and the series $\sum_{n=1}^{\infty} y_n$ is convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is also convergent.

Dirichlet's Test: If (x_n) is a decreasing sequence with $\lim_{n\to\infty} x_n = 0$, and if the partial sums (S_n) of $\sum_{n=1}^{\infty} y_n$ are bounded, then the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent.

(and absolutely convergence).

Exercises: Test for the convergence of the following series:

① $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ ② $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+2}$ ③ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n)}{n}$

④ $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{(n+1)^{n+1}}$ ⑤ $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ ⑥ $\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^n}{n^n}$.

⑦ $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ ⑧ $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ ⑨ $\sum_{n=1}^{\infty} \frac{n!}{e^{+n^2}}$

⑩ $\sum_{n=1}^{\infty} \frac{n^n}{e^n}$ ⑪ $\sum_{n=1}^{\infty} e^{-\ln(n)}$ ⑫ $\sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}\right)$

Power Series

Notation: $\mathbb{N}_0 = \mathbb{N} \cup \{0\} \rightarrow$ the set of all non-negative integers.

Definition: Given a sequence $(a_n)_{n\in\mathbb{N}_0}$ of real numbers and a point $x_0 \in \mathbb{R}$, a series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

or

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

is called the power series around the point x_0.

* In particular, if $x_0 = 0$ then $\sum_{n=0}^{\infty} a_n x^n$ is called the power series around the point 0 (or origin).

* Note that $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at $x = x_0$.

Abel's Theorem: If the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at a point $a \neq x_0$, then it converges absolutely at every x with $|x-x_0| < |x_0 - a|$.

Corollary: If the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ diverges at a point $b \neq x_0$, then it diverges at every x with $|x-x_0| > |x_0 - b|$.

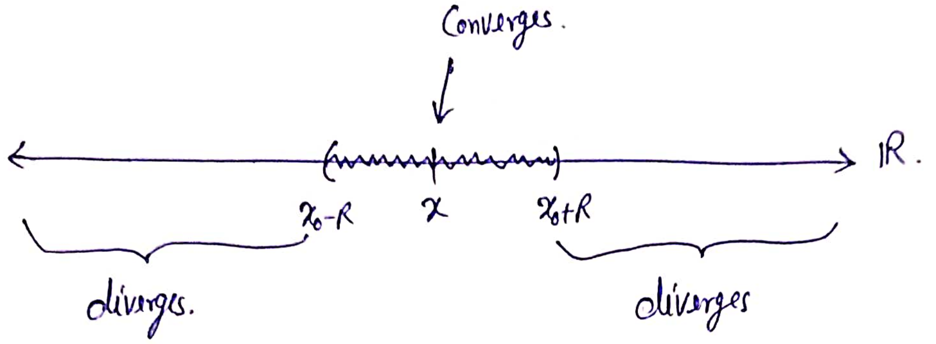
Example: Suppose $\sum_{n=0}^{\infty} a_n (x-2)^n$ converges at $x=1$. Can we say the series also converges at $x = 4$?

Soln: By Abel's theorem, the series converges at every x with $|x-2| < |2-1|$ or $1 < x < 3$. So, we can not conclude the convergence of the given power series at $x = 4$.

Definition: A number $R \geq 0$ is called the radius of convergence for the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ if

① the series converges for $|x-x_0|<R$.

② the series diverges for $|x-x_0|>R$.



*If $|x-x_0|=R$ then the power series may converge or diverge. So, we have check it manually.

Definition: The domain of convergence of a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is the set

$$D = \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ converges at } x \right\}.$$

* By Abel's theorem, the domain of a power series is either the singleton set $\{x_0\}$ or an interval.

* $R=0 \iff$ the power series converges only at $x=x_0$. ($D=\{x_0\}$)

$R=\infty \iff$ the power series converges at every $x \in \mathbb{R}$. ($D=\mathbb{R}$).

$0<R<\infty \iff$ $|x-x_0|<R \rightarrow$ converges, $|x-x_0|>R \rightarrow$ diverges $\left(D = (x_0-R, x_0+R) \text{ or } [x_0-R, x_0+R] \text{ or } (x_0-R, x_0+R] \text{ or } [x_0-R, x_0+R) \right)$

Theorem: Consider the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$, then

① $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

② $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$.

Proof can be done by ratio test and root test.

Examples: ① $\sum_{n=1}^{\infty} \frac{x^n}{n} \rightarrow$ Find domain of convergence.

Soln: Here $a_n = \frac{1}{n}$. So,

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1. \Rightarrow R = 1.$$

Therefore the radius of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is 1. So, the power series converges for $|x|<1$ and diverges for $|x|>1$.

Note that at $x=1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and, at $x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Hence, the domain of convergence is $[-1, 1)$. or $\{x \in \mathbb{R} : -1 \leq x < 1\}$.

② Find the domain of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Soln: Here, $a_n = \frac{1}{n!}$. So,

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\Rightarrow R = \infty.$$

So, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for every $x \in \mathbb{R}$.

The domain of convergence $= \mathbb{R}$.

③ Find the domain of the power series $\sum_{n=0}^{\infty} n!\, x^n$.

Soln: Here, $a_n = n!$. So,

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}(n+1) = \infty \Rightarrow R=0.$$

Hence, the domain of convergence is $\{0\}$.

④ Find the domain of convergence of the power series $\sum_{n=0}^{\infty} \frac{1}{(n+1)x^n}$.

Soln: Let $y = \frac{1}{x}$ then the power series reduces to the form.

$$\sum_{n=0}^{\infty} \frac{y^n}{(n+1)}.$$

So,
$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{1/n+2}{1/n+1}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right) = 1. \Rightarrow R=1.$$

$\Rightarrow$ The power series converges for $|y|<1 \Rightarrow \left|\frac{1}{x}\right|<1 \Rightarrow |x|>1$.

At, $x=1$, $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

At $x=-1$, $\sum_{n=0}^{\infty} \frac{1}{(n+1)(-1)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges.

Hence, the domain of convergence $= (-\infty,-1] \cup (1,\infty)$.

Exercises: ⓐ Find the domain of converges of the following power series:

① $\sum_{n=1}^{\infty} \frac{(-1)^n}{n x^n}$ ② $\sum_{n=1}^{\infty} \frac{n(x+5)^n}{(2n+1)^3}$ ③ $\sum_{n=1}^{\infty} \frac{2^n \sin^n x}{n^2}$

④ $\sum_{n=1}^{\infty} \frac{x^n}{n4^n}$ ⑤ $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$ ⑥ $\sum_{n=1}^{\infty} x^n$.

ⓑ If R is the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$, then show that the radius of convergence of $\sum n a_n x^{n-1}$ is also R.

Theorem: Let $R>0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ and let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad |x-x_0| < R.$$

Then f is infinitely differentiable on (x_0-R, x_0+R) and

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad \forall\, n \in \mathbb{N}.$$

Definition: If f is infinitely differentiable in a neighbourhood of a point x_0, and if the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

has positive radius of convergence, then this series is called the Taylor series of f around the point x_0.

* If $x_0 = 0$, then Taylor series of f is called the Maclaurin series of f.

Example: Find the Maclaurin series of $\cos x$.

Soln: Let $f(x) = \cos x$. Then

$$f(0) = \cos(0) = 1.$$

$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$

$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$

$$f^{(3)}(x) = \sin x \Rightarrow f^{(3)}(x) = 0$$

$$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(x) = 1$$

$$f^{(5)}(x) = -\sin x \Rightarrow f^{(5)}(0) = 0$$

$$f^{(6)}(x) = -\cos x \Rightarrow f^{(6)}(0) = -1$$

$$\vdots$$

Therefore, the Maclaurin series of $\cos x$ is given by

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n$$

$$= f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

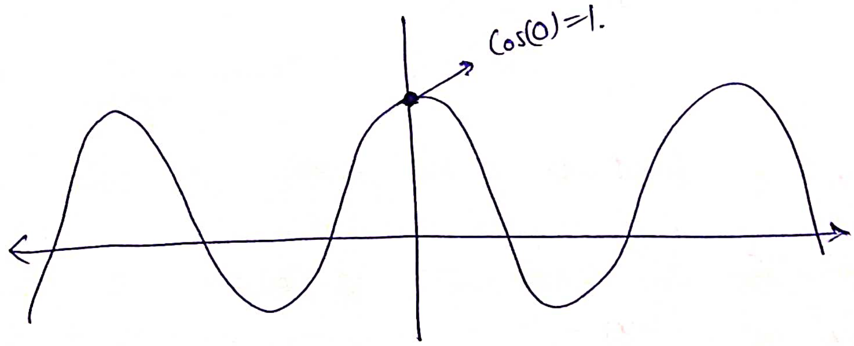
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

* Note that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n = \underbrace{f(0)}_{\substack{\text{Control on} \\ \text{function value} \\ \text{at } 0}} + \underbrace{f'(0)\frac{x}{1!}}_{\substack{\text{Control on} \\ \text{derivative at } 0}} + \underbrace{f''(0)\frac{x^2}{2!}}_{\substack{\text{Control on} \\ 2^{nd} \text{ derivative} \\ \text{at } 0}} + \cdots$$

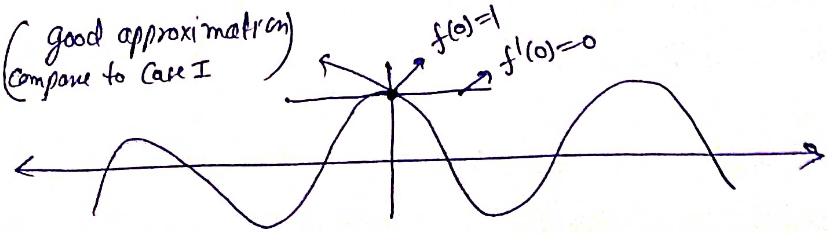
For example, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Case I: $\cos x \approx 1$ $\quad (f(0) = 1)$.

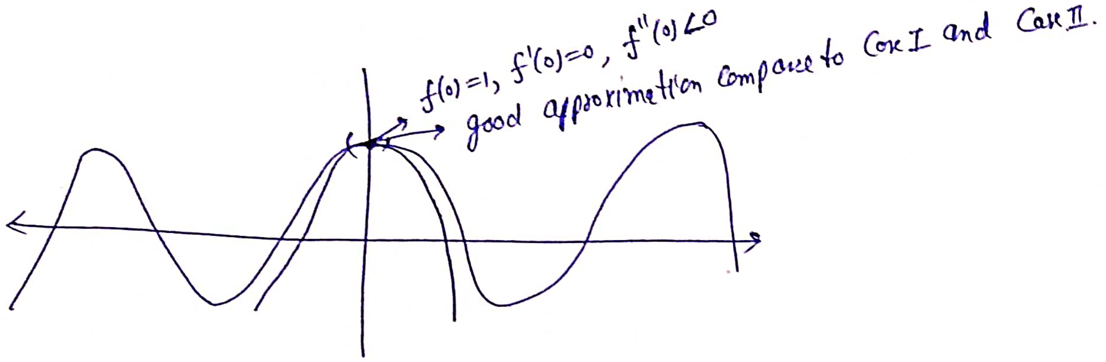


Case II: $\cos x \approx 1$ $\quad (f(0) = 1,\ f'(0) = 0)$

(good approximation compare to Case I)



Case III: $\cos x \approx 1-\frac{x^2}{2!}$ $\left(f(0)=1,\ f'(0)=0,\ f''(0)=-1<0 \right)$.



Similarly, if we take more powers of x^n, the approximation become good in a neighbourhood of 0.

* In the similar manner, Taylor series is used to get a get a good approximation of the function value in a neighbourhood of x_0.

Example: Find the Taylor series of e^x around $x=1$.

Soln: Let $f(x)=e^x$. Then

$$f(1) = e = f'(1) = f''(1) = f^{(3)}(1) = \cdots$$

Therefore, Taylor series of e^x around $x=1$ is

$$e^x = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n = f(1) + f'(1)\frac{(x-1)}{1!} + f''(1)\frac{(x-1)^2}{2!} + f^{(3)}(1)\frac{(x-1)^3}{3!} + \cdots$$

$$= e + e(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + \cdots$$

$$= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right].$$

Another way to define Taylor Series:

Taylor's Formula: Suppose f is defined and has derivatives $f'(x), \ldots, f^{(n+1)}(x)$ in $I \subseteq \mathbb{R}$ and $x_0 \in I$. Then, for every $x \in I$, there exists ξ_x between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)\frac{(x-x_0)}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + R_n(x),$$

where $R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x) \longrightarrow$ Remainder term.

If $f(x)$ can be expressed as an infinite series for all x in a neighbourhood of x_0, then such a series is called the Taylor series around the point x_0.

Theorem: If f is infinitely differentiable and $|f^{(k)}(x)| < M \ \forall k$, then $f(x)$ can be represented as Taylor's series.

Exercises: Find Taylor series for the following functions:

① $\cos x$ around $x=1$.

② $\sin x$ around $x=0$

④ e^x around $x=0$

⑤ $\frac{1}{1+x}$ around $x=0$

⑥ $\log(1+x)$ around $x=0$.

⑦ $\frac{1}{1+x^2}$ around $x=0$

⑧ $\frac{x^2}{1+x^3}$ around $x=0$

⑨ $\frac{x^2}{(1+x^3)^2}$ around $x=0$

⑩ e^x around $x=2$.

Function of One Variable

Definition: Let $D \subseteq \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is said to be a limit point of D (or cluster point) if and only if for any $\delta > 0$,

$$D \cap \{x \in \mathbb{R} : 0 < |x - x_0| < \delta\} \neq \phi.$$

Examples: ① The interval $[a,b]$ is the set of all limit points of the intervals (a,b), $(a,b]$, $[a,b)$, and $[a,b]$.

② 0 is the only limit point of $D = \{\frac{1}{n} : n \in \mathbb{N}\}$.

③ The set $\mathbb{N}$ and $\mathbb{Z}$ has no limit points.

