Probability solutions

Graham Middle School Math Olympiad Team

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Let's do complimentary counting: the product is odd if both multiplicands are odd. The probability that a chip from the first box odd is $\frac{2}{3}$, the same for the second box. So the probability that two chips are odd is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. The probability that the product is even is $1 - \frac{4}{9} = \boxed{\frac{5}{9}}$.

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We can ignore the 4 other classmates because they aren't relevant. We can treat Abby and Bridget as a pair, so there are $\binom{6}{2} = 15$ total ways to seat them. If they sit in the same row, there are $2 \cdot 2 = 4$ ways to seat them. If they sit in the same column, there are 3 ways to seat them. Thus our answer is $\dfrac{4+3}{15} = \boxed{\dfrac{7}{15}}$.

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The product can only be 0 if one of the numbers is 0. Once we chose 0, there are 5 ways we can choose the second number, or $6-1$. There are $\binom{6}{2}$ ways we can choose 2 numbers randomly, and that is 15. So, $\frac{5}{15} = \frac{1}{3}$ so the answer is $\boxed{\frac{1}{3}}$.

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E7. On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

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The number of people wearing caps and sunglasses is $\frac{2}{5} \cdot 35 = 14$. So then 14 people out of the 50 people wearing sunglasses also have caps. $\frac{14}{50} = \boxed{\frac{7}{25}}$

EXERCISE 8

E8. A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

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Assume that after you draw the three red chips in a row without drawing both green chips, you continue drawing for the next turn. The last/fifth chip that is drawn must be a green chip because if both green chips were drawn before, we would've already completed the game. So technically, the problem is asking for the probability that the "fifth draw" is a green chip. This probability is symmetric to the probability that the first chip drawn is green, which is $\frac{2}{5}$. So the probability is $\boxed{\frac{2}{5}}$.

CHALLENGE PROBLEMS 1-2

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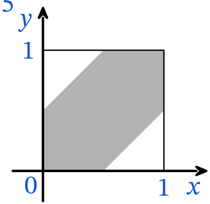

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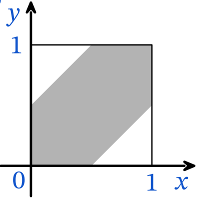

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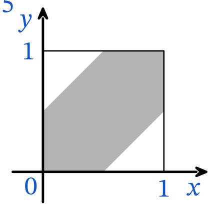

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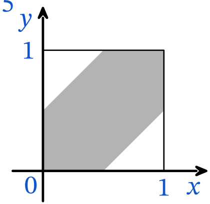

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There are two ways the contestant can win.
Case 1: The contestant guesses all three right.
This can only happen $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$ of the time.
Case 2: The contestant guesses only two right.
We pick one of the questions to get wrong, 3, and this can happen $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$ of the time. Thus, $\frac{2}{27} \cdot 3 = \frac{6}{27}$.
So, in total the two cases combined equals $\frac{1}{27} + \frac{6}{27} = \boxed{\frac{7}{27}}$.

CHALLENGE PROBLEMS 3-4

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The minimum number that can be shown on the face of a die is 1, so the least possible sum of the top faces of the 7 dice is 7.
In order for the sum to be exactly 10, 1 to 3 dices numbers on the top face must be increased by a total of 3.
There are 3 ways to do so: 3, $2+1$, and $1+1+1$
There are 7 for Case 1, $7 \cdot 6 = 42$ for Case 2, and $\frac{7 \cdot 6 \cdot 5}{3!} = 35$ for Case 3.
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When a number's unit's digit is 1, then any power to this number will also end in 1 (since 1^n for any n is always 1), so we have 20 choices for 11.
When a number's unit's digit is 3, then 3^{4n} for any n will produce a number ending with 1. So, $20 \div 4 = 5$ choices for 13.
5^n always ends in 5, so there are 0 possibilities for 15.
When a number's unit's digit is 7, then this is also the same thing with 3, so we have 5 choices.
When a number's unit's digit is 9, then 9^{2n} will produce a number ending in 1, so we have $20 \div 2 = 10$ possibilities.
Hence, we have a total of $5 \cdot 20 = 100$ ways, so the probability is $\frac{20+5+0+5+10}{100} = \frac{40}{100} = \boxed{\frac{2}{5}}$.