Number Theory 101 solutions

Graham Middle School Math Olympiad Team

E1. What is the remainder when 301×349 is divided by 9?

The remainder when 301 divisible by 9 is $3+0+1=4$ and the remainder when 349 divisible by 9 is $(3+4+9)-9=7$. The remainder of $4 \times 7 = 28$ divisible by 9 is $(2+8)-9 = \boxed{1}$.

E2. Find the GCD and LCM of 42 and 98.

Using prime factorization:
$42 = 2 \times 3 \times 7$ and $98 = 2 \times 7^2$, so
$\gcd(42, 98) = 2 \times 7 = 14$, and
$\operatorname{lcm}(42, 98) = 2 \times 3 \times 7^2 = 294$.
Using Euclidean algorithm:

$$\mathbf{98} = \mathbf{42} \times 2 + 14,$$

$$\mathbf{42} = \boxed{\mathbf{14}} \times 3 + 0.$$

and

$$\operatorname{lcm}(42, 98) = \frac{42 \times 98}{\gcd(42, 98)} = \frac{42 \times 98}{14} = \boxed{294}.$$

E3. The GCD of two numbers A and B is 7. What are the possible values of GCD of $15 \cdot A$ and $35 \cdot B$?

The possible values are shown in the table:

	$7^2 \nmid A$	$7^2 \mid A$
$3 \nmid B$	$\boxed{35}$	$\boxed{245}$
$3 \mid B$	$\boxed{105}$	$\boxed{735}$

$x|y$ means x divides y, and $x \nmid y$ means x not divides y.

E4. The number 6545 can be written as the product of a pair of positive two-digit integers. What are these two integers?

The prime factorization of 6545 is $5 \times 7 \times 11 \times 17$, since $5 \times 7 \times 11 > 100$ and $7 \times 17 > 100$, the only possible way to form a pair of two-digit integers is $5 \times 17 = \boxed{85}$ and $7 \times 11 = \boxed{77}$.

PROBLEMS 5-8

E5. What is the smallest prime factor of $11^7 + 7^5$?

The smallest prime number is 2, and it is a factor of $11^7 + 7^5$, because both 11^7 and 7^5 are odd and their sum is even.

E6. The four-digit number $A55B$ is divisible by 36. What is the sum of A and B?

$36 = 9 \times 4$, so $A55B$ should be divisible by 9 and 4. If $A55B$ is divisible by 9, so $A + 5 + 5 + B$ is divisible by 9, so $A + B$ is either 8 or 17. And since $5B$ is divisible by 4, B can be either 2 or 6. For both cases the sum $A + B$ can't be 17, so the only option for $A + B$ is $\boxed{8}$.

E7. What is the sum of the digits of $\dfrac{10^{25}+8}{9}$?

$10^{25} = \underbrace{99\cdot 9}_{\text{25 nines}} + 1$, so

$$\frac{10^{25}+8}{9} = \frac{\underbrace{99\cdot 9}_{\text{25 nines}} + 9}{9} = \underbrace{11\cdot 1}_{\text{25 ones}} + 1 = \underbrace{11\cdot 1}_{\text{24 nines}}2.$$

And the sum of the digits is $24 + 2 = \boxed{26}$.

E8. Find the GCD of $2n + 13$ and $n + 7$ by Euclid's algorithm.

$$\mathbf{2n + 13} = (\mathbf{n + 7}) \times 2 - 1,$$

$$\mathbf{n + 7} = \boxed{\mathbf{1}} \times (n + 7) + 0.$$

CHALLENGE PROBLEMS 1-3

C1. The positive integers A, B, $A-B$, and $A+B$ are all prime numbers. What is the sum of these four primes?

Since A, B, and $A+B$ are all prime, that means two of them should be odd, so B should be 2. One of the numbers $A-2$, A, and $A+2$ should be divisible by 3 since all of them have different reminders when divisible by 3. So we got $A-B=3$, $A=5$, and $A+B=7$ and the sum is $2+3+5+7=\boxed{17}$.

C2. Show that every prime greater than 3 must be of the form $6n+1$ or $6n-1$ for a positive integer n.

