



# Number Theory 101 solutions

Graham Middle School Math Olympiad Team



$$\sqrt{x} = 3, 14$$
$$3 \times 3 = 9$$



## PROBLEMS 1-4

**E1.** What is the remainder when  $301 \times 349$  is divided by 9?

The remainder when 301 is divisible by 9 is  $3 + 0 + 1 = 4$  and the remainder when 349 is divisible by 9 is  $(3 + 4 + 9) - 9 = 7$ . The remainder of  $4 \times 7 = 28$  is divisible by 9 is  $(2 + 8) - 9 = \boxed{1}$ .

**E2.** Find the GCD and LCM of 42 and 98.

Using prime factorization:

$42 = 2 \times 3 \times 7$  and  $98 = 2 \times 7^2$ , so

$\gcd(42, 98) = 2 \times 7 = 14$ , and

$\text{lcm}(42, 98) = 2 \times 3 \times 7^2 = 294$ .

Using Euclidean algorithm:

$$98 = 42 \times 2 + 14,$$

$$42 = \boxed{14} \times 3 + 0.$$

and

$$\text{lcm}(42, 98) = \frac{42 \times 98}{\gcd(42, 98)} = \frac{42 \times 98}{14} = \boxed{294}.$$

**E3.** The GCD of two numbers  $A$  and  $B$  is 7. What are the possible values of GCD of  $15 \cdot A$  and  $35 \cdot B$ ?

The possible values are shown in the table:

	$7^2 \nmid A$	$7^2 \mid A$
$3 \nmid B$	<div>35</div>	<div>245</div>
$3 \mid B$	<div>105</div>	<div>735</div>

$x|y$  means  $x$  divides  $y$ , and  $x \nmid y$  means  $x$  not divides  $y$ .

**E4.** The number 6545 can be written as the product of a pair of positive two-digit integers. What are these two integers?

The prime factorization of 6545 is  $5 \times 7 \times 11 \times 17$ , since  $5 \times 7 \times 11 > 100$  and  $7 \times 17 > 100$ , the only possible way to form a pair of two-digit integers is  $5 \times 17 = \boxed{85}$  and  $7 \times 11 = \boxed{77}$ .

## PROBLEMS 5-8

**E5.** What is the smallest prime factor of  $11^7 + 7^5$ ?

The smallest prime number is 2, and it is a factor of  $11^7 + 7^5$ , because both  $11^7$  and  $7^5$  are odd and their sum is even.

**E6.** The four-digit number  $A55B$  is divisible by 36. What is the sum of  $A$  and  $B$ ?

$36 = 9 \times 4$ , so  $A55B$  should be divisible by 9 and 4. If  $A55B$  is divisible by 9, so  $A + 5 + 5 + B$  is divisible by 9, so  $A + B$  is either 8 or 17. And since  $5B$  is divisible by 4,  $B$  can be either 2 or 6. For both cases the sum  $A + B$  can't be 17, so the only option for  $A + B$  is 8.

**E7.** What is the sum of the digits of  $\frac{10^{25} + 8}{9}$ ?

$$10^{25} = \underbrace{99 \cdot 9}_{25 \text{ nines}} + 1, \text{ so}$$

$$\frac{10^{25} + 8}{9} = \frac{\underbrace{99 \cdot 9}_{25 \text{ nines}} + 9}{9} = \underbrace{11 \cdot 1}_{25 \text{ ones}} + 1 = \underbrace{11 \cdot 1}_{24 \text{ nines}} 2. \text{ And}$$

the sum of the digits is  $24 + 2 = \span style="border: 1px solid black; padding: 0 5px;">26.$

**E8.** Find the GCD of  $2n + 13$  and  $n + 7$  by Euclid's algorithm.

$$2n + 13 = (n + 7) \times 2 - 1,$$

$$n + 7 = \span style="border: 1px solid black; padding: 0 5px;">1 \times (n + 7) + 0.$$

## CHALLENGE PROBLEMS 1-3

**C1.** The positive integers  $A$ ,  $B$ ,  $A - B$ , and  $A + B$  are all prime numbers. What is the sum of these four primes?

Since  $A$ ,  $B$ , and  $A + B$  are all prime, that means two of them should be odd, so  $B$  should be 2. One of the numbers  $A - 2$ ,  $A$ , and  $A + 2$  should be divisible by 3 since all of them have different remainders when divisible by 3. So we got  $A - B = 3$ ,  $A = 5$ , and  $A + B = 7$  and the sum is  $2 + 3 + 5 + 7 = \boxed{17}$ .

**C2.** Show that every prime greater than 3 must be of the form  $6n + 1$  or  $6n - 1$  for a positive integer  $n$ .

$6n$  and  $6n + 3$  can't be primes because they are divisible by 3;

$6n + 2$  and  $6n + 4$  can't be primes because they are divisible by 2.