Theorem: Let $D \subseteq \mathbb{R}$. A point $x_0 \in \mathbb{R}$ is a limit point of D if and only if there exists a sequence (x_n) in $D \setminus \{x_0\}$ such that $x_n \to x_0$ as $n \to \infty$.

Limit of a Function at a Point: Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$ be a function and $x_0 \in \mathbb{R}$ be a limit point of D. We say that $l \in \mathbb{R}$ is a limit of $f(x)$ as x approaches x_0 if for every $\varepsilon > 0$, there exists $\delta > 0$ (depends on ε) such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta.$$

or

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Notation: $f(x) \to l$ as $x \to x_0$

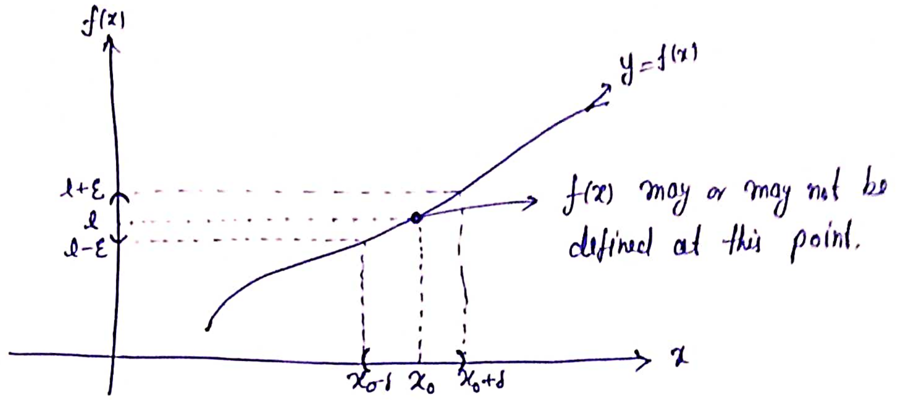
or

$$\lim_{x \to x_0} f(x) = l.$$

* Note that if $N_\delta = \{x : 0 < |x - x_0| < \delta\}$ then

$$f(N_\delta) \in (l - \varepsilon, l + \varepsilon).$$

* The function may not be defined at x_0.



Theorem: The limit of a function is always unique.

Examples: ① Show that $\lim_{x\to x_0} f(x) = x_0^2$, where $f(x) = x^2$.

Soln: Let $\varepsilon > 0$ be given. Consider

$$|f(x) - x_0^2| = |x^2 - x_0^2| = |x + x_0|\,|x - x_0|$$
$$= |x - x_0 + 2x_0|\,|x - x_0|$$
$$\le (|x - x_0| + 2|x_0|)\,|x - x_0|.$$

Note that if $|x - x_0| \le 1$ then $(|x-x_0| + 2|x_0|)\,|x - x_0| < (1 + 2|x_0|)\,|x - x_0| < \varepsilon$.

Choose $\delta = \min\left\{1, \frac{\varepsilon}{1+2|x_0|}\right\}$, we have

$$|f(x) - x_0^2| < \varepsilon \quad \text{whenever } |x - x_0| < \delta.$$

② Show that $\lim_{x\to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$.

Soln: Let $f(x) = \frac{x^3-4}{x^2+1}$. Consider

$$\left|f(x) - \frac{4}{5}\right| = \left|\frac{x^3-4}{x^2+1} - \frac{4}{5}\right| = \left|\frac{5x^3-4x^2-24}{5(x^2+1)}\right| = \left|\frac{5x^2+6x+12}{5(x^2+1)}\right|\,|x-2|.$$

Note that if $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow \left|\frac{5x^2+6x+12}{5(x^2+1)}\right| \le \frac{45+18+12}{5\times 2} = \frac{75}{10} = \frac{15}{2}.$

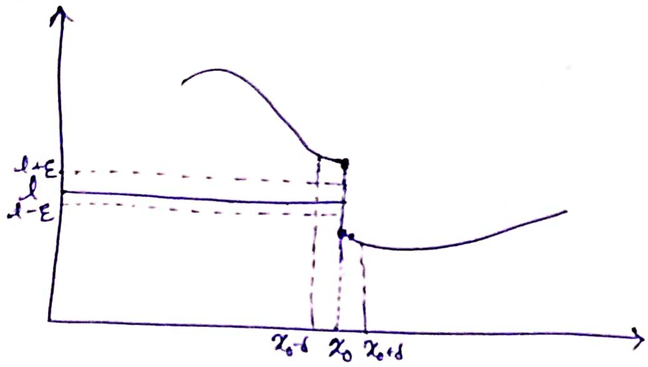
Choose $\delta = \min\{1, 2\varepsilon/15\}$, we have

$$\left| f(x) - \frac{4}{5} \right| < \varepsilon \qquad \text{whenever } |x-2| < \delta.$$

* $\lim_{x \to x_0} f(x)$ does not exists $\Rightarrow$ There exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$, there is at least one $x_\delta \in (x_0 - \delta, x_0 + \delta)$ such that $f(x_\delta) \notin (l - \varepsilon, l + \varepsilon)$

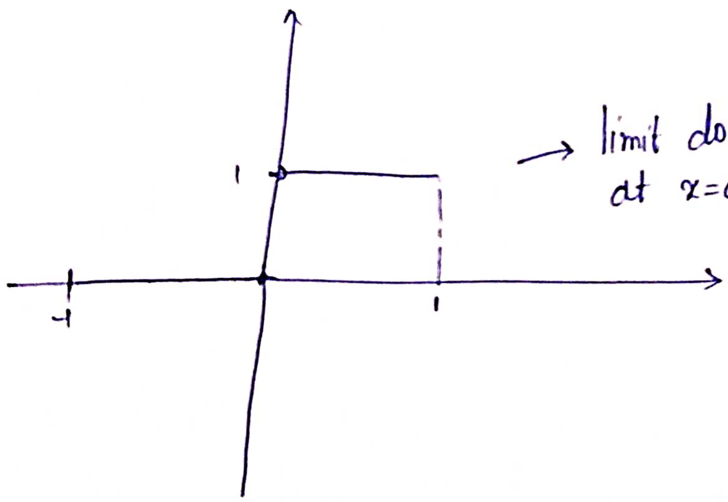
or

$|f(x_\delta) - l| \geq \varepsilon$.



Example: Let $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 < x \leq 1 \end{cases}$$



$\rightarrow$ limit does not exists at $x = 0$.

Theorem: If $\lim_{x \to x_0} f(x) = l$, then for every sequence (x_n) in D with $x_n \to x_0$, we have $f(x_n) \to l$.

Example: Show that $\lim_{x\to 0} \sin(1/x)$ does not exists.

Soln: Let $x_n = \dfrac{1}{(2n+1)\frac{\pi}{2}} \to 0$ as $n \to \infty$. But

$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n \not\to$ does not converge to any point.

This implies $\lim_{x\to 0} \sin(1/x)$ does not exists.

Squeeze Theorem: Let $D \subseteq \mathbb{R}$, $f, g, h : D \to \mathbb{R}$, and $x_0 \in \mathbb{R}$ be a limit point of D. If $f(x) \leq g(x) \leq h(x)$ for all $x \in D$, $x \neq x_0$, and if $\lim_{x \to x_0} f(x) = l = \lim_{x \to x_0} h(x)$

then $\lim_{x \to x_0} g(x) = l$.

Examples: ① $\lim_{x \to 0} \sin x = 0$

Soln:

$$\begin{array}{ccc} -x \leq & \sin x & \leq x \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \quad \forall\, x \geq 0.$$

$$\Downarrow$$

$\sin x \to 0$ as $x \to 0$.

② $\lim_{x \to 0} \cos x = 1$.

Soln: Note that

$$\begin{array}{ccc} 1 - \frac{1}{2} x^2 \leq & \cos x & \leq 1 \\ \downarrow & & \downarrow \\ 1 & & 1 \end{array}$$

$$\Downarrow$$

$$\lim_{x \to 0} \cos x = 1.$$

Definition: Let $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ be a limit point of D. Then

① We say that $f(x)$ has the left limit $l \in \mathbb{R}$ as x approaches to x_0 from left if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x_0 - \delta < x < x_0.$$

In this case, we write $\lim_{x \to x_0^-} f(x) = l$.

② We say that $f(x)$ has the right limit $l \in \mathbb{R}$ as x approaches to x_0 from right if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x_0 < x < x_0 + \delta.$$

In this case, we write $\lim_{x \to x_0^+} f(x) = l$.

* $$\lim_{x \to x_0^-} f(x) = \lim_{h \to 0} f(x_0 - h)$$

$$\lim_{x \to x_0^+} f(x) = \lim_{h \to 0} f(x_0 + h).$$

* If $\lim_{x \to x_0} f(x) = l$ exists then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x).$$

Limit at $\pm\infty$:

Definition: Let $f: (a, \infty) \to \mathbb{R}$, $a > 0$, then $\lim_{x \to \infty} f(x) = l$ if for every $\varepsilon > 0$, there exists $M > a$ (depends on ε) such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x > M.$$

Definition: Let $f : (-\infty, a) \to \mathbb{R}$, $a < 0$, then $\lim_{x \to -\infty} f(x) = l$ if for every $\varepsilon > 0$, there exists $m < a$ such that

$$|f(x) - l| < \varepsilon \quad \text{whenever} \quad x < m.$$

Limit $\pm\infty$:

Definition: $\lim_{x \to x_0} f(x) = +\infty$ if for every $M > 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow f(x) > M.$$

Definition: $\lim_{x \to x_0} f(x) = -\infty$ if for every $m < 0$, there exists $\delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow f(x) < m.$$

Definition: $\lim_{x \to \infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha > 0$ such that

$$x > \alpha \Rightarrow f(x) > M.$$

Definition: $\lim_{x \to \infty} = -\infty$ if for every $m < 0$, there exists $\alpha > 0$ such that

$$x > \alpha \Rightarrow f(x) < m.$$

Definition: $\lim_{x \to -\infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha < 0$ such that

$$x < \alpha \Rightarrow f(x) > M.$$

Definition: $\lim_{x \to -\infty} f(x) = -\infty$ if for every $m < 0$, there exists $\alpha < 0$ such that

$$x < \alpha \Rightarrow f(x) < m.$$

Exercises: ① Show that $\lim_{x \to x_0} (x) = x_0$.

② Show that $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$, $x \in \mathbb{R} \setminus \{0\}$.

③ Show that $\lim_{x \to 0} \frac{1}{x}$ does not exists, $x \in \mathbb{R} \setminus \{0\}$.

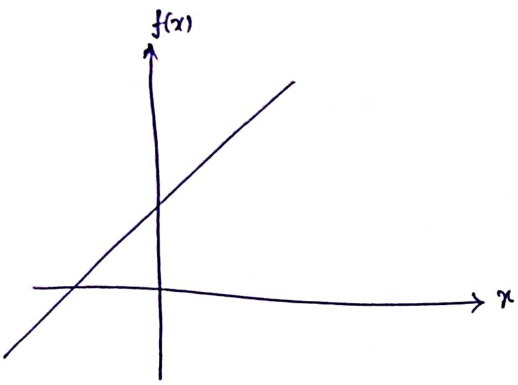
④ Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

⑤ Show that $\lim_{x \to 3} \frac{x}{4x-9} = 1$.

Consider the following functions:

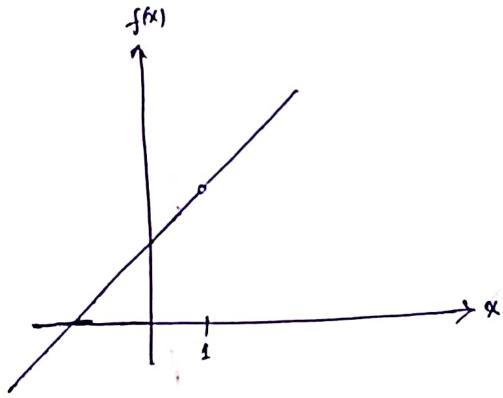
① $f(x) = x+1$

Here, $\lim_{x \to 1} f(x) = 2$



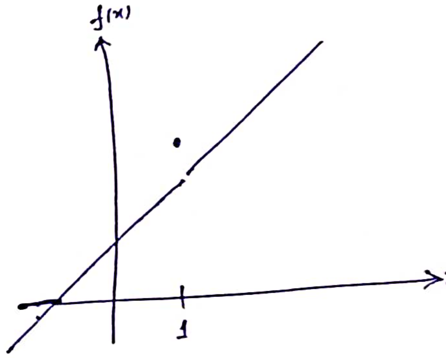
② $g(x) = \frac{x^2-1}{x-1}$

Here, $\lim_{x \to 1} g(x) = 2$



③ $h(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1. \end{cases}$

Here, $\lim_{x \to 1} h(x) = 2$.



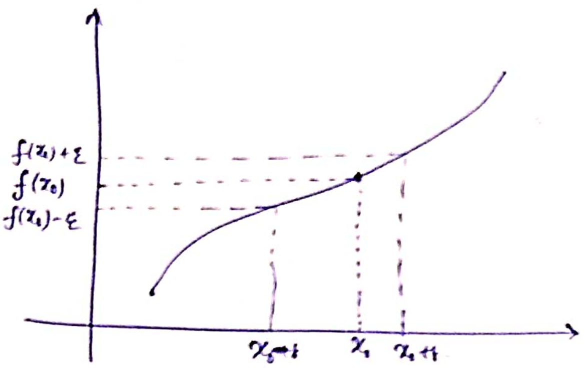
Definition: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Then f is said to be continuous at a point $x_0 \in D$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

* The function f is said to be continuous on D if it is continuous at every point in D.

* The function is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$



* If f is continuous at x_0, then for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Examples: ① Prove that $f(x) = |x - x_0|$, $x \in \mathbb{R}$, $x_0 \in \mathbb{R}$ (given) is continuous on $\mathbb{R}$.

