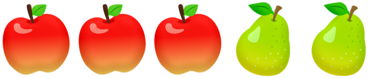
Combinatorics 101

Graham Middle School Math Olympiad Team

RULE OF SUM

Consider the following Kindergarten problem:

There are two plates, one with 3 apples and one with 2 pears. You can take *only one piece of fruit*. How many options do you have?



A solution is fairly simple. You can take one of 3 apples or one of 2 pears, so there are

$$3 + 2 = 5$$

different options for you to take one piece of fruit.

We have just used an operation called the **rule of sum** or **addition principle**. Stated simply, it is the intuitive idea that

if we have a number of ways of doing something and b number of ways of doing another thing and we can not do both at the same time, then there are $a + b$ ways to choose one of the actions.

Of course, this principle may be extended for many options, for example, if you have 5 plates with 10 apples on each, there are

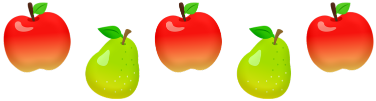
$$10 + 10 + 10 + 10 + 10 = 50$$

options to pick one apple.

RULE OF PRODUCT

Now we will grow a little more, and here is the problem from 3rd grade:

There are two plates, one with 3 apples and one with 2 pears. You can take *one apple* and *one pear*. How many options do you have?



A solution is also not that hard. You can take one of 3 apples, so you have 3 options for the first choice, then for each choice, you have 2 options to select one pear. So you have

$$3 \times 2 = 6$$

different options for you to take one apple and one pear.

We've just used the **rule of product** or **multiplication principle**. Stated simply, it is the idea that

if there are a ways of doing something and b ways of doing another thing, then there are $a \times b$ ways of performing both actions.

Of course, this principle may be extended for many options, for example, if you have 5 plates with 10 apples on each, there are

$$10 \times 10 \times 10 \times 10 \times 10 = 100,000$$

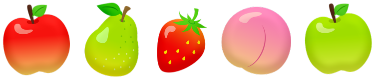
options to pick one apple from each plate.

Both the *rule of sum* and the *rule of product* are called ***basic counting principles***.

PERMUTATIONS

Now move to a problem worth 5th-grade.

There are 5 different fruits on a plate: *a red apple*, *a pear*, *a strawberry*, *a peach*, and *a green apple*. You have to take **all of them** *one by one*. In how many ways can you do that?



The first fruit can be any of these, so we have 5 options to take it.
There are 4 fruits left, and now we have 4 options.
Then 3 fruit left and we have 3 options.
For 2 leftover fruits, we have only 2 options and for the last fruit, we have only 1 option to choose from.
By the **rule of product**, there are

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

ways to pick 5 fruits one by one.

In combinatorics,

a **permutation** is an ordering of a list of objects. The *permutation* of all objects in the list is also called an **arrangement**.

To find a number of permutations of n distinct objects (denoted as P_n), we need, according to the *rule of product*, to multiply the following numbers: $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$. This product is called the **factorial** and denoted by $n!$. So for any positive integer n,

$$n! = 1 \times 2 \times \ldots \times (n-1) \times n,$$

In words:

$n!$ is the product of all positive integers from 1 to n.

We also define $0! = 1$, because there is one permutation of the empty list of objects. It is the empty permutation.

RULE OF DIVISION

Let's simplify the previous problem.

There are 5 different fruits on a plate: *a red apple*, *a pear*, *a strawberry*, *a peach*, and *a green apple*. You have to **take** 3 **of them** all *one by one*. In how many ways can you do that?

The solution would be similar to the previous one, except we will stop on 3rd step. So there are

$$5 \times 4 \times 3 = 60$$

ways to pick 3 fruits one by one.
Another way to think about the solution, that considers all possible ways to pick two remaining fruits as one option. So we will get the same result by the following operation:

$$\frac{5 \times 4 \times 3 \times 2 \times 1 \text{ total number of ways}}{2 \times 1 \text{ ways to get remainder}} = \frac{5!}{2!} = 60.$$

The second approach uses the **rule of division**:

if *some* options are considered the same, we can **divide** the *total number* of options by the *number of options* we considered the same.

