Verbal olympiad solutions

Graham Middle School Math Olympiad Team

PROBLEMS 1-2

P1. A duckling and a gosling were participating in a triathlon competition. The distances for running, swimming, and flying was the same. Duckling ran, swam, and flew with the same speed. The gosling ran twice as slow as the duckling but swam twice as fast as the duckling. Who should fly faster and by how much, so they start and finish at the same time?

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Answer: The gosling flew twice as fast as the duckling.
Let the duckling complete each part in 1 hour.
Then gosling was running for 2 hours and was swimming for $\frac{1}{2}$. As a result she had $3-2-\frac{1}{2}=\frac{1}{2}$ hours for flying to finish in the same time as duckling.

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Between each taller and shorter witches all other witches are sorted by height, so in this line, all other witches are ”in-between”. From each taller witch to the left and to the right there are lines of witches in descendant order, so between two taller witches exactly one shorter witch. Since these three taller witches form 3 pairs in this circle, there are 3 shorter witches in the circle. So we have $19-3-3=13$ ”in-between” witches.

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$$AB = 1,\ BC = 2,\ CD = 3,\ DE = 4,$$
$$EF = 5,\ FG = 6,\ GA = 7.$$

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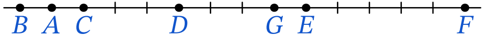
The circle $A - B - C - D - E - F - G - A$ has a length $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ and visit both the left and the right points. So the maximum distance between two furthest point is $\frac{28}{2} = 14$.
The possible configuration of the points is shown in the picture:



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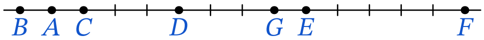

The uniqueness of this configuration is followed in that points between the two furthest points should be aligned from left to right, so 14 should be the sum of several consecutive numbers. Only one such sequence exists: $2 + 3 + 4 + 5 = 14$.

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So our *assumption* is incorrect and there should be two different numbers of one color with the difference of a square of an integer.