Note Title 02-10-2020

MA 323

Lecture # 08

§ Processes with Jumps:

It has been observed that the empirical distribution of log of price process is leptokurtic, i.e., it has high peak and heavy tails.

Accordingly, we discuss models to include jumps.

Jump- Diffusion Model:

Merton introduced the first models with both jump and diffusion terms, which can be specified by the SDE:

$$\frac{dS(t)}{S(t-)} = \mu\, dt + \sigma\, dW(t) + dJ(t)$$

Where $J(t)$ is a process independent of $W(t)$ with piecewise constant sample paths, and given by,

$$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1)$$

Where $Y_1, Y_2, \ldots Y_{N(t)}$ are random variables and $N(t)$ is a counting process. The meaning of this is that there are random arrival times $0 < \tau_1 < \tau_2 < \cdots\cdots$

and $N(t) = \sup\{n \mid \tau_n \leq t\}$ counts the number of arrivals in $[0, t]$. The symbol $dJ(t)$ stands for the jump in J at time t.

The size of this jump is $Y_j - 1$, if $t = \tau_j$ and 0 if t is not coincident with any of the τ_j.

We now need to specify the meaning of $S(t)$ when a jump is involved.

Accordingly, the value just before a potential jump is $S(t-)$ defined as $S(t-) = \lim_{u \uparrow t} S(u)$ (Left Hand Limit)

Therefore, we can write,

$$dS(t) = \mu S(t-)\,dt + \sigma S(t-)\,dW(t) + S(t-)\,dJ(t).$$

The meaning of this is that the increment $dS(t)$ in S "at" t is dependent on the value of S, just before a potential jump at t and not on the value after the jump. The jump in S at time t is $S(t) - S(t-)$, which is 0 unless J jumps at t, that is, $t = \tau_j$, for some j.

The jump at τ_j is

$$S(\tau_j) - S(\tau_j-) = S(\tau_j-)\left[J(\tau_j) - J(\tau_j-)\right]$$

$$= S(\tau_j-)(Y_j - 1)$$

Hence, $S(\tau_j) = S(\tau_j-)\,Y_j$.

Consequently, this shows that Y_j are the ratios of the asset price before and after a jump.

This also explains why we wrote $Y_j - 1$ instead of Y_j in the definition of $J(t)$.

Obviously the Y_j values must be positive.

Note that: $\log S(\tau_j) = \log S(\tau_j^-) + \log Y_j$ (Additive in Log)

Now the solution to the jump diffusion SDE is given by:

$$S(t) = S(0)\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)} \prod_{j=1}^{N(t)} Y_j$$

We now bring forth the distributional assumptions about the jump process $J(t)$, and the simplest model for $N(t)$ is the Poisson process with rate λ.

As a result, the interarrival times i.e., $\tau_{j+1} - \tau_j$ are independent and have a common exponential distribution

$$P(\tau_{j+1} - \tau_j \leq t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

We further assume that the Y_j are i.i.d and independent of both $N(t)$ and $W(t)$.

Finally, in Merton's model, Y_j (size of the jump) is lognormally distributed.

We now consider two approaches to simulating the jump diffusion model, namely (1) Simulating at fixed dates and (2) Simulating jump times.

(1) Simulating at fixed dates: In this approach, we simulate the process at a fixed set of dates $0 = t_0 < t_1 < \dots < t_n$.

We assume that N is a Poisson process, that $Y_1, Y_2, \dots$ are i.i.d and that N, W and $\{Y_1, Y_2, \dots\}$ are mutually independent.

No assumption is made about a specific distribution for Y_j.

To simulate $S(t)$ at time $t_1, t_2, \dots, t_n$ we use the recursive relation

$$S(t_{i+1}) = S(t_i)\, e^{(\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma(W(t_{i+1}) - W(t_i))} \prod_{j=N(t_i)+1}^{N(t_{i+1})} Y_j$$

with the product over j being equal to 1 if $N(t_{i+1}) = N(t_i)$.