So the only way for primes is to have the form  $6n + 1$  and  $6n - 1$ .

**C3.** If  $p$ ,  $q$  and  $r$  are prime numbers such that their product is 19 times their sum, find  $p^2 + q^2 + r^2$ .

Since  $pqr = 19(p + q + r)$ , one of the numbers should be 19. So  $pq = p + q + 19$ .

$pq - p - q + 1 = 20$ , so

$$(p - 1)(q - 1) = 20.$$

Let's take a look at different factorization of 20:

**Case 1:**  $(p - 1)(q - 1) = 1 \times 20$ ,  $p = 2$ ,  $q = 21$ .

**Case 2:**  $(p - 1)(q - 1) = 2 \times 10$ ,  $p = 3$ ,  $q = 11$ .

**Case 3:**  $(p - 1)(q - 1) = 4 \times 5$ ,  $p = 5$ ,  $q = 6$ .

Only in case 2  $p$  and  $q$  are primes, so 3, 11, and 19 is the only possible options for  $p$ ,  $q$ , and  $r$  in some order.

$$p^2 + q^2 + r^2 = 3^2 + 11^2 + 19^2 = \boxed{491}.$$

## CHALLENGE PROBLEM 4

**C4.** Let  $a$ ,  $b$ ,  $c$ , and  $d$  be positive integers such that  $\gcd(a, b) = 24$ ,  $\gcd(b, c) = 36$ ,  $\gcd(c, d) = 54$ , and  $70 < \gcd(d, a) < 100$ . Which of the following must be a divisor of  $a$ : 5, 7, 11, 13, or 17?

Notice that

$$\begin{aligned}\gcd(a, b, c, d) &= \\ &= \gcd(\gcd(a, b), \gcd(b, c), \gcd(c, d)) = \\ &= \gcd(24, 36, 54) = 6,\end{aligned}$$

so  $\gcd(d, a)$  must be a multiple of 6.

If  $\gcd(d, a)$  is multiple of  $2^2$ , then  $\gcd(c, d) \neq 54$ , since  $d$  and  $c$  divisible by 4.

If  $\gcd(d, a)$  is multiple of  $3^2$ , then  $\gcd(a, b) \neq 24$ , since  $a$  and  $b$  both divisible by 9.

The only answer choice that gives a value between 70 and 100 when multiplied by 6 is 13.

EMPTY

## TEAM ATTACK 3 SOLUTIONS

**TA1.** Number  $1A2$  should be divisible by  $11$ , so  $1 + 2 = A$  or  $1 + 2 = A \pm 11$ . Since  $A$  is a digit, the only option is  $A = 3$ .

**TA2.** Abe selects green with probability  $1/2$  and Bob matches with probability  $1/4$ , so the probability that both selected green is  $1/8$ . The probability that both select red is  $1/2 \times 1/2 = 1/4$ . The total probability is  $1/8 + 1/4 = 3/8$ .

**TA3.** From the first condition,  $n$  should have prime factors  $2$  and  $3$  in power  $1$ . From the second condition, because  $126 = 2 \times 3 \times 3 \times 7$   $n$  should have prime factor of  $7$  and no other prime factors. So  $n = 2 \times 3 \times 7 = 42$ .

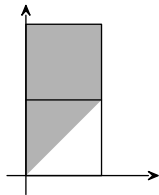
**TA4.**  $7 + 4 + A + 5 + 2 + B + 1$  should be divisible by  $3$ , so  $A + B$  has a remainder  $2$  when divisible by  $3$ .  $3 + 2 + 6 + A + B + 4 + C$  is divisible by  $3$ , so  $A + B + C$  should have the remainder  $0$  when divisible by  $3$ , so  $C$  should have the remainder  $1$  when divisible by  $3$ . The largest such digit is  $7$ .

**TA5.** To have all rolls different we have  $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$  options, and we have  $6$  options when all rolls are the same. There are  $6^5$  options to roll the dice  $5$  times. So the desired probability is

$$\frac{6 \times 120 + 6}{6^5} = \frac{121}{6^4} = \frac{121}{1296}.$$

**TA6.** Draw Chloé's number on the  $x$  axis and Laurent's on the  $y$  axis. The probability is  $3/4$ .

**TA7.** Since we are looking for an integer value, each of the prime numbers  $2$ ,  $3$ , and  $5$  occur as factors an even number of times, so  $2$ ,  $3$ , and  $5$  to split with half of the factors in the numerator canceling half in the denominator. The prime number  $7$  occurs only once in the expression, so it looks like the best we can possibly do is  $7$ .



$$\left( \left( \left( 1 \div ((2 \div 3) \div 4) \right) \div ((5 \div 6) \div 7) \right) \div 8 \right) \div (9 \div 10) = 7$$