So if we have n objects and want to arrange k of them in a row, there are $\frac{n!}{(n-k)!}$ ways to do this.
This is also known as a k-permutation of n, and is denoted by P_k^n or $P(n, k)$.

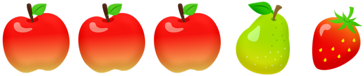
The rule of division also works in the case when we need to put 5 fruits not in a line, but in a circle. In this case, these 5 permutations are the same:

so the total number of ways to put 5 elements in a circle is $\frac{5!}{5} = 24$.

PERMUTATIONS WITH IDENTICAL ELEMENTS

Now get back to the problem from Kindergarten

There is a plate with 3 apples and with 1 pear and 1 strawberry. In how many ways can you take all of the pieces of fruit *one by one*, if all apples are considered the same?



We already know, that there are $5! = 120$ total orders if all apples are different. But consider the options, when apples are at places 1, 2, and 3. There is a total $3! = 6$ cases when we can put apples in these places. That means for every order, there are 5 other orders that are the same when all apples are the same because we can't distinguish the 6 ways we can arrange apples from one another. So the total number of permutations is

$$\frac{5!}{3!} = \frac{120}{6} = 20.$$

There is a plate with 3 apples and 2 strawberries. In how many ways can you take all of the fruit *one by one*, if selections of each type of fruit are the same?

We already know, that there are $5! = 120$ total orders if all apples and strawberries are different. We just saw that there are $3!$ identical arrangements of apples. In addition, we now have $2!$ identical arrangements of strawberries. So the total number of permutations is:

$$\frac{5!}{3! \times 2!} = \frac{120}{12} = 10.$$

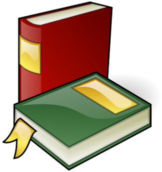
In general, if there are total N objects, and n_a, n_b, n_c, ... are the number of copies of the same objects, then there are $\frac{N!}{n_a! n_b! n_c! \ldots}$ permutations in the set.

PERMUTATIONS WITH RESTRICTIONS

Sometimes additional restrictions may be imposed on a problem. Consider this problem

There are 3 algebra books and 2 geometry books. In how many ways can you put them on a shelf, so given *subject books are kept together*?

First, we need to decide which subject will go first. There are $2!$ options for that. Then we need to decide on an order of algebra books, we have $3!$ options. Similarly, for geometry books, there are $2!$ options. So the total number of ways is



$$2! \times 3! \times 2! = 2 \times 6 \times 2 = 24.$$

When additional restrictions are imposed, the situation is transformed into a problem about **permutations with restrictions**.

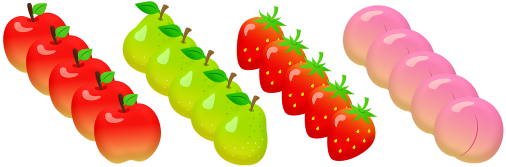
Most commonly, the restriction is that only a small number of objects are to be considered, meaning that not all the objects need to be ordered. Other common types of restrictions include restricting the type of objects that can be adjacent to one another or changing the ordering mechanism from a line to another configuration (e.g. a round table instead of a line, or a keychain instead of a ring).

Problems of this form are perhaps the most common in practice.

PERMUTATIONS IN PROBLEMS

Now instead of a plate, we have an unlimited supply of fruits

There is 4 bags with fruits: *apples*, *pears*, *peaches* and *strawberries*? You need to pick 5 fruits *one by one*. In how many ways you can do it?



Let's consider the first fruit. We have 4 options to choose from. For the second fruit, we also have the same 4 options and so on. So, the total number of ways is

$$\underbrace{4 \times 4 \times 4 \times 4 \times 4}_{5 \text{ times}} = 4^5 = 1024.$$

Sometimes permutations are not always about multiplication or division of factorials. Keep an eye on the exact wording of the problem and use the *rules of sum*, *product*, and *division* with your best judgment.

Example: find number of divisors of 60.

Let's write 60 as a product of prime numbers:

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3^1 \times 5^1.$$

Any divisor of 60 must be some combination of these prime numbers. Let's count all combinations of prime factors of a divisor:

The divisor may have 2 in the power of 0, 1, or 2 (2 to the power of 0 is also counted, because the divisor may be odd, like 15, for example).

