



# Probability

Graham Middle School Math Olympiad Team



$$\sqrt{x} = 3, 14$$
$$3 \times 3 = 9$$



## PROBABILITY

An **outcome** is a possible result of an experiment.

We say "the outcome of throwing the dice was 4." The experiment was throwing the die and its result (outcome) was that the face with 4 pips on it came up.



Probability problems are either based on *countable* outcomes or on outcomes that are not countable. The later case is called *geometric probability* or *continuous probability*.

If the number of outcomes is countable and if all outcomes have the same probability, then the **probability** of desired outcome is:

$$P = \frac{\text{(number of desired outcomes)}}{\text{(total number of events)}}.$$

**Example:** We write each of the names of all 30 students in a math club on a card and we draw a card at random. If there are 14 sixth-graders, 9 seventh-graders, and 7 eighth-graders in this club, what is the probability of the name on the card being that of a sixth-grader?

The total number of *outcomes* is all students in the club. The desired *outcomes* are all sixth-graders. So *probability* is

$$P = \frac{14}{30} = \frac{7}{15}.$$

In the *geometric* case, the **probability** of a desired outcome is calculated as the ratio:

$$P = \frac{\text{(desired area)}}{\text{(total area)}}.$$

Probabilities in any case are always between 0 and 1, inclusive.

## ADDITION RULE FOR PROBABILITY

What is the probability of choosing *an ace or a king* from a full deck of 52 cards?

A card that we got from the deck of cards may not be both an ace and a king. So we have 4 *oucomes* of getting an ace and 4 *oucomes* of getting a king. The result number of *desired oucomes* is 8. The *total number of possible oucomes* is 52. The total probability is

$$P = \frac{4 + 4}{52} = \frac{8}{52} = \frac{2}{13}.$$

We also may count the probability in other way. The *probability* of getting an ace is 4/52 and the *probability* of getting a king is also 4/52. Since both cases are good for us, the total probability is

$$P = \frac{4}{52} + \frac{4}{52} = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}.$$



If two events are *oucomes* of one experiment and they *may not happen together*, then *probability* of any of this event is **sum of their probabilities**

$$P(A \text{ or } B) = P(A) + P(B).$$

What is the probability of choosing a *multiple of 4* or a *multiple of 5* from a numbers from 1 to 100?

The probability of getting a multiple of 4 is 25/100, the probability of getting a multiple of 5 is 20/100. But there are multiples of 20, which we counted twice. To make up for this case, we need to subtract the probability of getting a multiple of 20, which is 5/100. So total probability would be

$$P = \frac{25}{100} + \frac{20}{100} - \frac{5}{100} = \frac{25 + 20 - 5}{100} = \frac{40}{100} = \frac{2}{5}.$$

So the *probability of two events* of one experiment

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

## MULTIPLICATION RULE FOR PROBABILITY

If the experiment "*throw a die and toss a coin*" is performed, what is the probability of the event "*a 4 and a tails*" to happen?



Let's first throw a die. If 4 didn't happen, we don't need to toss a coin, the desired event will not happen in any case. So we need to toss a coin only when the die rolls 4. The 4 has a probability of  $\frac{1}{6}$  and, in this case only, we toss a coin and with a probability of  $\frac{1}{2}$  we will get tails. So the overall probability would be

$$P = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

If two events are *independent* the **probability** of both of them happen is *a multiple of their probabilities*

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

The **conditional probability** of an event  $E$  that depends on another event  $F$  to happen is denoted

$$P(E | F)$$

and is probability of  $E$  to happen if we know that  $F$  has already happened.

We used that rule to solve the problem "*throw a die and toss a coin*".

If two events are *dependent* the **probability** of both of them happen is

$$P(A \text{ and } B) = P(A) \cdot P(B | A).$$

What is the probability of choosing *an ace and then a king* from a full deck of 52 cards?

The probability to choose an ace is  $\frac{4}{52}$ . Then we selecting a king from the deck without one card, so the probability to choose a king is  $\frac{4}{51}$ . The total probability is

$$P(A \text{ and then } K) = P(A) \cdot P(K | A) = \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}.$$

## USING THE NEGATIVE CASE

All probable outcomes of one experiment is equal to all experiment. That mean that probability of all outcomes is

$$P(\text{All outcomes}) = 1.$$

What is the probability if throwing the dice and not to get 4.

There are 5 outcomes that is desired: 1, 2, 3, 5, and 6. So total probability would be

$$P(\text{not throwing 4}) = 5 \cdot \frac{1}{6} = \frac{5}{6}.$$

From other hand there is only one event that is not desired, so we can calculate the same probability as

$$\begin{aligned} P(\text{not throwing 4}) &= \\ &= P(\text{all possible results}) - P(\text{throwing 4}) = \\ &= 1 - \frac{1}{6} = \frac{5}{6}. \end{aligned}$$

The **complement event** of a *desired event*  $E$ , denoted as  $\overline{E}$ , is the event that  $E$  does not occur. Since for any event we know whether it is desired or not

$$P(\overline{E}) = 1 - P(E).$$

Sometimes that simplifies the solution.

We are draw 4 cards from a full deck of cards. What is the probability to get *at least one ace*?

If we try to solve the problem using the *conditional probability* we find out that number of cases is pretty big. But we may count the probability to *not get any ace*.

$$P(\text{no ace}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} = \frac{38916}{54145} \approx 72\%.$$

And probability to get *an ace with 4 cards* is

$$P(\text{at least one ace}) = 1 - \frac{38916}{54145} = \frac{15229}{54145} \approx 28\%.$$

## THE BIRTHDAY PROBLEM

How many people need to attend a party until there is a 50% chance that at least two guests share a birthday?

It is easier to calculate the probability that no two people share a birthday.

Let's start with the first guest. He doesn't share a birthday with anyone. So the probability is  $365/365$ .

The second guest may share a birthday with the first guest with probability  $364/365$ .

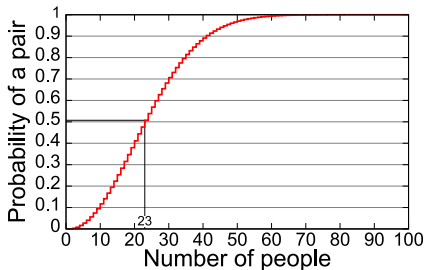
The third one doesn't share a birthday with previous guests with probability  $363/365$  and so on. For  $n$  guests

$$\begin{aligned}P(\text{no shared birthday}) &= \\&= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - n + 1}{365} = \\&= \frac{365!}{(365 - n)!} \cdot \frac{1}{365^n}.\end{aligned}$$

Using formula

$$\begin{aligned}P(\text{have a shared birthday}) &= \\&= 1 - P(\text{no shared birthday})\end{aligned}$$

we can calculate, using computer, the probability for any number of guests



So we may see that group of 23 people has probability of 50.7% to have a shared birthday, and group of 70 people has probability of 99.9% to share a birthday.