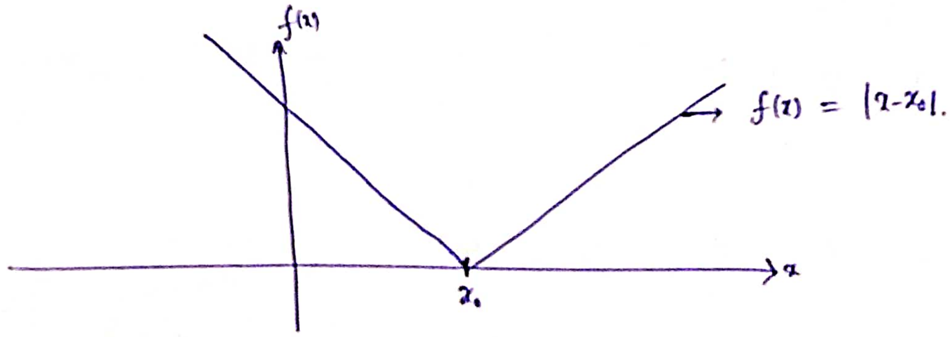
Soln: Consider

$$\left| f(x) - f(a) \right| = \left| |x - x_0| - |a - x_0| \right| \le \left| (x - x_0) - (a - x_0) \right| = |x - a| < \varepsilon$$

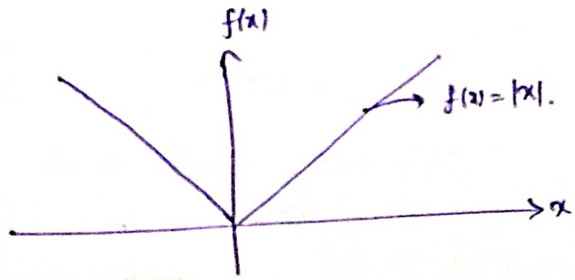
Choose $\delta = \varepsilon$, we have

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever } |x - a| < \delta.$$

Since $a \in \mathbb{R}$ is arbitrary, $f(x)$ is continuous on $\mathbb{R}$.



In particular, $f(x) = |x|$ is continuous on $\mathbb{R}$.



(2) Show that $f(x) = \sqrt{x}, \ x \geq 0$ is continuous on $\mathbb{R}$.

Soln: Let $\varepsilon > 0$ be given.

Case I: If $x_0 = 0$ then

$$|f(x) - f(0)| = \sqrt{x} < \varepsilon$$

Choose $\delta = \varepsilon^2$, we have

$$|f(x) - f(0)| < \varepsilon, \quad \text{whenever } |x| < \delta.$$

$\Rightarrow f$ is continuous at 0.

Case II: If $x_0 \neq 0$ then

$$|f(x) - f(x_0)| = |\sqrt{x} - \sqrt{x_0}| \times \frac{|\sqrt{x} - \sqrt{x_0}|}{|\sqrt{x} + \sqrt{x_0}|} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$$

Choose $\delta = \varepsilon\sqrt{x_0}$, we have

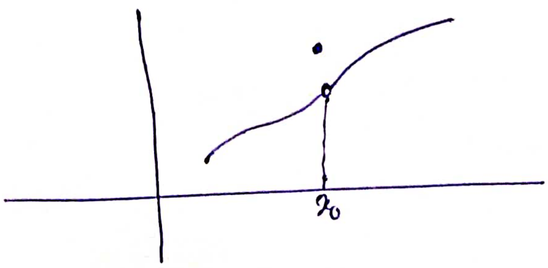
$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever } |x - x_0| < \delta.$$

Hence, f is continuous on $\mathbb{R}$.

Types of Discontinuity:

① Removable Discontinuity: If both limits exist (and equal) but not equal to the function value, that is,

$$\lim_{x \to x_0} f(x) \neq f(x_0)$$



Ex: $f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1. \end{cases}$

② **Discontinuity of First kind:** If left-hand and right-hand limit exist but not equal, that is,

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x).$$

Ex: $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

③ **Discontinuity of Second kind:** If left-hand and right-hand limits do not exists.

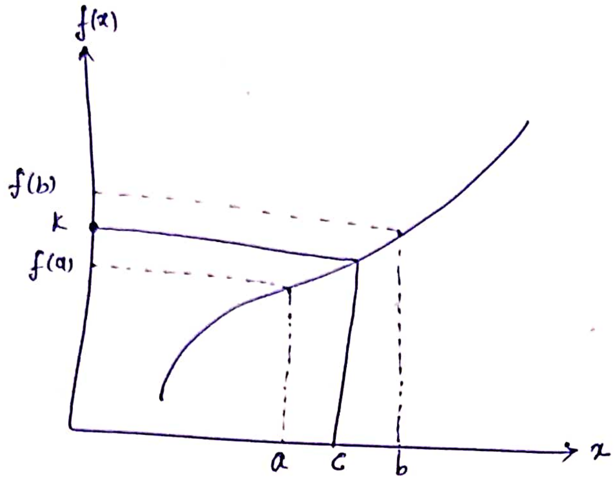
Ex: $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

④ **Mixed Discontinuity:** Either left-hand or right-hand (only one) limit exists.

Ex: $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ \sin(1/x) & \text{if } x > 0. \end{cases}$

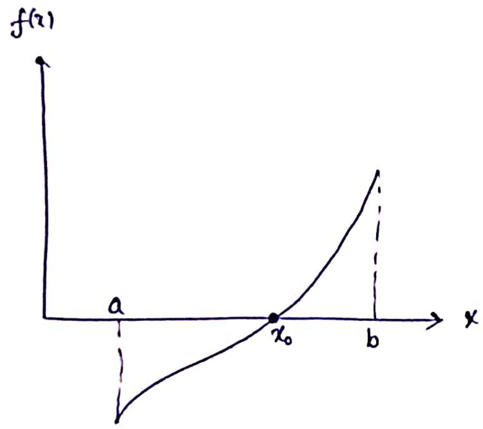
Theorem: Suppose $f: [a,b] \to \mathbb{R}$ is continuous then f is bounded function.

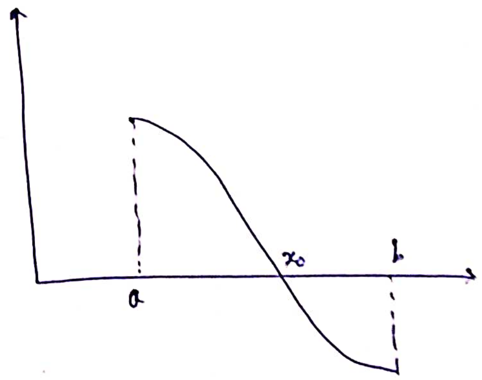
Intermediate Value Theorem (IVT): Let I be an interval and $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

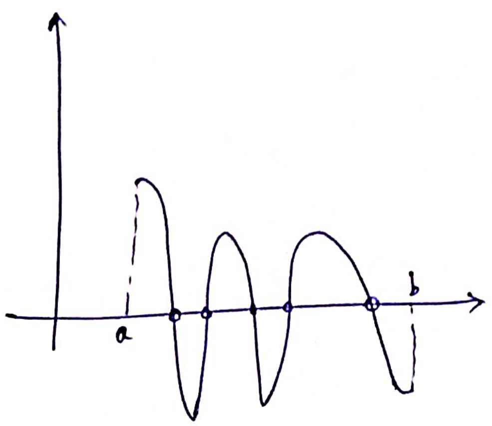


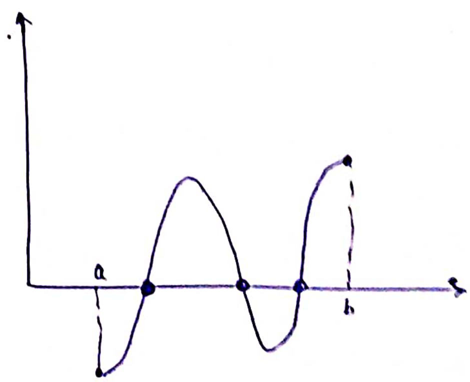
* There can be more than one such point, for example, $f(x) = x^2$, $[-1, 1]$.

Corollary: Let I be an interval and $f: I \to \mathbb{R}$ be continuous on I such that $f(a)f(b) < 0$, for $a, b \in I$. Then there exists $x_0 \in I$ such that $f(x_0) = 0$.









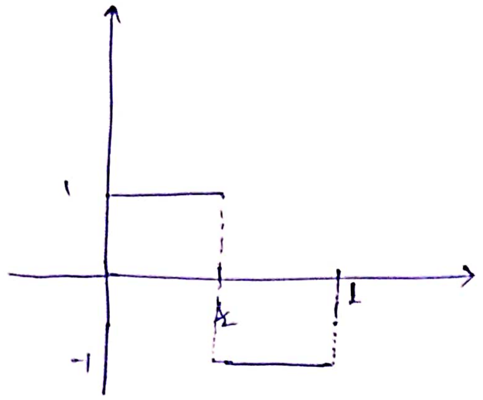
* Continuity is sufficient:

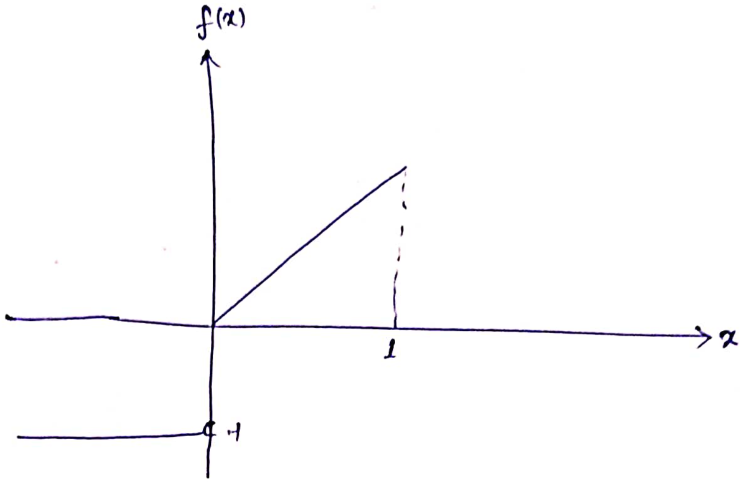
Example: ① $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x \le 1. \end{cases}$$

② $f : [-1,1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ x & 0 \le x \le 1. \end{cases}$$





Example: Consider the function $f(x) = x^3 + \sin x + 1$, $x \in \mathbb{R}$. Does there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$?

Soln: Note that $f(0) = 1 > 0$

and $f(-2) = -8 + \sin(-2) + 1 = -8 - \sin(2) + 1 < 0$.

Hence, by IVT, there exists $-2 < x_0 < 0$ such that $f(x_0) = 0$.

Corollary: Let f be a continuous function defined on an interval. Then range of f is an interval.

Corollary: Suppose f is a continuous function defined on closed and bounded interval I. Then its range is a closed and bounded interval.

Exercise: Check the continuity of the following functions at $x = 0$

① $f(x) = \sin(x)$ ② $f(x) = \cos x$ ③ $f(x) = \begin{cases} \frac{\sin x}{x} & , \text{if } x \ne 0 \\ 1 & , \text{if } x = 0 \end{cases}$

④ $f(x) = 1/x$ on $(0,1]$ ⑤ $f(x) = x^{1/k}$, $x \ge 0$, $k \in \mathbb{N}$.

Definition: Let I be an interval, $f: I \to \mathbb{R}$ be a function and $x_0 \in I$. Then f is said to be differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

In this case, the value of the limit is called the derivative of f at x_0 and is denoted by $f'(x_0)$.

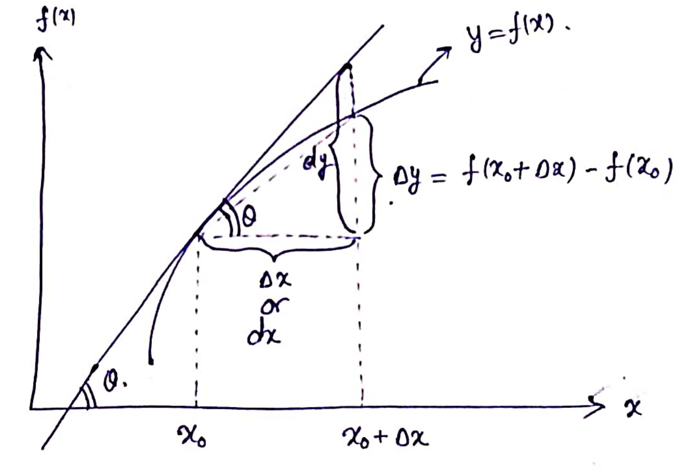
* f is differentiable at x_0 $\Leftrightarrow$ $\displaystyle\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists.

or

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ exists.}$$

* f is differentiable at x_0 $\Leftrightarrow$ For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$ whenever $0 < |x - x_0| < \delta$

Notation: $f'(x_0)$ or $\left.\frac{d}{dx} f(x)\right|_{x = x_0}$



$$\tan\theta = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

Here, (1) Δx and Δy measure changes along the function $f(x)$
(2) dx and dy measure changes along the tangent line.

* Left-hand derivative: $Lf'(x_0) = \displaystyle\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}$

* Right-hand derivative: $Rf'(x_0) = \displaystyle\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$

Example: Show that $f(x) = x^2$ is differentiable on $\mathbb{R}$ and find its derivative.

Soln: Let x_0 be an arbitrary point in $\mathbb{R}$. Note that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{x^2-x_0^2}{x-x_0} = 2x_0.$$

Therefore, $f'(x) = 2x \quad \forall\, x \in \mathbb{R}$.

Theorem: Every differentiable function is continuous.

Proof:

$$\lim_{x \to x_0} (f(x)-f(x_0)) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \cdot (x-x_0) = f'(x_0) \cdot 0 = 0$$

$$\Rightarrow \quad \lim_{x \to x_0} f(x) = f(x_0)$$

Hence, f is continuous at x_0 and x_0 is an arbitrary point and so, f is continuous on $\mathbb{R}$.

* The converse of above theorem is not true:

Example: $f(x) = |x|$.

At $x=0$,

$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{|x|}{x}$$

Here,

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

$\Rightarrow$ f is not differentiable at 0 but $\lim_{x \to 0} f(x) = 0 = f(0)$. So, f is continuous at 0.

