Electromagnetic Wave Theory

Lecture Notes: - Dr. Shyam

Vector Calculus

Del Operator ($\vec{\nabla}$)

$$\boxed{\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$

Vector field $\quad \vec{A} = \vec{A}(x, y, z)$

Scalar field $\quad V = V(x, y, z)$

Gradient/action of $\vec{\nabla}$ on scalar field

$$Grad(V) = \vec{\nabla} V = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) V$$

$$\vec{\nabla} V = \hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$$

Vector field

Two types of actions of $\vec{\nabla}$ on vector fields

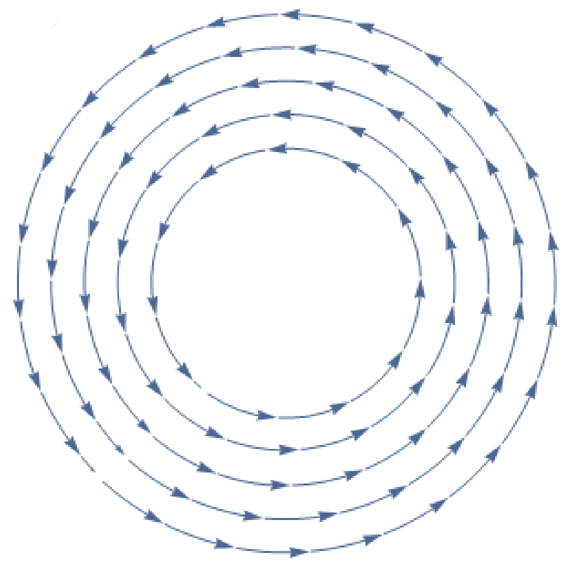
1. Curl of a vector field

$$\text{Let } \vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$$

$$curl\,(\vec{A}) = \vec{\nabla}\times\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right)$$

$$curl\,(\vec{A}) = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Vector field



Curl of a vector field signifies the existence of rotational characteristics in the field

If $\vec{\nabla}\times\vec{A} = 0$, then field is irrotational or Conservative.

1. Divergence of a vector field

Let $\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$

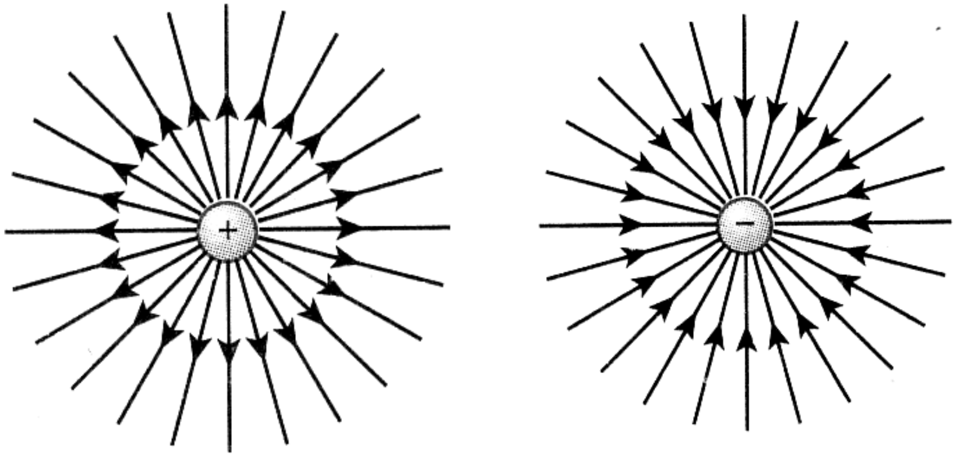
$$Div\,(\vec{A}) = \vec{\nabla}.\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right)$$

$$Div\,(\vec{A}) = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Scalar field

Divergence of vector is related with the flux density or the outflow/inflow of field lines

If $\vec{\nabla}.\vec{A} = 0$, then field is rotational or solenoidal or exists in closed loops.