The divisor may have 3 in the power of 0 or 1.

The divisor may have 5 in the power of 0 or 1.

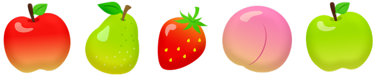
So, the total number of divisors of 60 would be

$$(2+1)(1+1)(1+1) = 12.$$

COMBINATIONS

Let's return back to fruits

There are 5 different fruits on a plate: *a red apple*, *a pear*, *a strawberry*, *a peach*, and *a green apple*. In how many ways can you take 3 of them from the plate if *order doesn't matter*?



First, we know, that there are $P_3^5 = 5 \times 4 \times 3 = 60$ ways to select 3 fruit *one by one*, but in our case order is not important. So for every 3 given fruits, there are $3! = 6$ ways to select them one by one. That means that each way to take 3 fruits was counted 6 times. As a result, the total number of ways would be

$$\frac{5!}{2! \times 3!} = \frac{120}{2 \times 6} = 10.$$

A **combination** is a way of choosing elements from a set in which *order does not matter*.

To find a number of combinations of k unordered elements from set of n distinct objects (denoted as C_k^n or $C(n, k)$), first we need to find P_k^n. Then, according to the *rule of division*, we need to divide the number by the number of ways to rearrange k elements, which is P_k. Combining everything we find, that

$$C_k^n = \frac{n!}{k!(n-k)!}.$$

The most common notation of the number of combinations is a *binomial coefficient*:

$$C_k^n = \binom{n}{k}.$$

It reads: "n *choose* k".

AND BEYOND

Combinatorial problems may look similar, but you always need to be aware of specific conditions. Let's consider this set of problems.

There are 4 sets of objects

- 4 *cups* (all different),
- 4 *glasses* (all the same),
- 7 *teaspoons* (all different),
- 10 *sugar cubes* (all the same).

And we need to count the number of ways to put:

a) teaspoons into cups;

b) sugar cubes into cups;

c) teaspoons into glasses;

d) sugar cubes into glasses.

Even though problems look the same, solutions are significantly different.

Let's look at the solutions:

a) This problem may be solved using permutations with identical elements, which we covered in these slides, (every spoon may be put in 4 different cups), so the answer is $4^7 = 16,384$.

b) This problem may be solved using a technique called stars and bars, we will cover it in a future lesson, calculated as $\binom{13}{3} = 286$.

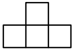
c) To calculate the solution we need to use Stirling numbers of the second kind and the *rule of sum*:

$$\left\{ {7 \atop 1} \right\} + \left\{ {7 \atop 2} \right\} + \left\{ {7 \atop 3} \right\} + \left\{ {7 \atop 4} \right\} = 1 + 63 + 301 + 350 = 715.$$

d) Here we need to calculate the number of Young tableaux with 10 cells and 4 boxes, which is far beyond middle school math. For this particular problem it is easier to just do the ***case bashing*** (head down and write out all cases) to find out that answer is 23. We'll practice case bashing in a future lesson as well.

EXERCISES

1. What is $\dfrac{100!}{98!}$?
2. The shape below is made up of four squares. How many ways can we shade some of the squares such that the shaded squares form exactly 1 polygon? (At least one of the squares must be shaded.)



3. How many 5×6 rectangles can fit in an 11×11 square such that the 5×6 rectangles do not overlap? (The rectangles may be rotated.)
4. How many four-digit numbers have exactly one digit 5?
5. How many four-digit numbers have exactly one digit 5 and no other digits equal each other?
6. How many different ways are there to place a white and a black rook on a chessboard so that they do not attack each other?
7. How many different ways are there to place a black and white king on a chessboard so that they do not attack each other?
8. In the arrangement of letters and numerals below, by how many different paths can one spell $MATH$? Beginning at the M in the middle, a path allows only moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.

	H	T	H	
H	T	A	T	H
T	A	M	A	T
H	T	A	T	H
	H	T	H	

CHALLENGE PROBLEMS

1. All the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What is the number in the center?
2. Each vertex of a cube is to be labeled with an integer 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?
3. A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)
4. There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?