Exercise: Determine where each of the following function from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative

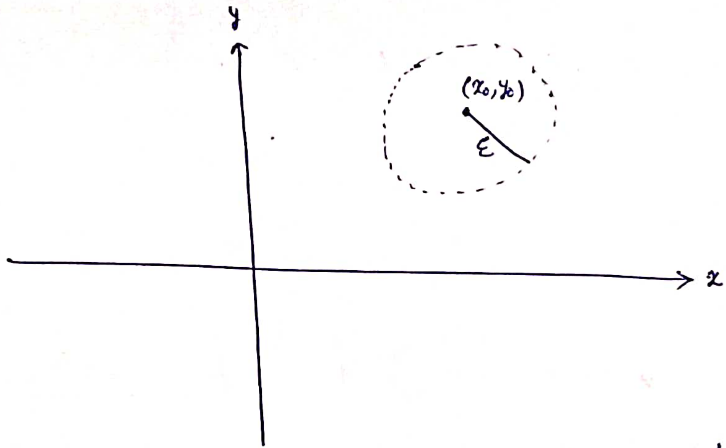
1. $f(x) = |x| + |x+1|$
2. $f(x) = x^3$
3. $f(x) = x^{1/3}$
4. $f(x) = x|x|$.
5. $f(x) = x^m \sin(1/x), m>1$
6. $f(x) = x^2 \sin(\frac{1}{x^2})$
7. $g(x) = |\sin x|$
8. $f(x) = \frac{1}{x}, x \neq 0$.

Functions of Several Variables

Definitions: Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be any point.

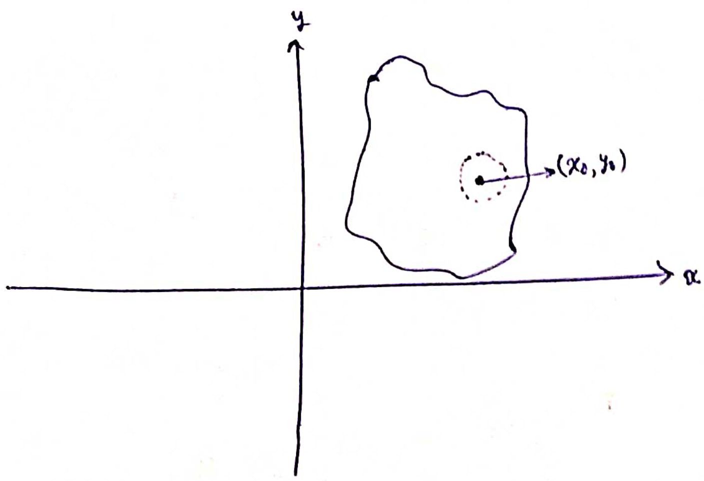
① An ε-disk around (x_0, y_0) is the set of all points $(x, y) \in \mathbb{R}^2$ whose distance from (x_0, y_0) is less than ε. That is,

$$\varepsilon\text{- disk around } (x_0, y_0) = \left\{ (x, y) \in \mathbb{R}^2 : |(x,y) - (x_0, y_0)| < \varepsilon \right\}$$
$$= \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon \right\}$$

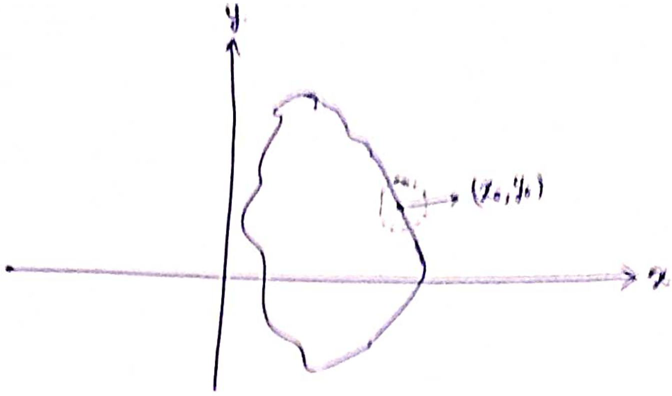


* ε-disk around (x_0, y_0) is also called ε- neighbourhood of the point (x_0, y_0).

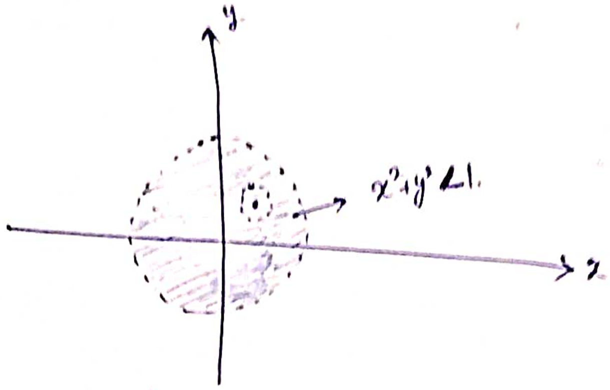
② A point (x_0, y_0) is said to be an interior of D if and only if some ε-disk around (x_0, y_0) is contained in D.



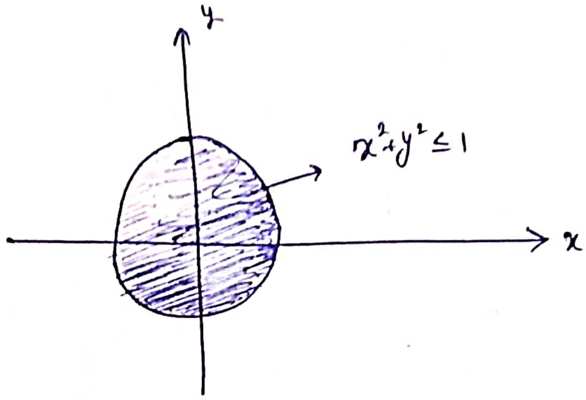
③ A point (x_0, y_0) is said to be a boundary point of D if and only if every ε-disk around (x_0, y_0) contains points from D and points not from D.



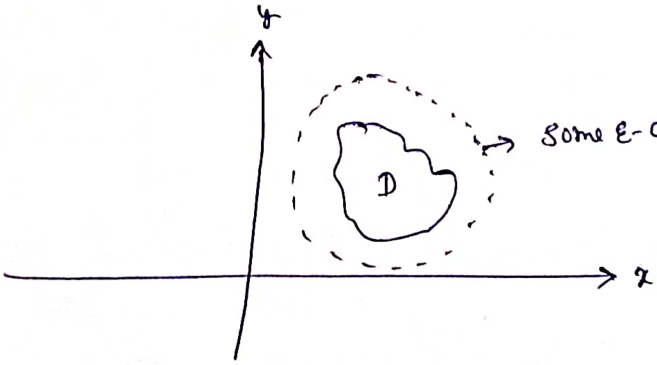
④ D is said to be an open subset of $\mathbb{R}^2$ if and only if all points of D are its interior points.



⑤ D is said to be a closed subset of $\mathbb{R}^2$ if and only if D^c is open.



⑥ D is said to be a bounded subset of $\mathbb{R}^2$ if and only if D is contained in some ε- disk (around some point)



* Try to relate the above definitions with a subset $D \subseteq \mathbb{R}$.

* $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R},\ i = 1, 2, \dots, n\}$

Definition: Let $D \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$. A function of several variables is a mapping

$$f : D \to \mathbb{R}.$$

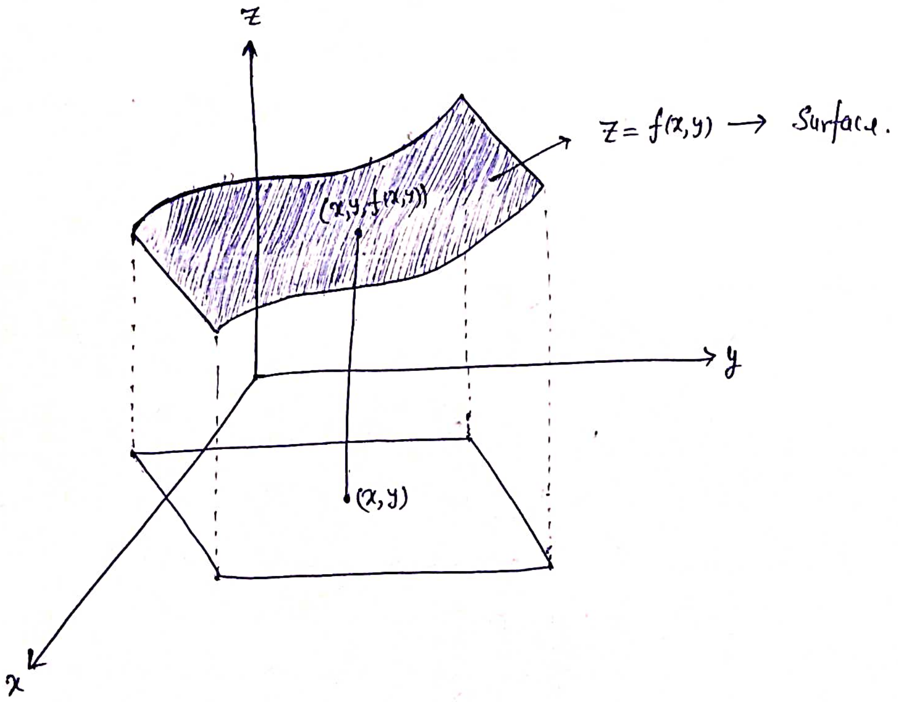
* $D =$ Domain of $f = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \dots, x_n) \text{ is defined}\}$.
* Range of $f = \{f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in D\}$.
* In this course, we deal with the functions of two and three variables.

Functions of Two Variables: Let $D \subseteq \mathbb{R}^2$. A function of two variables is a mapping

$$f : D \to \mathbb{R}.$$

That is, according to some given rule $f(x,y)$, we have a real value $z = f(x,y)$ for each point (x,y) of a certain part of xy-plane.

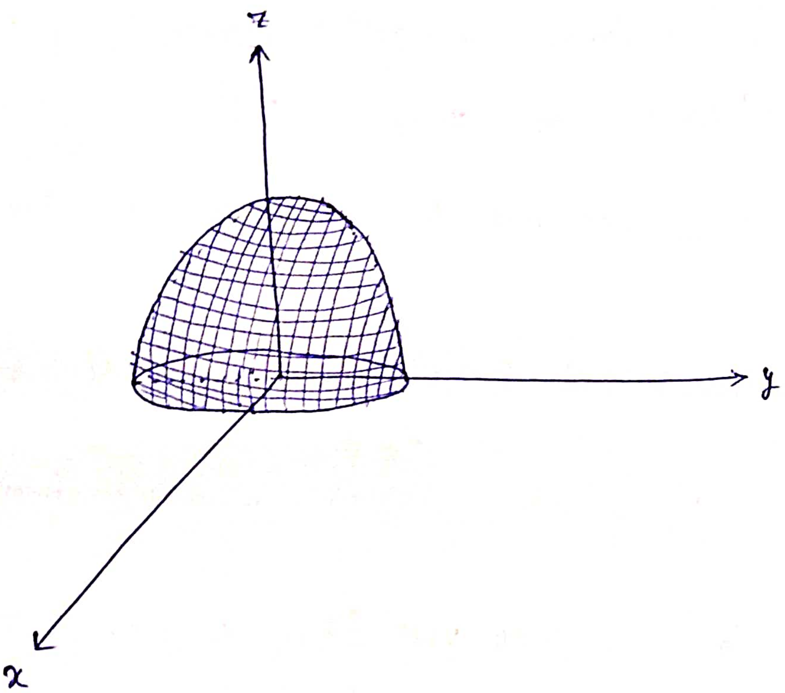
* $D =$ Domain of $f = \{(x,y) \in \mathbb{R}^2 : f(x,y) \text{ is defined}\}$.
* Range of $f = \{z = f(x,y) : (x,y) \in D\}$.
* Here, x and y are independent variables and z is a dependent variable.



Example: $z = f(x,y) = \sqrt{1-x^2-y^2}$

D = Domain of $f = \{(x,y) \in \mathbb{R}^2 ; f(x,y) \text{ is defined}\} = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$.

Range of $f = \{z = f(x,y) : (x,y) \in D\} = \{z \in \mathbb{R} : 0 \le z \le 1\}$.

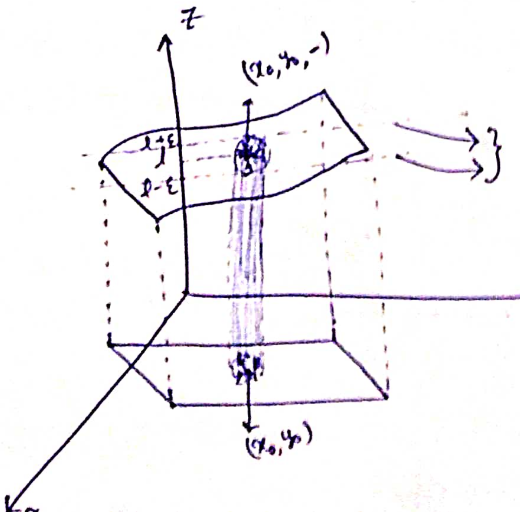


Limit: Let $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a function. We say $f(x,y)$ has a limit l as (x,y) approaches to (x_0,y_0) if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|f(x,y) - l| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta.$$

* The number l is called the limit of $f(x,y)$ as $(x,y) \to (x_0,y_0)$.