$6n$ and $6n+3$ can't be primes because they are divisible by 3;
$6n+2$ and $6n+4$ can't be primes because they are divisible by 2.
So the only way for primes is to have the form $6n+1$ and $6n-1$.

C3. If p, q and r are prime numbers such that their product is 19 times their sum, find $p^2+q^2+r^2$.

Since $pqr = 19(p+q+r)$, one of the numbers should be 19. So $pq = p+q+19$.
$pq-p-q+1 = 20$, so

$$(p-1)(q-1) = 20.$$

Let's take a look at different factorization of 20:
Case 1: $(p-1)(q-1) = 1\times 20$, $p=2$, $q=21$.
Case 2: $(p-1)(q-1) = 2\times 10$, $p=3$, $q=11$.
Case 3: $(p-1)(q-1) = 4\times 5$, $p=5$, $q=6$.
Only in case 2 p and q are primes, so 3, 11, and 19 is the only possible options for p, q, and r in some order.

$$p^2+q^2+r^2 = 3^2+11^2+19^2 = \boxed{491}.$$

CHALLENGE PROBLEM 4

C4. Let a, b, c, and d be positive integers such that $\gcd(a,\ b) = 24$, $\gcd(b,\ c) = 36$, $\gcd(c,\ d) = 54$, and $70 < \gcd(d,\ a) < 100$. Which of the following must be a divisor of a: 5, 7, 11, 13, or 17?

Notice that

$$\begin{aligned} &\gcd(a,\ b,\ c,\ d) = \\ &\quad = \gcd(\gcd(a,\ b),\ \gcd(b,\ c),\ \gcd(c,\ d)) = \\ &\qquad = \gcd(24,\ 36,\ 54) = 6, \end{aligned}$$

so $\gcd(d,\ a)$ must be a multiple of 6.
If $\gcd(d,\ a)$ is multiple of 2^2, then $\gcd(c,\ d) \neq 54$, since d and c divisible by 4.
If $\gcd(d,\ a)$ is multiple of 3^2, then $\gcd(a,\ b) \neq 24$, since a and b both divisible by 9.
The only answer choice that gives a value between 70 and 100 when multiplied by 6 is $\boxed{13}$.

EMPTY

TEAM ATTACK 3 SOLUTIONS

TA1. Number $\underline{1A2}$ should be divisible by 11, so $1+2=A$ or $1+2=A\pm 11$. Since A is a digit, the only option is $A=3$.

TA2. Abe selects green with probability $1/2$ and Bob matches with probability $1/4$, so the probability that both selected green is $1/8$. The probability that both select red is $1/2\times 1/2=1/4$. The total probability is $1/8+1/4=3/8$.

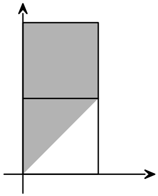
TA3. From the first condition, n should have prime factors 2 and 3 in power 1. From the second condition, because $126=2\times 3\times 3\times 7$ n should have prime factor of 7 and no other prime factors. So $n=2\times 3\times 7=42$.

TA4. $7+4+A+5+2+B+1$ should be divisible by 3, so $A+B$ has a reminder 2 when divisible by 3. $3+2+6+A+B+4+C$ is divisible by 3, so $A+B+C$ should have the reminder 0 when divisible by 3, so C should have the reminder 1 when divisible by 3. The largest such digit is 7.

TA5. To have all rolls different we have $6\cdot 5\cdot 4\cdot 3\cdot 2$ options, and we have 6 options when all rolls are the same. There are 6^5 options to roll the dice 5 times. So the desired probability is

$$\frac{6\times 120+6}{6^5}=\frac{121}{6^4}=\frac{121}{1296}.$$

TA6. Draw Chloé's number on the x axis and Laurent's on the y axis. The probability is $3/4$.



TA7. Since we are looking for an integer value, each of the prime numbers 2, 3, and 5 occur as factors an even number of times, so 2, 3, and 5 to split with half of the factors in the numerator canceling half in the denominator. The prime number 7 occurs only once in the expression, so it looks like the best we can possibly do is 7.

$$\left(\left(\left(1\div((2\div 3)\div 4)\right)\div((5\div 6)\div 7)\right)\div 8\right)\div(9\div 10)=7$$