Integral Notations

Open Line integral	$\int$
Closed Line integral	$\oint$
Open Surface integral	$\iint$
Closed Surface integral	$\oiint$
Volume integral	$\iiint$

Circulation of $\vec{A} = \oint \vec{A}.\vec{dl}$

$$curl\,(\vec{A}) = \vec{\nabla} \times \vec{A} = \frac{\oint \vec{A}.\vec{dl}}{S}$$

Stoke's Theorem

$$\boxed{\oint \vec{A}.\vec{dl} = \iint (\vec{\nabla} \times \vec{A}).\vec{dS}}$$

Flux of $\vec{A} = \oiint \vec{A}.\vec{dS}$

$$\text{Div}\,(\vec{A}) = \vec{\nabla}.\vec{A} = \frac{\oiint \vec{A}.\vec{dS}}{V}$$

Gauss Divergence Theorem

$$\boxed{\oiint \vec{A}.\vec{dS} = \iiint (\vec{\nabla}.\vec{A})dV}$$

Maxwell's Equations

Maxwell compiled four fundamental equations of Electricity and Magnetism and named them as Maxwell's Equations

1. Maxwell's First Equation

(Gauss Law in Electrostatics)

$$\boxed{\phi = \frac{q}{\epsilon_o}}$$

$$\phi = \oiint \vec{E}.d\vec{S}$$

$$q = \iiint \rho dV$$

$$\oiint \vec{E}.d\vec{S} = \frac{\iiint \rho dV}{\epsilon_o}$$

$$\vec{D} = \epsilon_o \vec{E}$$

$$\boxed{\oiint \vec{D}.d\vec{S} = \iiint \rho dV}$$

Integral Form

$$\oiint \vec{D}.d\vec{S} = \iiint \rho dV$$

$$\iiint (\vec{\nabla}.\vec{D}) dV = \iiint \rho dV$$

$$\vec{\nabla}.\vec{D} = \rho$$

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o}}$$

Differential Form

Characteristics

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o}}$$

1. Time independent equation

2. Describes Gauss Law in Electricity

3. Relates E with q or ρ

4. Signifies that q or ρ is the source for space variable E

2. Maxwell's Second Equation
(Gauss Law in Magnetism)

$$\boxed{\phi_m = 0}$$

$$\oiint \vec{B}.d\vec{S} = 0$$

$$\boxed{\oiint \vec{H}.d\vec{S} = 0}$$

$$\vec{B} = \mu_o \vec{H}$$

Integral Form

$$\iiint (\vec{\nabla}.\vec{B})dV = 0$$

$$\boxed{\vec{\nabla}.\vec{B} = 0}$$

$$\vec{\nabla}.\vec{H} = 0$$

Differential Form

Characteristics

$$\boxed{\vec{\nabla}.\vec{B} = 0}$$

1. **Time independent equation**
2. **Describes Gauss Law in Magnetism**
3. **Signifies that there is no source or sink for magnetic field line or magnetic field lines always exist in closed loops or magnetic monopole doesn't exist**

3. Maxwell's Third Equation

(Faradays Law of EMI)

$$\xi = -\frac{d\phi_m}{dt}$$

$$\oint \vec{E}.d\vec{l} = -\frac{d \iint \vec{B}.d\vec{S}}{dt}$$

$$\oint \vec{E}.d\vec{l} = -\iint \frac{\partial}{\partial t}\vec{B}.d\vec{S}$$

Integral Form

$$\iint (\vec{\nabla} \times \vec{E}).d\vec{S} = -\iint \frac{\partial}{\partial t}\vec{B}.d\vec{S}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}}$$

Differential Form

Characteristics

1. **Time dependent equation**
2. **Describes Faraday's Law of EMI**
3. **Relates *E* with *B***
4. **Signifies that the space variable *E* can be generated with time variation in *B***

4. Maxwell's 4th Equation

(Modified Ampere circuital

$$\oint \vec{B}.d\vec{l} = \mu_o(I_c + I_D)$$

$$I_c = \frac{dq}{dt} = \iint \vec{J}.d\vec{S}$$

$$I_D = \epsilon_o \frac{d\phi_e}{dt} = \epsilon_o \frac{d \iint \vec{E}.d\vec{S}}{dt}$$

$$\oint \vec{B}.d\vec{l} = \mu_o\left(\iint \vec{J}.d\vec{S} + \frac{d \iint \vec{D}.d\vec{S}}{dt}\right)$$

$$\boxed{\oint \vec{B}.d\vec{l} = \mu_o \iint \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).d\vec{S}}$$

Integral Form

$$\iint (\vec{\nabla} \times \vec{B}).d\vec{S} = \mu_o \iint \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).d\vec{S}$$

$$\vec{\nabla} \times \vec{B} = \mu_o\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