Notation: $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l.$



These are planes $z = l-\varepsilon$ and $z = l+\varepsilon$

Ex: Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0.$

Soln: Let $\varepsilon > 0$ be given. Consider

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} < \delta < \varepsilon,$$

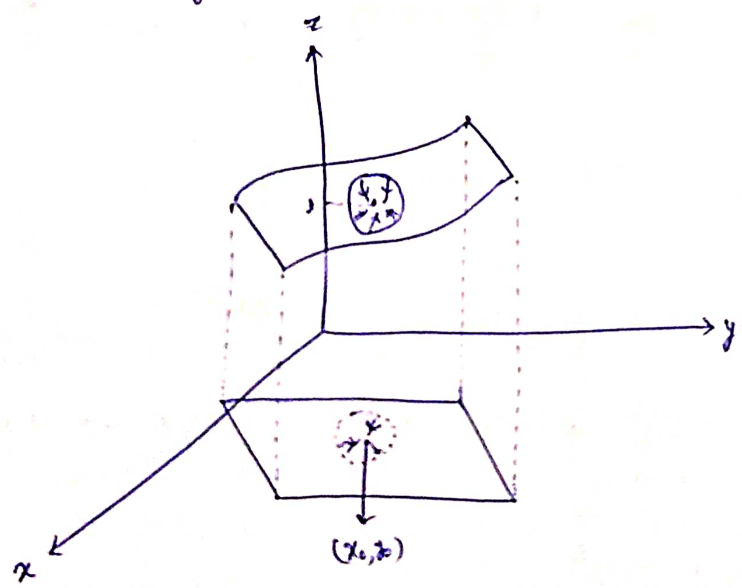
Choose $\delta \leq \varepsilon$, we have

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta.$$

Hence,

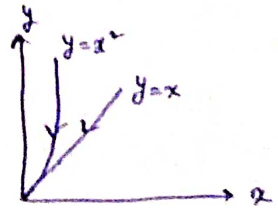
$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

* Note that $(x,y) \to (x_0, y_0)$ along infinite number of paths and the limit is unique along all the paths.



So, if the limit is dependent on a path, then the limit does not exists.

Example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$



Soln:

Along $y = x$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \dfrac{x^3}{x^2(x^2+1)} = \lim\limits_{x\to 0} \dfrac{x}{x^2+1} = 0$

Along $y = x^2$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0} \dfrac{x^4}{2x^4} = \dfrac{1}{2}$

$\Rightarrow$ limit does not exists.

* To prove the limit does not exists, the following paths may help in practice:

ⓐ along x-axis (that is, $y=0$)

ⓑ along y-axis (that is, $x=0$)

ⓒ along $y=mx$

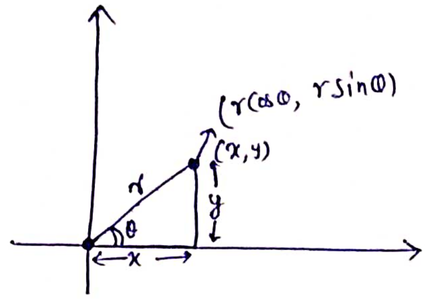
ⓓ along $y^{\alpha}=mx^{\beta}$, for suitable values of α and β.

Alternative Approach: Change of Coordinate system from Cartesian to Polar, that is

$$x=r\cos\theta \text{ and } y=r\sin\theta.$$

Then $\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = l.}$

* Note that $r\to 0$ and Changing θ containing all paths. Do not confuse this with $y=mx$ path.



Example: ① Compute $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.

Soln: Put $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2} = \lim_{r\to 0} r\cos^2\theta\sin\theta = 0.$$

② Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exists.

Soln: Put $x=r\cos\theta$ and $y=r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta \rightarrow \text{depends on } \theta.$$

Therefore, limit does not exists.

Exercises: Find (or show)

① $\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)=0$

② $\lim_{(x,y)\to(0,0)} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(3x+6y)}$

③ $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$

④ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

⑤ $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

⑥ $\lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)$

Continuity: A function $z = f(x,y)$ is said to be continuous at a point (x_0, y_0) if

ⓐ $f(x,y)$ is defined at (x_0, y_0)

ⓑ $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

ⓒ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

ε-δ Definition: A function $z = f(x,y)$ is said to be continuous at (x_0, y_0) if for every $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ such that

$$|f(x,y) - f(x_0,y_0)| < \varepsilon \qquad \text{whenever} \qquad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta.$$

* If a function f is continuous at every point in a domain D, then it is said to be continuous in D.

Removable Discontinuity: If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists and $f(x_0,y_0)$ is defined but

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq f(x_0,y_0).$$

Examples: ① Show that

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Soln: Let $\varepsilon > 0$ be given. Consider

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leq \left| \frac{3(x^2+y^2)y}{x^2+y^2} \right| = 3|y| \leq 3\sqrt{x^2+y^2} < 3\delta = \varepsilon.$$

Choose $\delta = \varepsilon/3$, we have $\left| \dfrac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$ whenever $\sqrt{x^2+y^2} < \delta$.

(2) Discuss the continuity of

$$f(x,y) = \begin{cases} \dfrac{2x^4+3y^4}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

at $(0,0)$.

Soln: Put $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^4+3y^4}{x^2+y^2} = \lim_{r\to 0} \frac{2r^4\cos^4\theta + 3r^4\sin^4\theta}{r^2} = 0.$$

Hence, $f(x,y)$ is continuous at $(0,0)$.

(3) Check the continuity of

$$f(x,y) = \begin{cases} \dfrac{(x-y)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

at $(0,0)$.

Soln: Choosing the path $y = mx$,

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2} = \frac{1-m}{1+m^2} \longrightarrow \text{depends on } m.$$

Hence, the limit does not exists, and so, the function is discontinuous at $(0,0)$.

Remark: Changing the polar coordinate does not always help and the transformation may tempt us to false conclusion.

Example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$.

Soln: Put $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{r\cos^4\theta + \sin^2\theta}$$

Case 1: if we fix θ, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = 0$.

Case 2: if $r\sin\theta = r^2\cos^2\theta$ $(y = x^2)$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2} = \lim\limits_{r\to 0} \dfrac{r^2\cos^4\theta}{2r^2\cos^4\theta} = \dfrac{1}{2}$.

So, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist.

Exercises: Discuss the continuity of the following functions:

① $f(x,y) = \begin{cases} \dfrac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ $\to$ at $(0,0)$.

② $f(x,y) = \begin{cases} \dfrac{x^4y^4}{(x^4+y^4)^3}, & \text{if } (x,y) \neq (0,0) \\ 0. & \text{if } (x,y) = (0,0) \end{cases}$ $\to$ at $(0,0)$

③ $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\tan(xy)}, & \text{if } xy \neq 0 \\ 0 & \text{if } xy \neq 0 \end{cases}$ $\to$ at $(0,0)$

④ $f(x) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3} & \text{if } x^3+y^3 \neq 0 \\ 0, & \text{elsewhere} \end{cases}$ $\to$ at $(0,0)$

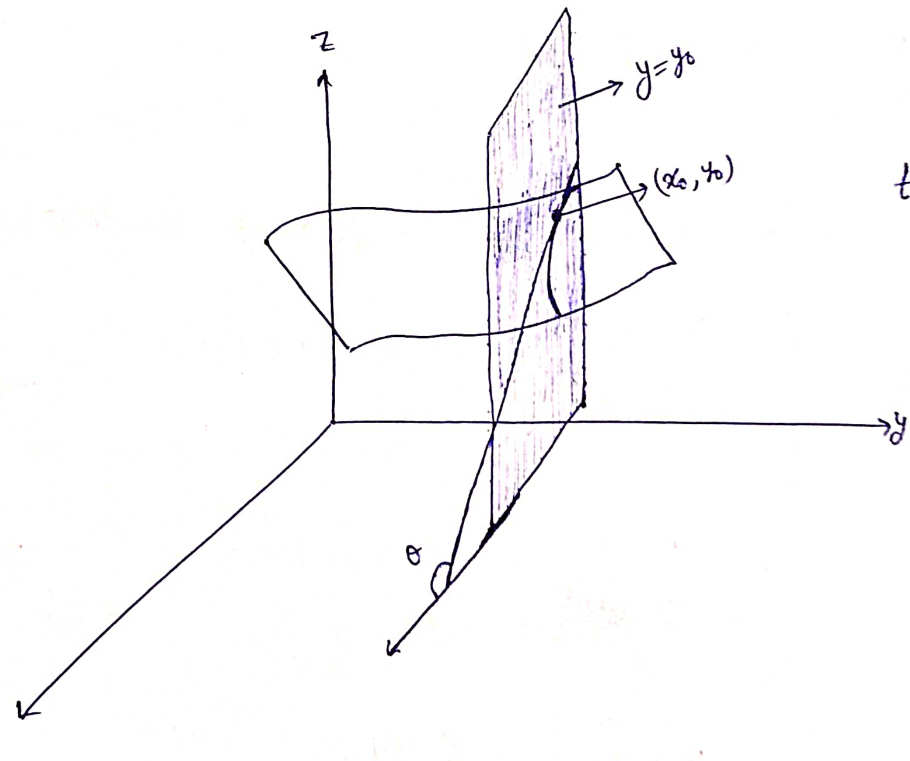
⑤ $f(x) = \begin{cases} \dfrac{e^{-1/x^2+y^2}}{x^4+y^4}, & \text{if } (x,y) \neq (0,0) \\ 0. & \text{elsewhere} \end{cases}$ $\to$ at $(0,0)$

Partial Derivatives: The usual derivatives of a function of several variables with respect to one of the independent variables keeping all other independent variables as constant is called the partial derivatives of the function with respect to that variable.

Let $z = f(x,y)$, $(x,y) \in \mathbb{R}^2$, $z \in \mathbb{R}$.

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} = \left.\frac{d}{dx} f(x,y_0)\right|_{x=x_0}.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0,y_0) = \lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k} = \left.\frac{d}{dy} f(x_0,y)\right|_{y=y_0}.$$



$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

Example: Find the partial derivatives of $f(x,y) = ye^{-x}$ at a point (x,y).

Soln:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{ye^{-(x+h)} - ye^{-x}}{h} = ye^{-x} \lim_{h\to 0} \frac{e^{-h}-1}{h} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} = \lim_{k\to 0} \frac{(y+k)e^{-x} - ye^{-x}}{k} = \lim_{k\to 0} e^{-x} = e^{-x}$$

Remark: A function can have partial derivatives with respect to both x and y at a point without being continuous there. On the other hand a continuous function may not have partial derivatives.

Example: ① Show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right), & \text{if } x+y \neq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is [redacted] continuous at $(0,0)$ but its partial derivatives do not exists at $(0,0)$.

Soln: Let $\varepsilon > 0$ be given. Consider

$$|f(x,y) - 0| = \left|(x+y)\sin\left(\frac{1}{x+y}\right) - 0\right| \leq |x+y| \leq |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2},$$

where the last inequality can be seen as follows:

$$(|x| - |y|)^2 \geq 0$$

$$\Rightarrow \quad x^2 + y^2 \geq 2|x||y|$$

$$\Rightarrow \quad 2(x^2+y^2) \geq x^2 + y^2 + 2|x||y| = (|x| + |y|)^2$$

$$\Rightarrow \quad |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}.$$

Choose $\delta < \varepsilon/\sqrt{2}$, we have

$$|f(x,y) - 0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta.$$

Now, Consider

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h \sin(1/h)}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \to \text{does not exists.}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k \sin(1/k)}{k} = \lim_{k\to 0} \sin\left(\frac{1}{k}\right) \to \text{does not exists.}$$

(2) Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{elsewhere} \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives exists at $(0,0)$.

Soln: Choosing the path $y = mx$, we have.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} = \frac{m}{1+2m^2} \rightarrow \text{limit does not exists.}$$

Note that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0.$$

Sufficient Condition for Continuity in a Region R: If a function $f(x,y)$ has partial derivatives f_x and f_y everywhere in a region R and these derivatives everywhere satisfy the inequalities

$$|f_x(x,y)| < M \quad \text{and} \quad |f_y(x,y)| < M,$$

where M is independent of x and y, then $f(x,y)$ is continuous everywhere in R.

Sufficient Condition for Continuity at (x_0,y_0): If one of the first order partial derivatives exists and is bounded in the neighbourhood of (x_0,y_0) and the other exists at (x_0,y_0) then $f(x,y)$ is continuous at (x_0,y_0).

Second Order Partial Derivatives:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial x}(f_x)\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f_x(x_0+h,y_0)-f_x(x_0,y_0)}{h} = f_{xx}(x_0,y_0).$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial y}(f_y)\right|_{(x_0,y_0)} = \lim_{k\to 0}\frac{f_y(x_0,y_0+k)-f_y(x_0,y_0)}{k} = f_{yy}(x_0,y_0).$$

$$\left.\frac{\partial^2 f}{\partial y\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial y}(f_x)\right|_{(x_0,y_0)} = \lim_{k\to 0}\frac{f_x(x_0,y_0+k)-f_x(x_0,y_0)}{k} = f_{yx}(x_0,y_0).$$

(Mixed Derivatives)

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right|_{(x_0,y_0)} = \left.\frac{\partial}{\partial x}(f_y)\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h} = f_{xy}(x_0,y_0).$$

Example: Compute $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial x\partial y}$ at $(0,0)$ of

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$$

Soln: Consider

$$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0}\frac{hk(h^2-k^2)}{k(h^2+k^2)} = h$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = 0$$

Therefore,

$$f_{xy}(0,0) = \left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1.$$

Next, consider

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{hk(h^2-k^2)}{h(h^2+k^2)} = -k$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{yx}(0,0) = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1.$$

Note that $$f_{xy}(0,0) \neq f_{yx}(0,0).$$

<u>Mixed Derivative Theorem</u>: If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0) then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

<u>Exercises</u>: ① Let

$$f(x,y) = \begin{cases} \dfrac{2x^3 + 3y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Compute $f_x(0,0)$ and $f_y(0,0)$.

② Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and discuss the continuity of these partial derivatives.

③ Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & x \neq -y^2 \\ 0, & \text{elsewhere.} \end{cases}$$

Also, check the continuity of f_{xy} and f_{yx}.

