Arithmetic Sequences solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-3

E1. Find the value of the expression $100-98+96-94+92-90+\cdots+8-6+4-2.$

We can group each subtracting pair together: $(100-98)+(96-94)+(92-90)+\ldots+(8-6)+(4-2).$

After subtracting, we have: $2+2+2+\ldots+2+2=2(1+1+1+\ldots+1+1).$ There are 50 even numbers, therefore there are 25 pairs. Therefore the sum is $2\cdot 25=\boxed{50}$

E2. The sum of 25 consecutive even integers is $10,000$. What is the largest of these 25 consecutive integers?

Let n be the 13th consecutive even integer that's being added up. By doing this, we can see that the sum of all 25 $(n-2k)+\cdots+(n-4)+(n-2)+(n)+(n+2)+(n+4)+\cdots+(n+2k)=25n$. Now, $25n=10000$, so $n=400$. Remembering that this is the 13th integer, we wish to find the 25th, which is $400+2(25-13)=\boxed{424}$.

E3. If the 7th term of an arithmetic sequence is 24 and the 12th term is 48, then what is the 57th term?

Going from the 7th term to the 12th term of an arithmetic sequence requires adding the common difference to the 7th term a total of $12–7=5$ times. Therefore we know $48=24+5d5\Rightarrow 24=5d\Rightarrow d=24/5$. To get to the 57th term from the 7th term, the common difference of $24/5$ must be added $57-7=50$ times, resulting in $24+(24/5)(50)=24+240=\boxed{264}$.

EXERCISES 3-5

E4. The first two terms of an arithmetic sequence are 4, 10. What is the first term greater than 1000?

Knowing the first two terms of the arithmetic sequence are 4, 10 tells us the common difference is $10 - 4 = 6$. To get from 4 to 1000, we will need to add $1000 - 4 = 996$. This means we will have to start with 4 and add 6 a total of $996/6 = 166$ times. This will get us to exactly 1000. We will first exceed 1000 by starting with 4 and adding 6 a total of 167 times. The result is $4 + 6 \cdot 167 = \boxed{1006}$.

E5. The sequence an has the property that $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. It is also true that $a_0 = 4$ and $a_4 = 26$. What is the value of a_5?

$a_0 = 4;$

$a_1 = a_1;$ [an as yet unknown value]

$a_2 = a_1 + 2a_0 = a_1 + 8;$

$a_3 = a_2 + 2a_1 = a_1 + 8 + 2a_1 = 3a_1 + 8;$

$a_4 = a_3 + 2a_2 = 3a_1 + 8 + 2(a_1 + 8) = 5a_1 + 24 = 26,$

so $a_1 = \dfrac{26 - 24}{5} = \dfrac{2}{5}$ and $a_3 = \dfrac{6}{5} + 8 = \dfrac{46}{5};$

$$a_5 = a_4 + 2a_3 = 26 + 2 \times \frac{46}{5} = \frac{130 + 92}{5} = \boxed{\frac{222}{5}}.$$

EXERCISES 6-8

E6. The 2020th term of an arithmetic sequence is $\frac{21}{19}$ times the 2022nd term of the arithmetic sequence. What is the value of the 2024th term of this sequence divided by the first term of this sequence?

We can suppose that the 2020th term of the arithmetic sequence is 21 and the 2022nd term of the arithmetic sequence is 19. The common difference of this arithmetic sequence is $\frac{19-21}{2} = -1$. This means that the first term of the arithmetic sequence is $21 - 2019d = 2040$ and the 2024th term of the arithmetic sequence is $19 + 2d = 17$. The ratio of these terms is $\frac{17}{2040} = \boxed{\frac{1}{120}}$.

E7. What is the value of $1+2+3-4+5+6+7-8+\cdots+197+198+199-200$?

$1+2+3-4+5+6+7-8+\cdots+197+198+199-200 =$
$(1+2+3+4+\cdots+200)-2\cdot(4+8+12+\cdots+200) =$
$= \frac{(1+200)\cdot 200}{2} - 2\cdot 4\cdot \frac{(1+50)\cdot 50}{2} =$
$= 20100 - 10200 = \boxed{9900}$.

E8. What is the greatest number of consecutive integers whose sum is 45?

The problem says that they can be integers, not necessarily positive. Observe also that every term in the sequence $-44, -43, \cdots, 44, 45$ cancels out except 45. Thus, the answer is $\boxed{90}$ integers.

CHALLENGE PROBLEMS 1-2

CP1. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

Factor $345 = 3 \cdot 5 \cdot 23$.
Suppose we take an odd number k of consecutive integers, with the median as m. Then $mk = 345$ with $\frac{1}{2}k < m$. Looking at the factors of 345, the possible values of k are $3, 5, 15, 23$ with medians as $115, 69, 23, 15$ respectively.
Suppose instead we take an even number $2k$ of consecutive integers, with median being the average of m and $m+1$. Then $k(2m+1) = 345$ with $k \leq m$. Looking again at the factors of 345, the possible values of k are $1, 3, 5$ with medians $(172, 173), (57, 58), (34, 35)$ respectively.
Thus the answer is $\boxed{7}$.

CP2. 852 digits are used to number the pages of a book consecutively from page 1. How many pages are there in the book?

One-digit numbers contribute only 1 digit, two-digit numbers 2, and so on. Thus we treat each type separately. The 9 one-digit numbers each contribute their 1, for a total 9 digits; there are $852 - 9 = 843$ to go. The 90 two-digit numbers from 10 to 99 each contribute 2 digits, for a total of 180; $843 - 180 = 663$ to go. The 900 three-digit numbers from 100 to 999 contribute 3 digits each, for 2700 digits. Thus all the remaining 663 are three-digit numbers, so 221 $(= 663/3)$ are used. We add these 221 to the last two-digit, 99, so that the last number used is $99 + 221 = \boxed{320}$.

CHALLENGE PROBLEMS 3-4

CP3. For every n the sum of n terms of an arithmetic progression is $2n+3n^2$. What is the rth term of the sequence in terms of r?

The sum of the first n terms is always $\frac{n}{2}(2a+(n-1)d)$, so we have

$$\frac{n}{2}(2a+(n-1)d) = an + dn^2/2 - dn/2.$$

Thus $d/2 = 3$, so $d = 6$, and $a - d/2 = 2$, so $a = 5$. The rth term is thus

$$a + (r-1)d = 5 + 6(r-1) = \boxed{6r-1}.$$

CP4. A sequence of natural numbers is constructed by listing the first 4, then skipping one, listing the next 5, skipping 2, listing 6, skipping 3, and on the nth iteration, listing $n+3$ and skipping n. The sequence begins 1, 2, 3, 4, 6, 7, 8, 9, 10, 13. What is the 1000th number in the sequence?

If we list the rows by iterations, then we get

$1, 2, 3, 4$

$6, 7, 8, 9, 10$

$13, 14, 15, 16, 17, 18$ etc.

so that the 1000th number is the 16th number on the 42th row because

$$4+5+6+7+\cdots+44 = \frac{(4+44)\cdot 41}{2} = 984.$$

The last number of the 41th row (when including the numbers skipped) is $984 + (1+2+3+4+\cdots+41) = 1845$, (we add the $1-41$ because of the numbers we skip) so our answer is $1845 + 16 = \boxed{1861}$.