Alternatively, we can simulate $X(t) = \log(S(t))$ as:

$$X(t_{i+1}) = X(t_i) + \left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma\left(W(t_{i+1}) - W(t_i)\right)$$

$$+ \sum_{j=N(t_i)+1}^{N(t_{i+1})} \log Y_j$$

We can then exponentiate the simulated values of $X(t_i)$ to produce samples of $S(t_i)$.

Algorithm for Generating $X(t_{i+1})$.

(1) Generate $Z \sim N(0,1)$

(2) Generate $N \sim \text{Poisson}\left(\lambda(t_{i+1}-t_i)\right)$

If $N=0$, set $M=0$ and go to step (4)

(3) Generate $\log Y_1, \log Y_2, \dots \log Y_N$ from their common distribution and set $M = \log Y_1 + \log Y_2 + \cdots + \log Y_N$

(4) Set $X(t_{i+1}) = X(t_i) + \left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1}-t_i) + \sigma\sqrt{t_{i+1}-t_i}\, Z_i + M$

Note that here, we have used the property that $N(t_{i+1}) - N(t_i)$

has a Poisson distribution with mean $\lambda(t_{i+1} - t_i)$, and it is independent of increments of N over $[0, t_i]$.

(2) Simulating Jump Times: Observe that the preceding approach does not identify the times at which $S(t)$ jumps, and instead uses the Poisson distribution to generate the total number of jumps in each interval.

Accordingly, we now resort to the alternative approach of

Simulating the jump times, $\tau_1, \tau_2, \ldots$ explicitly.

From one jump to the next, $S(t)$ evolves like an ordinary geometric Brownian motion (since W and J have been assumed to be independent). This gives us

$$S(\tau_{j+1}^-) = S(\tau_j)\, e^{(\mu - \frac{1}{2}\sigma^2)(\tau_{j+1} - \tau_j) + \sigma(W(\tau_{j+1}) - W(\tau_j))}$$

and $S(\tau_{j+1}) = S(\tau_{j+1}^-)\, Y_{j+1}$

Taking logarithm on both sides, we obtain

$$X(\tau_{j+1}) = X(\tau_j) + \left(\mu - \frac{1}{2}\sigma^2\right)(\tau_{j+1} - \tau_j) + \sigma\left[W(\tau_{j+1}) - W(\tau_j)\right] + \log Y_{j+1}$$

Algorithm for Generating $X(t_{i+1})$:

1. Generate R_{j+1} from the exponential distribution with mean $\frac{1}{\lambda}$

(2) Generate $Z_{j+1} \sim N(0,1)$

(3) Generate $\log Y_{j+1}$

(4) Set $\tau_{j+1} = \tau_j + R_{j+1}$ and

$$X(\tau_{j+1}) = X(\tau_j) + \left(\mu - \frac{1}{2}\sigma^2\right) R_{j+1} + \sigma \sqrt{R_{j+1}}\, Z_{j+1} + \log Y_{j+1}.$$

Recall that R_{j+1} can be generated setting $R_{j+1} = -\dfrac{\log(U)}{\lambda}$

where $U \sim \mathcal{U}[0,1]$.

A COMBINED APPROACH

The two approaches to simulate $S(t)$ can be combined.

Suppose that we fix a date t in advance that we would like to include among the simulated dates. Suppose that it happens that $\tau_j < t < \tau_{j+1}$.

$$S(t) = S(\tau_j)\, e^{(\mu - \frac{1}{2}\sigma^2)(t - \tau_j) + \sigma(W(t) - W(\tau_j))}, \text{ and,}$$

$$S(\tau_{j+1}) = S(t)\, e^{(\mu - \frac{1}{2}\sigma^2)(\tau_{j+1} - t) + \sigma(W(\tau_{j+1}) - W(t))} Y_{j+1}.$$