④ Let

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & \text{if } x \neq y \\ 0, & \text{elsewhere.} \end{cases}$$

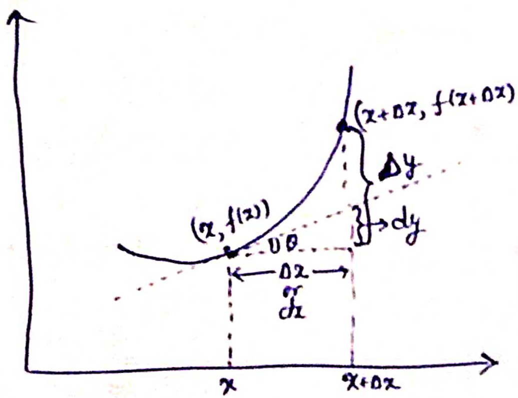
Show that existence of second order partial derivative does not imply continuity.

Differentiability and Differentials: A function $f(x)$ is said to be differentiable at a point x, if when x is given the arbitrary increment Δx, the increment Δy can be expressed in the form

$$\Delta y = f(x+\Delta x) - f(x) = A\,\Delta x + \varepsilon\,\Delta x,$$

where A is independent of Δx and $\varepsilon \to 0$ as $\Delta x \to 0$.

* $A\,\Delta x$ is called differential or total differential of y.



* Δx and Δy measure changes along the function.
* dx and dy measure changes along the tangent.

* $\tan\alpha = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

* Note that $f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{A\,\Delta x + \varepsilon\,\Delta x}{\Delta x}$

$$= A + \lim_{\Delta x \to 0} \varepsilon = A.$$

Hence, Total differential $= A\,\Delta x = \frac{dy}{dx} \cdot dx = dy$.

Theorem: The function $y = f(x)$ is differentiable at a point x_0 if and only if $f'(x_0)$ is finite.

Testing Differentiability: We can use the following ways to test the differentibility

① $\Delta y = A\,\Delta x + \varepsilon\,\Delta x = dy + \varepsilon\,\Delta x$

② $\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$

③ $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ exists.

Example: Show that the function $f(x) = x^2$ is differentiability.

Soln: Let $y = f(x) = x^2$. Consider.

$$\Delta y = f(x+\Delta x) - f(x) = (x+\Delta x)^2 - x^2 = x^2 + \Delta x^2 + 2x\,\Delta x - x^2$$

$$= \underbrace{2x}_{A}\,\Delta x + \underbrace{\Delta x}_{\varepsilon} \cdot \Delta x$$

This implies f is differentiable and its derivative is $2x$.

Also, note that

$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = 2x$ or $\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0$.

Differentiability of Two Variables: The function $z = f(x,y)$ is said to be differentiable at a point (x,y), if at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

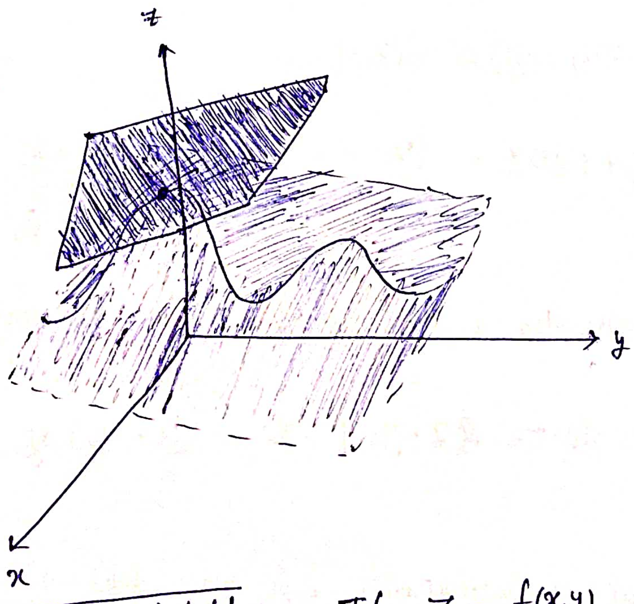
where a and b are independent of Δx and Δy, and ε_1 and ε_2 are functions of Δx and Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \varepsilon_1 = 0 = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \varepsilon_2 .$$

* $a\,\Delta x + b\,\Delta y$ is called total differential of z at the point (x,y) and is denoted by dz, that is,

$$dz = a\,\Delta x + b\,\Delta y = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy . \longrightarrow$$ Total Differential or Total Derivative.

* If Δx and Δy are sufficiently small, dz gives a close approximation to Δz.
* Differentiability of two variables approximate the function by a tangent plane near to the given point



(Increment Theorem)

* **Necessary Condition for Differentiability:** If $z = f(x,y)$ is differentiable then $f(x,y)$ is continuous and has partial derivatives with respect to x and y at the point (x,y) and that

$$a = f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \quad \text{and} \quad b = f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}.$$

Sufficient Condition for Differentiability: If the partial derivatives exists and continuous at the point (x,y) then the function is differentiable at (x,y)

* The function may not be differentiable at a point even if partial derivatives exist.

* A function may be differentiable even if f_x and f_y are not continuous.

Testing Differentiability:

① $\Delta z = a\,\Delta x + b\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

② $\lim\limits_{\Delta\rho \to 0} \dfrac{\Delta z - dz}{\Delta \rho} = 0, \qquad \Delta \rho = \sqrt{\Delta x^2 + \Delta y^2}$

③ $\lim\limits_{(h,k)\to(0,0)} \dfrac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}}$ exists.

Examples: ① Show that $z = x^2 + xy + xy^2$ is differentiable and write down its total differential.

Soln: Note that

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= \underbrace{(2x+y+y^2)}_{a}\Delta x + \underbrace{(x+2xy)}_{b}\Delta y + \underbrace{(\Delta x + \Delta y + 2y\,\Delta y)}_{\varepsilon_1}\Delta x + \underbrace{(x\,\Delta y + \Delta x\,\Delta y)}_{\varepsilon_2}\Delta y.$$

Hence, the given function is differentiable and it's total differential is

$$dz = (2x+y+y^2)\,dx + (x+2y)\,dy.$$

② [Continuous, Partial derivative exists but not differentiable]

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Soln! Note that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0} r\cos\theta\sin\theta = 0 \Rightarrow f \text{ is continuous at } (0,0)$$

and

$$\left.\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0 \\ f_y(0,0) &= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0 \end{aligned}\right\} \Rightarrow \text{Partial derivatives exists.}$$

Next,

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{(1+m^2)h^2} \qquad (k=mh)$$

$$= \frac{m}{1+m^2} \rightarrow \text{depends on } m \text{ along the path } k=mh.$$

$\Rightarrow$ limit does not exist

This implies, f is not differentiable at $(0,0)$.

(3) [Differentiable but f_x and f_y are not continuous]

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y)\neq(0,0) \\ 0. & \text{if } (x,y)=(0,0). \end{cases}$$

Soln! Note that

$$\left.\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h^2\cos\left(\tfrac{1}{|h|}\right) = 0 \\ f_y(0,0) &= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} k^2\cos\left(\tfrac{1}{|k|}\right) = 0 \end{aligned}\right\} \text{Partial Derivative exists.}$$

and

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + 2x \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Along x- axis,

$$\lim_{x\to 0} f_x(x,y) = \lim_{x\to 0} \left[\frac{x}{|x|} \sin\left(\frac{1}{|x|}\right) + 2x \cos\left(\frac{1}{|x|}\right)\right] \neq 0.$$

$\Rightarrow$ f_x is not Continuous at $(0,0)$.

Similarly, f_y is not Continuous at $(0,0)$.

Next, note that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - h f_x(0,0) - k f_y(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2} \cos\left(\frac{1}{\sqrt{h^2+k^2}}\right) = \underline{\underline{0}}.$$

Hence, f is differentiable at $(0,0)$.

<u>Exercises</u> ① [Continuous, Partial derivatives exist but not differentiable]

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

② [Differentiable but f_x and f_y are not Continuous]

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \cos\left(\frac{1}{y}\right) & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

③ Discuss the differentiability at origin of the functions:

ⓐ $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

ⓑ $f(x,y) = \begin{cases} y^3 \sin\left(\frac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Composite Function: Consider

$$z = f(x,y) \quad \text{——— (1)}$$

Let

$$\left.\begin{aligned} x &= \phi(t) \\ y &= \psi(t) \end{aligned}\right\} \text{— (A)} \qquad \text{or} \qquad \left.\begin{aligned} x &= \phi(u,v) \\ y &= \psi(u,v) \end{aligned}\right\} \text{— (B)}$$

The equations (1) & (A) or (1) & (B) are said to be composite function of t or u & v.

Chain Rule 1: Let $x(t)$ and $y(t)$ be differentiable functions. Let $f(x,y)$ have continuous first order partial derivatives. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Chain Rule 2: Let $f(x,y)$ have continuous first order partial derivatives. Suppose $x = x(u,v)$ and $y = y(u,v)$ are functions such that x_s, x_t, y_s and y_t are also continuous. Then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

Example: Let $z = e^x \sin y$, $x = st^2$ and $y = s^2 t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (e^x \sin y)t^2 + (e^x \cos y)(2st)$$

$$= t e^{st^2}\left(t \sin(s^2t) + 2s \cos(s^2t)\right).$$

Similarly,

$$\frac{\partial z}{\partial t} = s e^{st^2}\left(2t \sin(s^2t) + s \cos(s^2t)\right).$$

Exercises: ① Find z_x and z_y if $x^3+y^3+z^3+6xyz=1$

② Find $\frac{dy}{dx}$ if $y=y(x)$ is given by $y^2=x^2+\sin(xy)$

③ Find w_x if $w=x^2+y^2+z^2$ and $z=x^2+y^2$.

④ Given that $w=x^2+y^2+z^2$ and $z(x,y)$ satisfies $z^3-xy+yz+y^3=1$, evaluate $\frac{\partial w}{\partial x}$ at $(2,-1,1)$.

Definition: A function $f(x,y)$ is called homogeneous of degree n in a region $D\subseteq\mathbb{R}^2$ if for all $(x,y)\in D$, and for each positive λ,

$$f(\lambda x,\lambda y)=\lambda^n f(x,y)$$

Euler Theorem: Let D be a region in $\mathbb{R}^2$. Let $f: D\to\mathbb{R}$ have continuous first order partial derivatives. Then f is a homogeneous function of degree n if and only if

$$xf_x+yf_y=nf.$$

* Also, $x^2f_{xx}+2xy\,f_{xy}+y^2f_{yy}=n(n-1)f$.

Exercise: ① If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x\neq y$. Then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

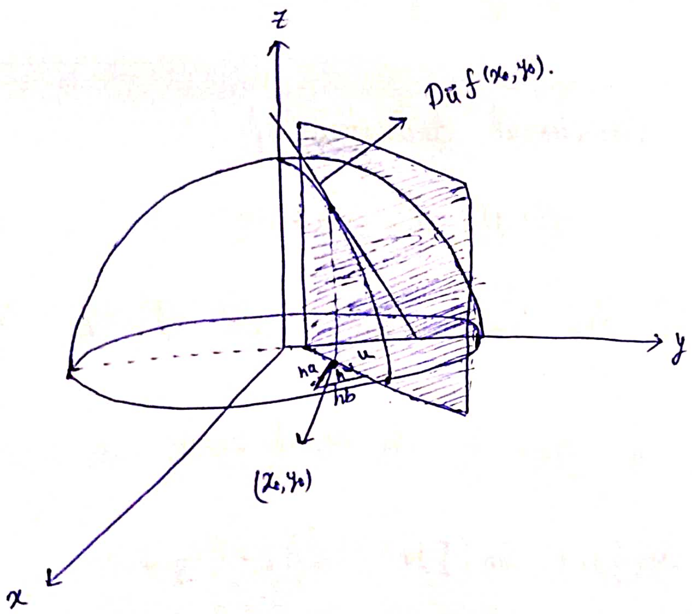
② Let $z=xy\,f\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$, where f and g are two times differentiable functions. The evaluate

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

Directional Derivatives: Let $f(x,y)$ be a function defined in a region D. The directional derivative of $f(x,y)$ in the direction of unit vector $u = a\hat{i} + b\hat{j}$ at $(x_0, y_0) \in D$ is given by

$$(D_u f)(x_0, y_0) = \left(\frac{df}{ds}\right)_u \Big|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0 + ha,\ y_0 + hb) - f(x_0, y_0)}{h}$$

* Note that f_x and f_y are rate of changes in $f(x,y)$ in the directions of x-axis and y-axis, respectively. However, $D_u f$ is the rate of change in $f(x,y)$ in the direction of a general unit vector $\vec{u}$.



Example: Find the derivative of $z = x^2 + y^2$ at $(1,2)$ in the direction $\vec{u} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$.

Soln:

$$(D_u z)(1,2) = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{\sqrt{2}},\ 2 + \frac{h}{\sqrt{2}}\right) - f(1,2)}{h} = \lim_{h \to 0} \frac{\sqrt{2}h + 2\sqrt{2}h}{h} = 3\sqrt{2}.$$

* Note that $f_x(1,2) \times \frac{1}{\sqrt{2}} + f_y(1,2) \times \frac{1}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$.

Theorem: Let $f(x,y)$ have continuous first order partial derivatives. Then $f(x,y)$ has a directional derivative at (x,y) in any direction $\hat{u} = a\hat{i} + b\hat{j}$ and is given by

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b = (f_x \hat{i} + f_y \hat{j}).(a\hat{i} + b\hat{j})$$
$$= \nabla f . \vec{u}.$$

* Note that $D_{\vec{u}} f(x_0,y_0) = \nabla f \big|_{(x_0,y_0)} \vec{u}.$
* Here, $\nabla f = f_x \hat{i} + f_y \hat{j}$ is called the gradient of f.

* ∇f is normal to the surface.

Caution: To apply above formula, we have assumed that f_x and f_y are continuous at (x_0, y_0), and $\vec{u}$ is a unit vector.

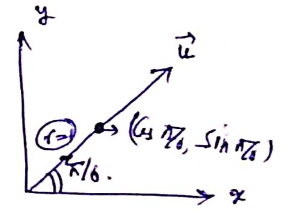
Example: Find the directional derivative of

$$f(x,y) = x^3 - 3xy + 4y^2$$

in the direction of the line that makes angle of $\pi/6$ with the x-axis.

