Number Theory 102 solutions

Graham Middle School Math Olympiad Team

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There are 28 multiples of 5 less or equal than 144, there are 5 multiples of $5^2 = 25$ less or equal than 144, and there is one multiple of $5^3 = 125$ less of equal than 144.

So $144!$ has $28 + 5 + 1 = \boxed{34}$ trailing zeros.

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The only way for $R + RR$ to be bigger than 100 is $R = 9$ and $BOW = 108$.So for 6 other digits F, A, I, N, T, and G we have only 6 other digits, so $F \times A \times I \times N \times T \times G = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$, and it is ended up with 0. So doesn't matter which digits are N and I the last digit of $F \times A \times I \times N \times T \times I \times N \times G$ is $\boxed{0}$.

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We can represent the amount of gold with g and the amount of chests with c. We can use the problem to make the following equations:

$$9c - 18 = g, \quad 6c + 3 = g$$

Therefore, $6c + 3 = 9c - 18$. This implies that $c = 7$. We therefore have $g = \boxed{45}$.

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We see that since QRS is divisible by 5, S must equal either 0 or 5, but it cannot equal 0, so $S = 5$. We notice that since PQR must be even, R must be either 2 or 4. However, when $R = 2$, we see that $T \equiv 2 \pmod 3$, which cannot happen because 2 and 5 are already used up; so $R = 4$. This gives $T \equiv 3 \pmod 4$, meaning $T = 3$. Now, we see that Q could be either 1 or 2, but 14 is not divisible by 4, but 24 is. This means that $Q = 2$ and $P = \boxed{1}$.

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Observe that $2022_{10} = 5616_7$. To maximize the sum of the digits, we want as many 6s as possible (since 6 is the highest value in base 7), and this will occur with either of the numbers 4666_7 or 5566_7. Thus, the answer is $4 + 6 + 6 + 6 = 5 + 5 + 6 + 6 = \boxed{22}$.

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Next, we use the fact that $19!$ is a multiple of both 11 and 9. Their divisibility rules tell us that $T+M \equiv 3 \pmod{9}$ and that $T-M \equiv 7 \pmod{11}$.

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Therefore the answer is $4+8+0=\boxed{12}$.

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The sum of purchased boxes should be divisible by 3, since if first customer purchased n pounds, they together purchased $3n$. Since remainder of sum of all 6 boxes is $0+1+0+1+2+1 \equiv 2 \pmod 3$, the box that left should weight $\boxed{20}$.
The purchase should be split 33 to 66 pounds, so first customer purchased boxes 15 and 18 and the second purchased 16, 19, and 31.