# Probability

Graham Middle School Math Olympiad Team

# PROBABILITY

An **outcome** is a possible result of an experiment.

We say "the outcome of throwing the dice was $4$." The experiment was throwing the die and its result (outcome) was that the face with $4$ pips on it came up.

Probability problems are either based on *countable* outcomes or on outcomes that are not countable. The later case is called *geometric probability* or *continuous probability*.

If the number of outcomes is countable and *if all oucomes have the same probability*, then the **probability** of desired oucome is:

$$P = \frac{\text{(number of desired outcomes)}}{\text{(total number of events)}}.$$

Example: We write each of the names of all $30$ students in a math club on a card and we draw a card at random. If there are $14$ sixth-graders, $9$ seventh-graders, and $7$ eighth-graders in this club, what is the probability of the name on the card being that of a sixth-grader?

The total number of *outcomes* is all students in the club. The desired *outcomes* are all sixth-graders. So *probability* is

$$P = \frac{14}{30} = \frac{7}{15}.$$

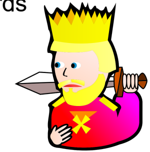
In the *geometric case*, the **probability** of a desired outcome is calculated as the ratio:

$$P = \frac{\text{(desired area)}}{\text{(total area)}}.$$

Probabilities in any case are always between $0$ and $1$, inclusive.

# ADDITION RULE FOR PROBABILITY

What is the probability of choosing *an ace* **or** *a king* from a full deck of $52$ cards?



A card that we got from the deck of cards may not be both an ace and a king. So we have $4$ *oucomes* of getting an ace and $4$ *oucomes* of getting a king. The result number of *desired oucomes* is $8$. The *total number of possible outcomes* is $52$. The total probability is

$$P = \frac{4+4}{52} = \frac{8}{52} = \frac{2}{13}.$$

We also may count the probability in other way. The *probability* of getting an ace is $4/52$ and the *probability* of getting a king is also $4/52$. Since both cases are good for us, the total probability is

$$P = \frac{4}{52} + \frac{4}{52} = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}.$$

If two events are *oucomes* of one experiment and they *may not happen together*, then *probability* of any of this event is **sum of their probabilities**

$$P(A \text{ or } B) = P(A) + P(B).$$

What is the probability of choosing *a multiple of* $4$ or *a multiple of* $5$ from a numbers from $1$ to $100$?

The probability of getting a multiple of $4$ is $25/100$, the probability of getting a multiple of $5$ is $20/100$. But there are multiples of $20$, which we counted twice. To make up for this case, we need to subtract the probability of getting a multiple of $20$, which is $5/100$. So total probability would be

$$P = \frac{25}{100} + \frac{20}{100} - \frac{5}{100} = \frac{25+20-5}{100} = \frac{40}{100} = \frac{2}{5}.$$

So the *probability of two events* of one experiment

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

# MULTIPLICATION RULE FOR PROBABILITY

If the experiment *"throw a die and toss a coin"* is performed, what is the probability of the event *"a $4$ and a tails"* to happen?


Let's first throw a die. If $4$ didn't happen, we don't need to toss a coin, the desired event will not happen in any case. So we need to toss a coin only when the die rolls $4$. The $4$ has a probability of $1/6$ and, in this case only, we toss a coin and with a probability of $1/2$ we will get tails. So the overall probability would be

$$P = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

If two events are *independent* the **probability** of both of them happen is *a multiple of their probabilities*

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

The **conditional probability** of an event $E$ that depends on another event $F$ to happen is denoted

$$P(E \mid F)$$

and is probability of $E$ to happen if we know that $F$ has already happened.

We used that rule to solve the problem *"throw a die and toss a coin"*.

If two events are *dependent* the **probability** of both of them happen is

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

What is the probability of choosing *an ace* **and then** *a king* from a full deck of $52$ cards?

The probability to choose an ace is $4/52$. Then we selecting a king from the deck without one card, so the probability to choose a king is $4/51$. The total probability is

$$P(A \text{ and then } K) = P(A) \cdot P(K \mid A) = \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}.$$

# USING THE NEGATIVE CASE

All probable outcomes of one experiment is equal to all experiment. That mean that probability of all outcomes is

$$P(\text{All outcomes}) = 1.$$

What is the probability if throwing the dice and not to get $4$.

There are $5$ outcomes that is desired: $1$, $2$, $3$, $5$, and $6$. So total probability would be

$$P(\text{not throwing } 4) = 5 \cdot \frac{1}{6} = \frac{5}{6}.$$

From other hand there is only one event that is not desired, so we can calculate the same probability as

$$\begin{aligned} &P(\text{not throwing } 4) = \\ &= P(\text{all possible results}) - P(\text{throwing } 4) = \\ &= 1 - \frac{1}{6} = \frac{5}{6}. \end{aligned}$$

The **complement event** of a *desired event* $E$, denoted as $\overline{E}$, is the event that $E$ does not occur. Since for any event we know whether it is desired or not

$$P(\overline{E}) = 1 - P(E).$$

Sometimes that simplifies the solution.

We are draw $4$ cards from a full deck of cards. What is the probability to get *at least one ace*?

If we try to solve the problem using the *conditional probability* we find out that number of cases is pretty big. But we may count the probability to *not get any ace*.

$$P(\text{no ace}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} = \frac{38916}{54145} \approx 72\%.$$

And probability to get *an ace with* $4$ *cards* is

$$P(\text{at least one ace}) = 1 - \frac{38916}{54145} = \frac{15229}{54145} \approx 28\%.$$

# THE BIRTHDAY PROBLEM

How many people need to attend a party until there is a $50\%$ chance that at least two guests share a birthday?

It is easier to calculate the probability that *no two people share a birthday*.
Les's start with the first guest. He don't share a birthday with anyone. So the probability is $1$.
The second guest may share a birthday with the first guest with probability $364/365$.
The third one don't share a birthday with previous guests with probability $363/365$ and so on.
For $n$ guests

$$P(\text{no shared birthday}) =$$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \frac{365-n+1}{365} =$$

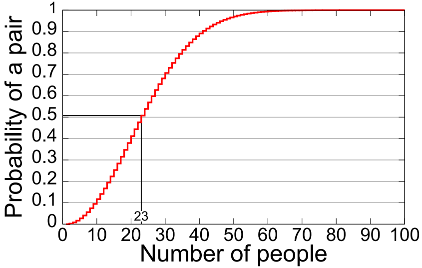
$$= \frac{365!}{(365-n)!} \cdot \frac{1}{365^n}.$$

Using formula

$$P(\text{there is a shared birthday}) =$$

$$= 1 - P(\text{no shared birthday})$$

we can calculate, using computer, the probability for any number of guests




So we may see that group of $23$ people has probability of $50.7\%$ to have a shared birthday, and group of $70$ people has probability of $99.9\%$ to share a birthday.