Soln. The direction is given by the unit vector

$$\vec{u} = \cos\left(\tfrac{\pi}{6}\right) i + \sin\left(\tfrac{\pi}{6}\right) j = \tfrac{\sqrt{3}}{2} i + \tfrac{1}{2} j.$$



Thus.

$$D_u f(x,y) = f_x(x,y) . \tfrac{\sqrt{3}}{2} + f_y(x,y) . \tfrac{1}{2}$$
$$= \tfrac{1}{2}\left[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y\right].$$

Theorem: Let $f(x,y)$ has continuous first order partial derivatives. The maximum value of the directional derivative $D_u f(x,y)$ is $|\nabla f|$ and it is achieved when the unit vector $\hat{u}$ has the same direction as that of ∇f.

* $f(x,y)$ increases most rapidly in the direction of its gradient.
* $f(x,y)$ decreases most rapidly in the opposite direction of its gradient.
* $f(x,y)$ remains constant in any direction orthogonal to its gradient.

Exercises: ① How much the value of $y \sin x + 2yz$ change if the point (x,y,z) moves 0.1 units from $(0,1,0)$ towards $(2,2,-2)$?

Answer: -0.067 units

② Find the directions in which the function

$$f(x,y) = \frac{x^2+y^2}{2}$$

Change most, least, and not at all, at the point $(1,1)$.

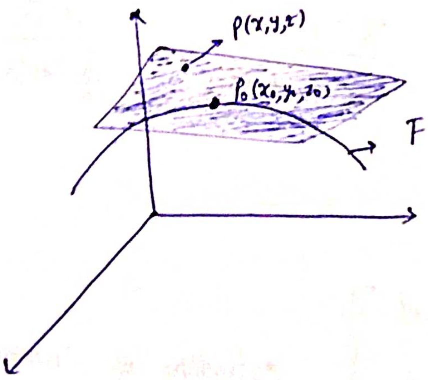
Answer: It is increase most at $(1,1)$ in the direction $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$

It is decrease most at $(1,1)$ in the direction $-\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$.

It does not change at $(1,1)$ in the direction $\pm\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$

Definition: A function of the form $F(x,y,z)=0$ is called the implicit function. And a function $z = f(x,y)$ is called explicit function.

Let $F(x,y,z)=0$ is an implicit function.



$$\vec{PP_0} = (x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}$$

$$\nabla f = f_x(x_0,y_0,z_0)\hat{i} + f_y(x_0,y_0,z_0)\hat{j} + f_z(x_0,y_0,z_0)\hat{k}$$

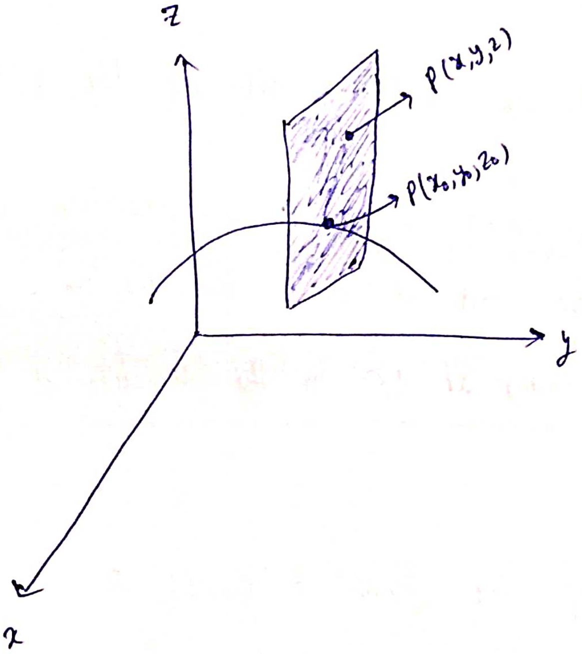
$$\nabla f \perp \vec{PP_0} \Rightarrow \boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)=0}$$

Equation of tangent plane at (x_0,y_0,z_0).

So, if $z = f(x,y)$ then $F(x,y,z) = f(x,y) - z = 0$ and the equation of tangent plane becomes

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0.$$

Next, for $F(x,y,z) = 0$, we have



$$\vec{PP_0} = (x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}$$

and $\nabla f = f_x(x_0,y_0,z_0)\hat{i} + f_y(x_0,y_0,z_0)\hat{j} + f_z(x_0,y_0,z_0)\hat{k}$

Note that $\nabla f \parallel \vec{PP_0} \Rightarrow \dfrac{x-x_0}{f_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{f_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{f_z(x_0,y_0,z_0)} = t$

So,

$$\left.\begin{aligned} x &= x_0 + f_x(x_0,y_0,z_0)t \\ y &= y_0 + f_y(x_0,y_0,z_0)t \\ z &= z_0 + f_z(x_0,y_0,z_0)t \end{aligned}\right\} \text{ equation of normal plane at } (x_0,y_0,z_0)$$

So, if $z = f(x,y)$ then $F(x,y,z) = f(x,y) - z = 0$ and the equation of normal plane becomes

$$x = x_0 + f_x(x_0,y_0)t$$
$$y = y_0 + f_y(x_0,y_0)t.$$
$$z = z_0 - t$$

Examples: ① Find the equation of tangent plane to $z = 3x^2 - xy$ at the point $(1,2,1)$.

Soln: Here, $z = f(x,y) = 3x^2 - xy$

$$f_x(1,2) = 6x - y\big|_{(1,2)} = 4$$

$$f_y(1,2) = -x\big|_{(1,2)} = -1$$

$$f(1,2) = 1$$

Hence, the equation of tangent plane is

$$4(x-1) - (y-2) - (z-1) = 0$$

$$\Rightarrow \quad 4x - y - z = 1.$$

② Find the parametric equations for the normal line to

$$x^2yz - y + z - 7 = 0$$

at the point $(1,2,3)$.

Soln: Here, $F(x,y,z) = x^2yz - y + z - 7 = 0.$

$$\nabla F = (2xyz,\ x^2z - 1,\ x^2y + 1)$$

$$\Rightarrow \nabla F\big|_{(1,2,3)} = (12, 2, 3)$$

Therefore,

$$x = 1 + 12t,\quad y = 2 + 2t,\quad z = 3 + 3t.$$

Exercises: ① Find the equation of normal line to $z = -x^2 - y^2 + 2$ at $(0,1)$.

$\boxed{\text{Answer: } x = 0,\ y = 1 - 2t,\ z = 1 - t}$

② Find the equation of tangent plane to $z = 4xy - x^4 - y^4$ at the point $(1,1,1)$.

$\boxed{\text{Answer: } z - 1 = 0.}$

Definition: Let $D \subseteq \mathbb{R}^2$ and $f: D \to \mathbb{R}$ be a function, and $(x_0, y_0) \in D$.

1. The function f is said to have a local maximum at $(x_0, y_0) \in D$ if

$$f(x,y) \leq f(x_0, y_0) \quad \forall\ (x,y) \in D_0 \cap D,$$

where D_0 is a neighbourhood of (x_0, y_0). In this case, $f(x_0, y_0)$ is called a local maximum value of f.

2. The function f is said to have a local minimum at $(x_0, y_0) \in D$ if

$$f(x,y) \geq f(x_0, y_0) \quad \forall\ (x,y) \in D_0 \cap D,$$

where D_0 is a neighbourhood of (x_0, y_0). In this case, $f(x_0, y_0)$ is called a local minimum value of f.

3. The function f is said to have a local extremum at $(x_0, y_0) \in D$ if it has a local maximum or local minimum at (x_0, y_0).

4. The function f is said to have absolute (global) maximum at (x_0, y_0) if

$$f(x,y) \leq f(x_0, y_0) \quad \forall\ (x,y) \in D.$$

In this case, $f(x_0, y_0)$ is called the absolute maximum value of f

5. The function f is said to have absolute (global) minimum at (x_0, y_0) if

$$f(x,y) \geq f(x_0, y_0), \quad \forall\ (x,y) \in D.$$

In this case, $f(x_0, y_0)$ is called the absolute minimum value of f.

(6) The function f is said to have an absolute extremum at $(x_0, y_0) \in D$ if it has an absolute maximum or absolute minimum at (x_0, y_0).

* Note that a local extremum point must be an interior point whereas an absolute extremum point need not be an interior point; it is allowed to be any point from D (may be boundary points).

Critical Point: An interior point (x_0, y_0) of D is a critical point of $f(x,y)$ if and only if either

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$$

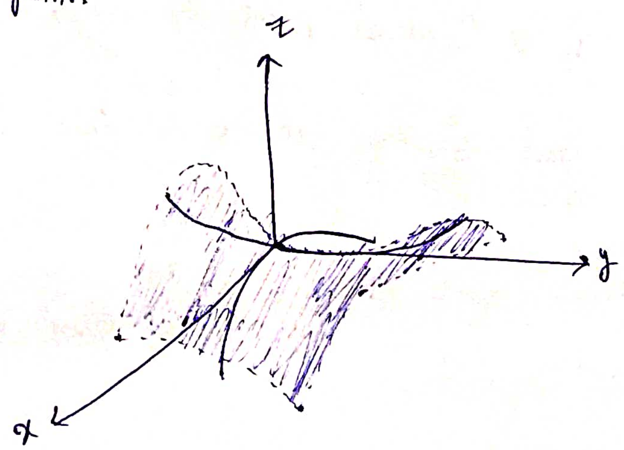
or

at least one of $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ does not exist.

Saddle Point: Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$ have continuous partial derivatives f_x and f_y. Let (x_0, y_0) be a critical point of $f(x,y)$. The point $(x_0, y_0, f(x_0, y_0))$ is called a saddle point of $f(x,y)$ if in every open disk centered at (x_0, y_0) and contained in D, there are points (x_1, y_1) and (x_2, y_2) such that

$$f(x_1, y_1) < f(x_0, y_0) < f(x_2, y_2)$$

* A critical point where f has no maxima or minima is called a saddle point.



Necessary Condition for a function to have extremum: Let $f(x,y)$ be continuous and have first order partial derivatives at a point (x_0, y_0). Then necessary conditions for the existence of an extreme value are

$$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$$

* The above condition is not sufficient.

Example: Find all critical points of the function

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$$

Soln:

$$f_x(x,y) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$f_y(x,y) = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow y = \pm 2$$

Therefore, critical points are $(\pm 1, \pm 2)$.

Exercise: Find all critical points of the function

$$f(x,y) = (x^2 - y^2)\, e^{-\frac{(x^2+y^2)}{2}}$$

$\boxed{\text{Answer: } (0,0),\ (\pm\sqrt{2}, 0),\ (0, \pm\sqrt{2})}$

Sufficient Condition for a function to have extremum: Let $f: D \to \mathbb{R}$ have continuous first and second order partial derivatives in an open disk centered at $(x_0, y_0) \in D$. Suppose (x_0, y_0) is a critical point of $f(x,y)$.

① If $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ at (x_0, y_0) then $f(x,y)$ has local maximum at (x_0, y_0)

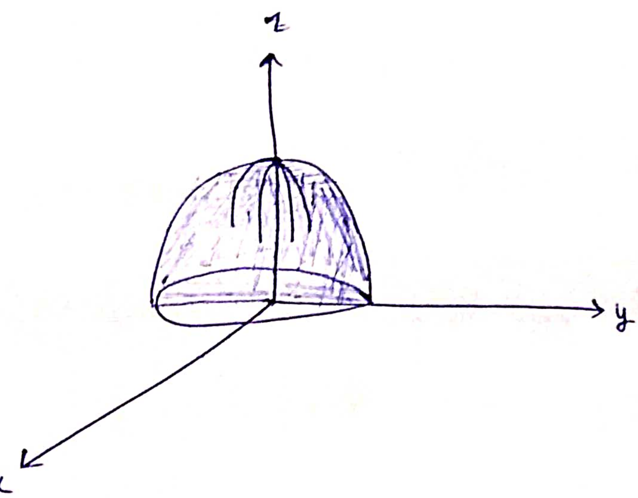
② If $f_{xx} f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at (x_0, y_0) then $f(x,y)$ has local minimum at (x_0, y_0).

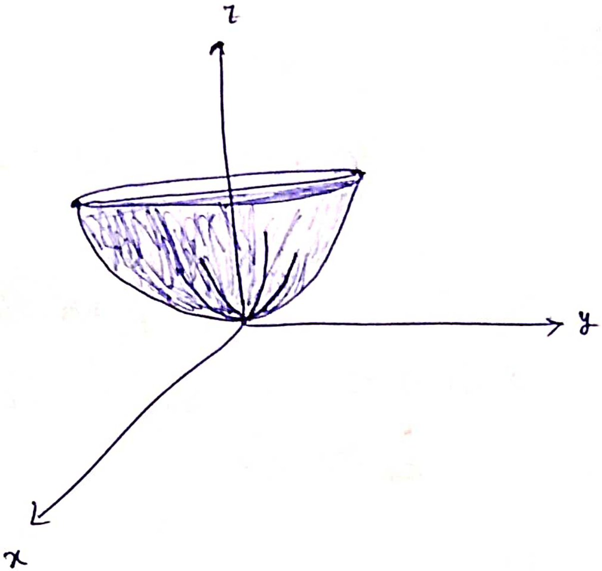
③ If $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (x_0, y_0) then $f(x,y)$ has saddle point at (x_0, y_0) (46)

④ If $f_{xx} f_{yy} - f_{xy}^2 = 0$, then nothing can be said, in general.

