Arithmetic sequencies

Graham Middle School Math Olympiad Team
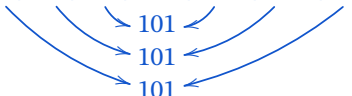
To keep his 3rd grade students busy, a teacher in 18th century Germany asked to find the sum of the numbers from 1 to 100. But one of the students in the class was 10-year-old Carl Friedrich Gauss. He instantly answered 5050. How did he do that? To sum the digits

$$1 + 2 + 3 + ... + 98 + 99 + 100$$

$$101$$
$$101$$
$$101$$

we pair terms as shown. So total, we have 50 pairs with a value 101. Then multiplying them together, we will get 5050.

What is the sum of first n numbers?

If n is even, there are $\frac{n}{2}$ pairs, each equal to $n+1$, and the sum is $\frac{n(n+1)}{2}$.

If n is odd, the middle term, which is equal to $\frac{n+1}{2}$, is unpaired, and there are $\frac{n-1}{2}$ paired terms that each sum to $n+1$. So in the odd case the total sum is $\frac{n+1}{2} + \frac{(n-1)(n+1)}{2}$ which also equals $\frac{n(n+1)}{2}$.

The sum of the counting numbers from 1 to n is equal to $\frac{n(n+1)}{2}$.

This formula is easy to remember as n is the number of terms, and $(n+1)/2$ is the average or arithmetic mean value of the terms in the series.

An **arithmetic sequence** is a sequence of numbers in which *consecutive terms* differ by the **same amount**.

For example

$$5, \ 11, \ 17, \ 23, \ 29, \ 35, \ 41.$$

An order of elements in a sequence matters. The sequence 1, 2, 3, 4 is not the same as the sequence 4, 3, 2, 1.

Arithmetic sequences are sometimes referred to as **arithmetic progressions**, and terms that form an arithmetic sequence are said to be "in arithmetic progression." For example 7, 9, 11, 13 are in arithmetic progression.

Sequence can also have *infinitely* many terms. For example *counting numbers* are *infinite arithmetic sequence*

$$1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ 10, \ 11, \ 12, \ ...$$

In an arithmetic progression, the *first number* in the series is called the **initial (first) term**.

The value by which *consecutive terms increase or decrease* is called the **common difference**.

Usually initial term denoted as a or a_1 and common difference as d.

Some examples

arithmetic sequence	a_1	d
1, 2, 3, 4, 5, 6, ...	1	1
5.5, 7.8, 10.1, 12.4, ...	5.5	2.3
4, 2, 0, -2, -4, -6, ...	4	-2
$\dfrac{2}{3}, \dfrac{5}{6}, 1, \dfrac{7}{6}, \dfrac{4}{3}, \ ...$	$\dfrac{2}{3}$	$\dfrac{1}{6}$
$x, \ 2x, \ 3x, \ 4x, \ 5x, \ ...$	x	x
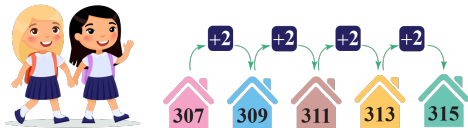
On February 1st, Alice starts reading a book and ends on page 10 (she read the forward and introduction). Then she reads 3 pages per day. How many pages will she read by February 15th?

She will read 3 pages for 14 days (from February 2nd till February 15th), so she will read total of

$$3 \times 14 = 42$$

pages after February 1st. So total she will read

$$42 + 10 = 52$$

pages.

The number of pages she has read by the end of the day from an arithmetic sequence

10, 13, 16, 19, 21, ...

And 15th term of this sequence is 52.

An initial term of an arithmetic series is a, and a common difference is d. Find a term on n^{th} place?

The second term is the initial term and the common difference, the third term is the second term plus the common difference, and so on.

Term = Initial Term +
+ (Common Difference×
× Number of steps from the initial term).

Or written as a formula for a general term of an arithmetic sequence:

$$a_n = a_1 + d \times (n - 1).$$

It is also worth mentioning, each term in the arithmetic series is an average of its neighbors.

The *sum* of the terms of an arithmetic sequence is called an **arithmetic series**.

Find the sum $7 + 11 + 15 + 19 + ... + 83 + 87$?

Let's do the trick done by Gauss in 3rd grade. We pair numbers: 7 with 87, 11 with 83, 15 with 79, and so on. Each pair has the sum 94, and we just need to find a total number of pairs.

How to know how many terms we have?
To get from 7 to 87 we need to add 4 20 times, which means that we have a total of 21 terms. That means we have 10 pairs and one element in the middle of the sequence. The element is on 11th place, so it should be equal to

$$7 + 4 \times (11 - 1) = 47.$$

The sum of the arithmetic series is

$$94 \times 10 + 47 = 987.$$

The **number of terms** in the arithmetic series is

$$\frac{\text{last term} - \text{first term}}{\text{increment}} + 1.$$

Instead of using elements in pairs, we can replace both terms with 2 average values. This will not change the sum, but all terms will become the same.