Differential Form

Characteristics

1. **Time dependent eqn.**
2. **Describes Modified Amper-Circuital law**
3. **Relates *H* with *D***
4. **Signifies that space variable *H* can be generated with time variation in *D and J* individually or jointly**

Maxwell's Equations

Differential Forms of Equations are

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o}$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Absence of magnetic monopole makes all the differences

But, In Free Space

$\boxed{\rho = 0}$ $\boxed{J = 0}$

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Clear similarities & parallelism between Electricity & Magnetism

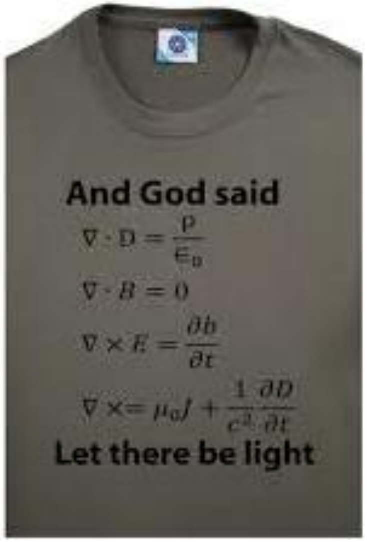
Significance of Maxwell's equations

Maxwell's equations are fundamental equations of electricity & magnetism, just like Newton's Laws in mechanics.

Newton's Laws require significant modifications in relativistic & microscopic limit, but Maxwell's equations are invariant in all regimes.

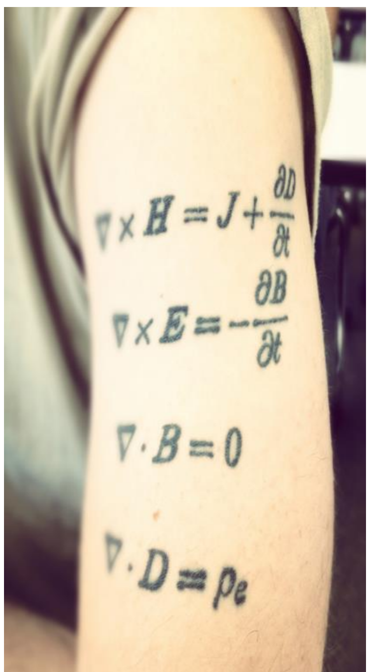
These equations find their immense applications in circuit analysis, communication systems, design and study of antenna, receivers, transmission channels, signal propagations etc.

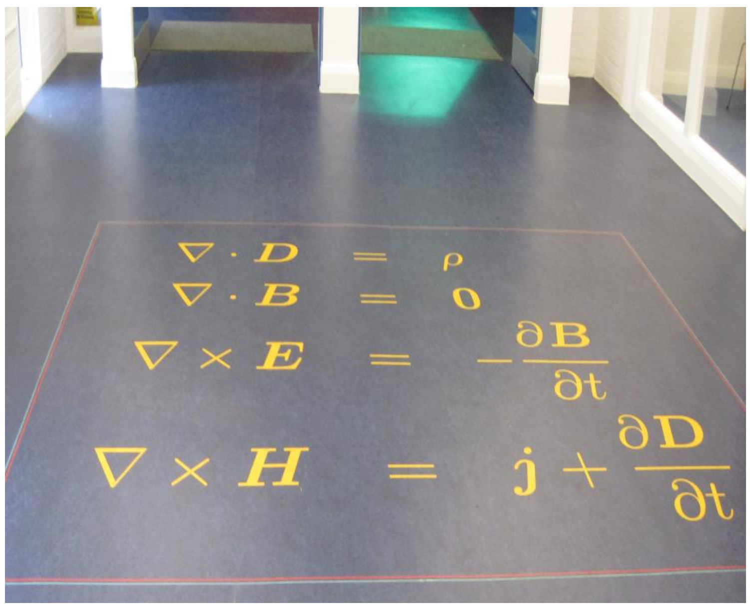
Symmetrical structures of these equations led Maxwell to predict the existence of EM wave