<u>Intution of Sufficient Condition</u>:

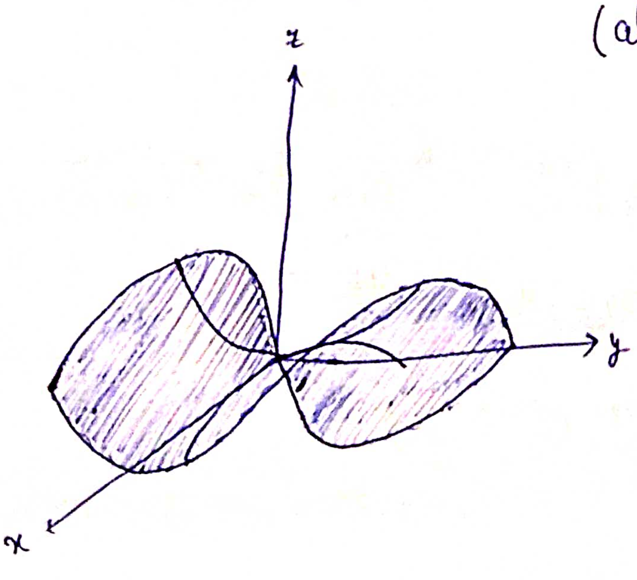
① $$\underbrace{f_{xx}}_{\text{(Concavity of } x)} \quad \underbrace{f_{yy}}_{\text{(Concavity of } y)} - \underbrace{f_{xy}^2}_{\text{(in diagonal direction)}} > 0 \Rightarrow f_{xx} \text{ and } f_{yy} \text{ have same sign.}$$





Both f_{xx} and f_{yy} have same sign (both positive or both negative).

② $f_{xx} f_{yy} - f_{xy}^2 < 0 \Rightarrow \underbrace{f_{xx} \text{ and } f_{yy} \text{ has oppositive sign.}}_{\text{(always saddle point)}}$



③ If f_{xx} and f_{yy} have same sign and $f_{xx} f_{yy} - f_{xy}^2 < 0 \Rightarrow$ saddle point.

That meas, the convexity or concavity does not dominated the direction convexity or concavity.

Examples: ① Find all critical points of

$$f(x,y) = x^3 - 6x^2 - 8y^2$$

and investigate their nature for local maximum/minimum and saddle point.

Soln:

$$f_x(x,y) = 0 \Rightarrow 3x^2 - 12x = 0 \Rightarrow x = 0, 4$$

$$f_y(x,y) = 0 \Rightarrow -16y = 0 \Rightarrow y = 0$$

Therefore, critical points are, $(0,0)$ and $(4,0)$.

Note that $f_{xx} = 6x - 12$, $f_{yy} = -16$, $f_{xy} = 0$.

At $(0,0)$,

$$f_{xx}(0,0) = -12, \quad f_{yy}(0,0) = -16, \quad f_{xy}(0,0) = 0$$

$$\Rightarrow \left. \begin{array}{l} f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) = 192 > 0 \\ \text{Also, } f_{xx}(0,0) < 0 \end{array} \right\} \Rightarrow (0,0) \text{ is a point of local maximum.}$$

At $(4,0)$,

$$f_{xx}(4,0) = 12, \quad f_{yy}(4,0) = -16, \quad f_{xy}(4,0) = 0$$

$$\Rightarrow f_{xx}(4,0)\, f_{yy}(4,0) - f_{xy}^2(4,0) = -192 < 0$$

$\Rightarrow (4,0)$ is a saddle point.

(2) Discuss local extremum of the function

$$f(x,y) = x^2 + y^2 - 3xy.$$

Soln: Note that

$$\left.\begin{aligned} f_x(x,y) = 0 &\Rightarrow 2x - 3y = 0 \\ f_y(x,y) = 0 &\Rightarrow 2y - 3x = 0 \end{aligned}\right\} \Rightarrow (0,0) \text{ is the critical point.}$$

Also,

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = -3.$$

So,

$$f_{xx}(0,0) = 2, \quad f_{yy}(0,0) = 2, \quad f_{xy}(0,0) = -3.$$

Next, Consider

$$f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0) = 4 - 9 = -5 < 0.$$

$\Rightarrow$ $(0,0)$ is the saddle point.

* Note here $f_{xx}(0,0) = 2 > 0$ and $f_{yy}(0,0) = 2 > 0$ but still, we have saddle point.

Exercises: (1) Discuss local extrema of the function

$$f(x,y) = (4x^2 + y^2)\, e^{-x^2 - 4y^2}$$

Answer:	$(0,0)$	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$(1,0)$	$(-1,0)$
	local minimum	saddle point	saddle point	local maximum	local maximum

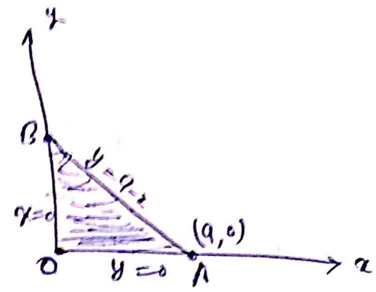
(2) Discuss local extremum of the function $f(x,y) = y^2 + x^2 y + x^4$.

Answer: $(0,0) \rightarrow$ test fails

Example: Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular plate in the first quadrant bounded by the lines $x=0$, $y=0$ and $y = 9-x$.



Soln: ① Interior Points:

$f_x(x,y) = 0 \Rightarrow 2-2x = 0 \Rightarrow x = 1$

$f_y(x,y) = 0 \Rightarrow 2-2y = 0 \Rightarrow y = 1$

$\Rightarrow (1,1)$ is an interior point.

Also, $f_{xx}(x,y) = -2$, $f_{yy}(x,y) = -2$, and $f_{xy}(x,y) = 0$

$$\Rightarrow \left.\begin{array}{l} f_{xx}(1,1)\, f_{yy}(1,1) - f_{xy}^2(1,1) = 4 > 0 \\ \text{and } f_{xx}(1,1) = -2 < 0 \end{array}\right\} f \text{ has local maximum at } (1,1).$$

② Boundary Points:

ⓐ Along OA: $f = 2 + 2x - x^2$

$f_x = 0 \Rightarrow 2 - 2x = 0 \Rightarrow \underline{x = 1}$

ⓑ along OB: $f = 2 + 2y - y^2$

$f_y = 0 \Rightarrow 2 - 2y = 0 \Rightarrow \underline{y = 1}$

ⓒ along AB: $f = -61 + 18x - 2x^2$

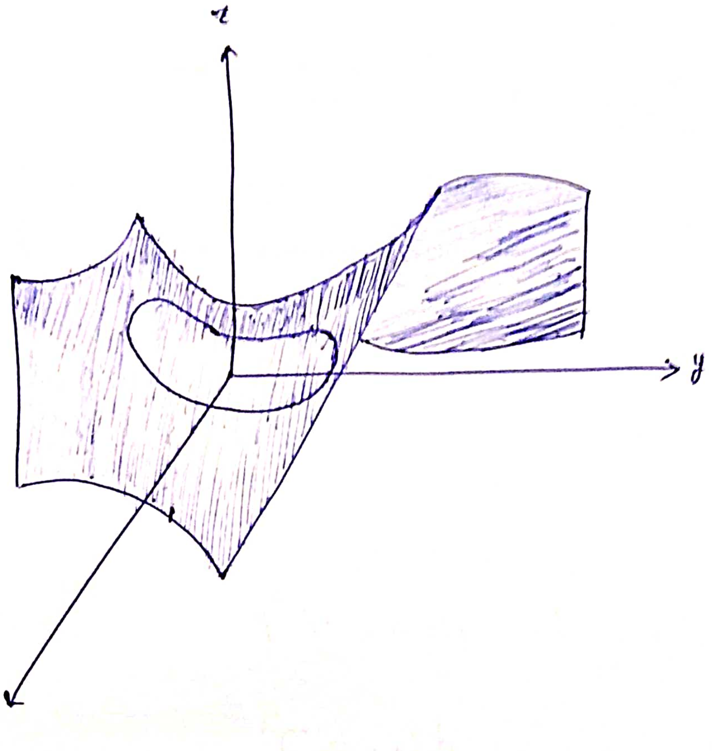
$f_x = 0 \Rightarrow x = \frac{9}{2} \Rightarrow \underline{y = \frac{9}{2}}$

Possible Points for extrema are as follows

(x,y)	$(1,1)$	$(0,0)$	$(1,0)$	$(9,0)$	$(0,1)$	$(0,9)$	$(\frac{9}{2},\frac{9}{2})$
$f(x,y)$	4	2	3	-61	3	-61	$-41/2$

$\Rightarrow$ The absolute maximum value is 4 and absolute minimum value is -61.

Method of Lagrange Multipliers: Find the maxima/minima of the function $z = f(x,y)$ with the constraint $g(x,y)=0$



Using chain rule

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

At the point of extrema

$$\frac{dz}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad \text{--- (1)}$$

The equation $g(x,y)=0$ is satisfied at any point and so at the point of extrema

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad \text{--- (2)}$$

We eliminate $\frac{dy}{dx}$ from (1) & (2), we get

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0, \qquad \text{where } \lambda = -\frac{f_y}{g_y}$$

Therefore,

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} &= 0 \end{aligned}\right\} \Rightarrow \boxed{\nabla f + \lambda \nabla g = 0}$$

and $\quad g(x,y) = 0$

<u>Working Rule</u>: Find maxima/minima of $z = f(x,y)$ with the constraint $g(x,y)=0$

Define an auxiliary equation

$$F(x,y,\lambda) = f(x,y) + \lambda\, g(x,y)$$

Necessary condition for extrema of F

$$\left.\begin{aligned} F_x = 0 &\Rightarrow f_x + \lambda g_x = 0 \\ F_y = 0 &\Rightarrow f_y + \lambda g_y = 0 \\ F_\lambda = 0 &\Rightarrow g(x,y) = 0 \end{aligned}\right\} \text{Compute the critical point using these equations.}$$

* The method of Lagrange's multiplier is used to obtain critical points where the max/min take place. We do not determine the nature of critical points (maxima/minima). No further test is required if we wish to find only absolute maximum and minimum.

* If we want to find maxima/minima for $z = f(x,y)$ with the constraints $g_1(x,y)=0$ and $g_2(x,y)=0$ then use

$$\left.\begin{aligned} f_x + \lambda g_{1x} + \mu g_{2x} &= 0 \\ f_y + \lambda g_{1y} + \mu g_{2y} &= 0 \\ g_1(x,y) &= 0 \\ g_2(x,y) &= 0 \end{aligned}\right\} \text{Compute the critical points using these equations.}$$

Examples: ① Find maximum/ minimum of the function

$$f(x,y) = x^2 - y^2 - 2x$$

in the region $x^2+y^2 \le 1$.

Soln: Local extrema in the interior $x^2+y^2<1$. Let

$$f(x,y) = x^2 - y^2 - 2x$$

$f_x(x,y)=0 \Rightarrow 2x-2=0 \Rightarrow x=1$
$f_y(x,y)=0 \Rightarrow -2y=0 \Rightarrow y=0$

$(1,0)$ is the critical point which does not belongs to $x^2+y^2<1$.

Next, local extrema on the boundary $x^2+y^2=1$.

Auxiliary function

$$F(x,y,\lambda) = (x^2-y^2-2x) + \lambda(x^2+y^2-1)$$

$F_x = 0 \Rightarrow x(1+\lambda) = 1$

$F_y = 0 \Rightarrow y(\lambda-1) = 0 \Rightarrow y=0$ or $\lambda=1$

$F_\lambda = 0 \Rightarrow x^2+y^2=1$

* If $y=0$ then $x=\pm 1$ and

$$\pm 1(1+\lambda) = 1 \Rightarrow \lambda = 0, -2.$$

So, $(1,0,0)$ and $(-1,0,-2)$ are required points.

* If $\lambda=1$ then $x=\frac{1}{2}$ and $y=\pm\frac{\sqrt{3}}{2}$

So, $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2})$ are required points.

Candidates for extrema: $(\pm 1, 0)$ $\left(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)$.

Therefore,

(x,y)	$(1,0)$	$(-1,0)$	$\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
$f(x,y)$	-1	3	$-3/2$

Hence, the maximum value of the function is 3 and minimum value of the function is $-\frac{3}{2}$.

② Find maximum and minimum values of the function

$$f(x,y) = x^2+y^2$$

in the region $(x-2)^2+(y-1)^2 \leq 20$.

Soln: Local extrema in the interior $(x-2)^2+(y-1)^2 < 20$.

$$\left.\begin{array}{l} f_x = 0 \Rightarrow x = 0 \\ f_y = 0 \Rightarrow y = 0 \end{array}\right\} \Rightarrow (0,0) \text{ is the critical point inside } (x-2)^2+(y-1)^2 \leq 20.$$

Local extrema on the boundary $(x-2)^2+(y-1)^2 = 20$.

$$F(x,y,\lambda) = (x^2+y^2) + \lambda\left((x-2)^2+(y-1)^2-20\right)$$

$F_x = 0 \Rightarrow x + \lambda(x-2) = 0 \Rightarrow x-2 = -\frac{2}{1+\lambda} \Rightarrow x = -2, 6$

$F_y = 0 \Rightarrow y + \lambda(y-1) = 0 \Rightarrow y-1 = -\frac{1}{1+\lambda} \Rightarrow y = -1, 3$

$F_\lambda = 0 \Rightarrow (x-2)^2+(y-1)^2 = 20 \Rightarrow \lambda+1 = \pm\frac{1}{2} \Rightarrow \boxed{\lambda = -\frac{1}{2}, -\frac{3}{2}}$

So, $(-2,-1)$ and $(6,3)$ are the required points.

(x,y)	$(0,0)$	$(2,1)$	$(6,3)$
$f(x,y)$	0	5	45

Hence, the maximum value is 45 and minimum value is 0.

Exercises: ① Find the maximum value of

$$f(x,y,z) = x + 2y + 3z$$

on the curve of intersection of the plane $g(x,y,z) = x - y + z - 1 = 0$ and the cylinder $h(x,y,z) = x^2 + y^2 - 1 = 0$.

$\boxed{\text{Answer: } 3 + \sqrt{29}}$

② Find the maximum of $w = xyz$ given that $xy + zx + yz = a$ for a given positive number a, and $x > 0$, $y > 0$, $z > 0$.

$\boxed{\text{Answer: } (a/3)^3}$

③ Determine the maximum value of $z = (x_1 \ldots x_n)^{\frac{1}{n}}$ provided that $x_1 + \cdots + x_n = a$, where a is a given positive number.

$\boxed{\text{Answer: } (a/n)}$