In an arithmetic series, the **average value** is

$$\frac{\text{first term} + \text{last term}}{2}.$$

This will leads us to formula for a sum of arithmetic series.

In an arithmetic series, the sum is

$$\text{average value} \times \text{number of terms}.$$

REPEATING GEOMETRIC PATTERN

A popular variety of question in middle school math contests involves series of geometric patterns. Consider this question from MOEMS in December 2020:
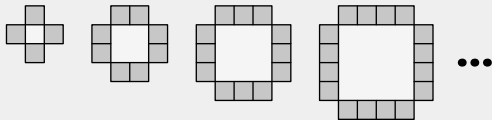
Each figure in the sequence shown is composed entirely of 1×1 shaded squares. If the pattern is continued, how many 1×1 shaded squares will there be altogether in the first 21 figures?

There are 4 squares in the first figure, 8 squares in the second figure, 12 squares in the third figure and so on.

With each successive figure, they are adding one square to each side, and there are 4 sides, so the number of squares for the nth figure in the series is $4n$. So we have figured out the pattern, and now have the general expression for the value of any term in this arithmetic series.

The total number of squares in the first 21 figures

$$4 + 8 + 12 + ... + 84 = 4(1 + 2 + 3 + ... + 21).$$

Using Gauss' rule for summing integers from 1 to n, the sum of the integers 1 to 21 is:

$$\frac{22 \times 21}{2} = 231.$$

The total number of squares is $4 \times 231 = 924$.

You don't always need to use a formula to calculate the sum of arithmetic series.

Find the sum of the sequence: $7 + 8 + ... + 106$?

When we replace 101 with 1, 102 with 2, and so on until 106 with 6 we will get the sum of the first 100 consecutive integers. The result must be the sum of the numbers from 1 to 100 $(= 5050)$ plus 600 for a total of 5650. The 600 is added because the first 6 terms of the series have been reduced to get the example that Gauss summed.

Consider the arithmetic series

$$7, \ 14, \ 21, \ ... \ , \ 693, \ 700.$$

What is the sum of the elements in the series?

We may see that each element in this series is seven times larger than the numbers $1 - 100$, so this sum must be $7\times$ larger than 5050. Well,

$$7 \times 5050 = 35350.$$

The sums of arithmetic series are usually denoted as S_n, where n is the number of terms.
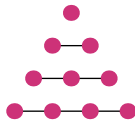It is also helpful to know some sums that are often used in olympiad problems.

Sums of arithmetic series of *counting numbers* 1, 2, 3, 4, $...$ are called the **triangular numbers**.
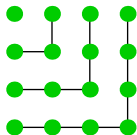
$$T_n = \frac{n(n+1)}{2},$$
$$T_1 = \mathbf{1}, \ T_2 = \mathbf{3}, \ T_3 = \mathbf{6}, \ T_4 = \mathbf{10},$$
$$T_5 = \mathbf{15}, \ T_{10} = \mathbf{55}, \ T_{100} = \mathbf{5050}.$$

Sums of arithmetic series of *odd number* 1, 3, 5, 7, $...$ are called the **square numbers**.

$S_n = n^2, \ S_1 = \mathbf{1}, \ S_2 = \mathbf{4}, \ S_3 = \mathbf{9},$
$S_4 = \mathbf{16}, \ S_5 = \mathbf{25}, \ S_{10} = \mathbf{100}, \ ... \ .$

The general name for that kind of series is the **figurate numbers**.

1. Find the value of the expression
   $100 - 98 + 96 - 94 + 92 - 90 + \cdots + 8 - 6 + 4 - 2.$

2. The sum of 25 consecutive even integers is $10,000$. What is the largest of these 25 consecutive integers?

3. If the 7th term of an arithmetic sequence is 24 and the 12th term is 48, then what is the 57th term?

4. The first two terms of an arithmetic sequence are 4, 10. What is the first term greater than 1000?

5. The sequence an has the property that $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. It is also true that $a_0 = 4$ and $a_4 = 26$. What is the value of a_5?

6. The 2020th term of an arithmetic sequence is $\dfrac{21}{19}$ times the 2022nd term of the arithmetic sequence. What is the value of the 2024th term of this sequence divided by the first term of this sequence?

7. What is the value of
   $1 + 2 + 3 - 4 + 5 + 6 + 7 - 8 + \cdots + 197 + 198 + 199 - 200$?

8. What is the greatest number of consecutive integers whose sum is 45?

1. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

2. 852 digits are used to number the pages of a book consecutively from page 1. How many pages are there in the book?

3. For every n the sum of n terms of an arithmetic progression is $2n + 3n^2$. What is the rth term of the sequence in terms of r?

4. A sequence of natural numbers is constructed by listing the first 4, then skipping one, listing the next 5, skipping 2, listing 6, skipping 3, and on the nth iteration, listing $n + 3$ and skipping n. The sequence begins 1, 2, 3, 4, 6, 7, 8, 9, 10, 13. What is the 1000th number in the sequence?