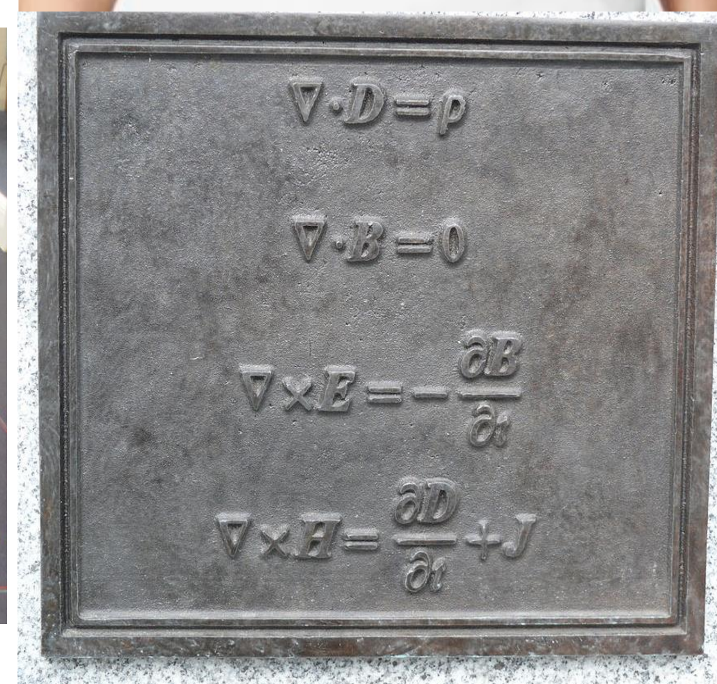

And God said
$\nabla \cdot D = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot B = 0$
$\nabla \times E = \frac{\partial b}{\partial t}$
$\nabla \times = \mu_0 J + \frac{1}{c^2}\frac{\partial D}{\partial t}$
Let there be light


And God Said
$\nabla \cdot \vec{D} = \rho_{\text{free}}$
$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\nabla \times \vec{H} = \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t}$
And then there was light.


In the beginning God said:
$\nabla \cdot B = 0$
$\nabla \cdot D = \rho_v$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = J + \frac{\partial D}{\partial t}$
and there was light


$\nabla \times H = J + \frac{\partial D}{\partial t}$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \cdot B = 0$
$\nabla \cdot D = \rho_e$


$\nabla \cdot \boldsymbol{D} = \rho$
$\nabla \cdot \boldsymbol{B} = 0$
$\nabla \times \boldsymbol{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \boldsymbol{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$


$\nabla \cdot D = \rho$
$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = \frac{\partial D}{\partial t} + J$

Electromagnetic Wave Equation

Every travelling wave with equation $y = r\sin(\omega t - kx)$ **travelling with** $v = \frac{\omega}{k}$**, obeys the following differential equation:**

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

<u>In Free Space</u>

<u>Maxwells eqns are:</u>

$$\vec{\nabla}.\vec{E} = 0$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{\nabla} \times \vec{B}$$

$$\boxed{\vec{a} \times \vec{b} \times \vec{c} = \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b})}$$

$$\vec{\nabla}(\vec{\nabla}.\vec{E}) - \vec{E}(\vec{\nabla}.\vec{\nabla}) = -\mu_o \frac{\partial}{\partial t}\vec{\nabla} \times \vec{H}$$

$$0 - \nabla^2\vec{E} = -\mu_o \frac{\partial}{\partial t}\frac{\partial \vec{D}}{\partial t}$$

$$\nabla^2\vec{E} = \mu_o \varepsilon_o \frac{\partial}{\partial t}\frac{\partial \vec{E}}{\partial t}$$

$$\boxed{\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

Here $$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

This represents a differential equation of a travelling wave of oscillations of $\vec{E}$ travelling with speed of light

Similarly, if we proceed in same manner for

$$\vec{\nabla} \times \vec{H} = \frac{\partial\vec{D}}{\partial t}$$

then we get another differential equation for a travelling wave of oscillations of $\vec{H}$ also travelling with speed of light, as:

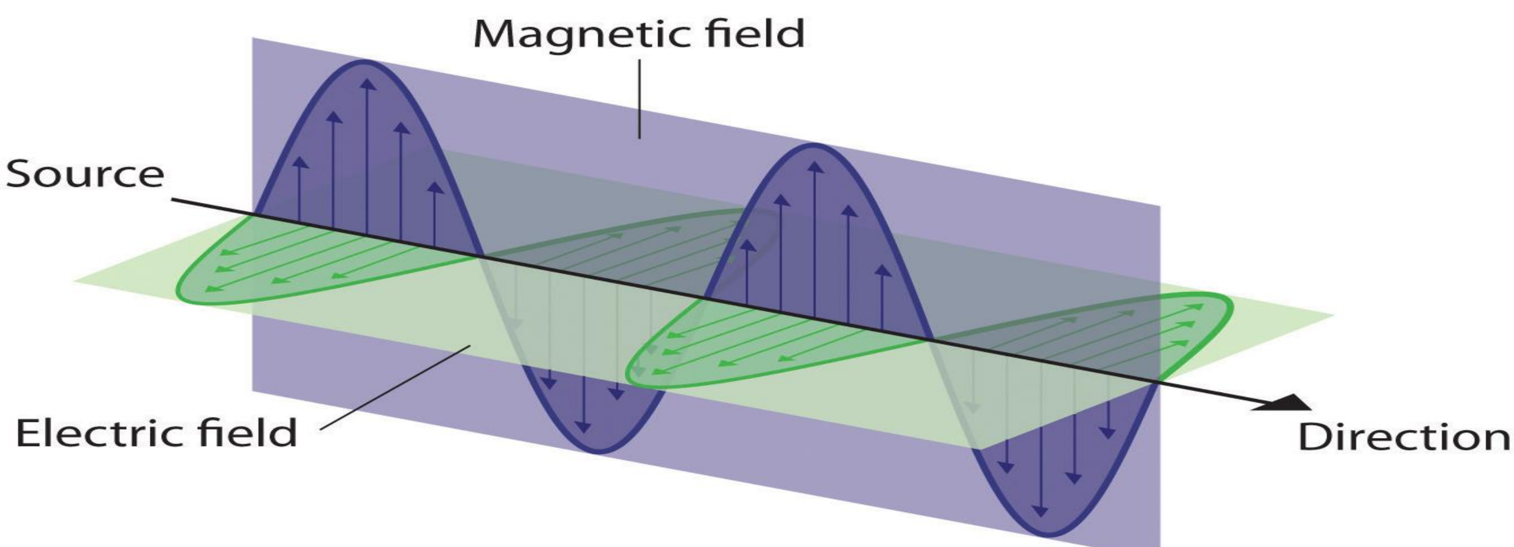
$$\boxed{\nabla^2\vec{H} = \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}}$$

Concept of Electromagnetic Waves

Maxwell's equations result into the following wave equations

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \qquad \nabla^2\vec{H} = \frac{1}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} \qquad c = \frac{1}{\sqrt{\mu_o\varepsilon_o}}$$

From these equations, Maxwell's concluded that variation in *E* with time generates *H* and vice versa. So oscillations in one of these vector generates a coupled wave of *E* and *H* such that these oscillation are not localized but travel in space with speed of light in the form of waves called *Electromagnetic waves*



Poynting Vector ($\vec{S}$)

It is defined as the cross product of $\vec{E}$ and $\vec{H}$

$$\vec{S} = \vec{E} \times \vec{H}$$

Physical Significance of $\vec{S}$

$$\vec{S} = \vec{E} \times \vec{H} = Ey\,\hat{j} \times Hz\,\hat{k} = Ey\,H_z\hat{i}$$

Units of $\vec{S}$ $\quad Vm^{-1}Am^{-1} = VA\,m^{-2} = W\,m^{-2} = Js^{-1}\,m^{-2}$

From the direction and units of $\vec{S}$ it is clear that Poynting vector represents the amount of EM energy flowing per sec per unit area in the direction of propagation of EM wave. Or it is power flux carried by EM waves

$$\boxed{S = \frac{Power}{Area}}$$

$$|\vec{S}| = |\vec{E} \times \vec{H}| = EH \qquad \mathbf{S} = EH$$

Average value of Poynting Vector

$$\langle \mathbf{S} \rangle = E_{rms} H_{rms} = \frac{E_o H_o}{2} = \frac{1}{2} c\varepsilon_o E_0{}^2 = \frac{1}{2} c\mu_o H_0{}^2$$

$$\langle \mathbf{S} \rangle = \frac{Average\ Power}{Area}$$

$$\mathbf{S} = EH = c\varepsilon_o E^2 = c\mu_o H^2 = \frac{1}{2} c\varepsilon_o E_0{}^2 + \frac{1}{2} c\mu_o H_0{}^2$$

i.e. total energy is equally distributed between electric field component and magnetic field component in EM wave

$$\vec{\mathbf{E}} = E_0 \sin(kx - \omega t)\hat{\mathbf{j}}, \qquad \vec{\mathbf{B}} = B_0 \sin(kx - \omega t)\hat{\mathbf{k}}$$