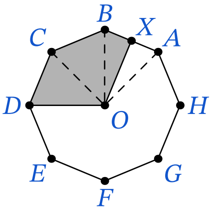
Angles in Circles solutions

Graham Middle School Math Olympiad Team

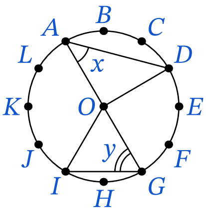
EXERCISES 1-2

E1. Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side $\overline{AB}$. What fraction of the area of the octagon is shaded?



Area of each triangle YOZ where Y and Z are neighbor points from A to H is $1/8$ of the area of the octagon. Since $DCBXO$ contains 2.5 of such triangles, so the total fraction is $2.5 \cdot \frac{1}{8} = \frac{5}{2} \cdot \frac{1}{8} = \boxed{\frac{5}{16}}$.

E2. The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y?

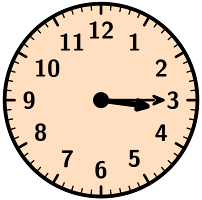


$\angle AOD = \frac{360°}{12} \cdot 3 = 90°$, and since $AO = OD$, $\angle OAD = \angle ADO = x$. So $2x + 90° = 180°$ and $x = 45°$. In similar way we got $\angle IOG = \frac{360°}{12} \cdot 2 = 60°$ and $2y + 60° = 180°$, so $y = 60°$.

$x + y = 45° + 60° = \boxed{105°}$.

EXERCISES 3-5

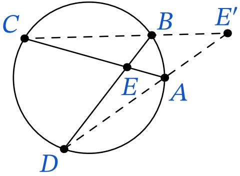
E3. What is the angle (less than $180°$) formed by the hands of a clock when the time is $3:15$?



The minute hand moved $15 \cdot \frac{360°}{60} = 90°$ from the top. The hours hand moved $3.25 \cdot \frac{360°}{12} = \frac{13}{4} \cdot 30° = 97.5°$. So the angle is $97.5° - 90° = \boxed{7.5°}$.

E4. Points A, B, C, and D are on a circle with center O in that order, and AC and BD meet at E. Given that $\angle AOB = 34°$, $\angle COD = 102°$, $\angle DOA = 109°$, find $\angle BEC$.

E5. Points A, B, C, and D are on a circle with center O in that order, and DA and CB meet at E'. Given that $\angle AOB = 34°$, $\angle COD = 102°$, $\angle DOA = 109°$, find $\angle AE'B$.



Problem 4:
$\angle ACB = \frac{1}{2}\angle AOB = 17°$ and $\angle DBC = \frac{1}{2}\angle COD = 51°$. So $\angle BEC = 180° - \angle ECB - \angle EBC = 180° - 17° - 51° = \boxed{112°}$.

In general $\angle BEC = \dfrac{\angle CEB + \angle DEA}{2}$.

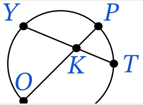
Problem 5:
$\angle CAD = \frac{1}{2}\angle COB = 51°$ and $\angle ACB = \frac{1}{2}\angle AOB = 17°$. So $\angle CAE' = 180° - \angle CAD = 129°$, and $\angle AE'B = 180° - 17° - 129° = \boxed{34°}$.

In general $\angle AE'B = \dfrac{\angle COD - \angle AOB}{2}$.

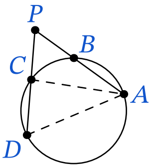
EXERCISES 6-8

E6. Chords $\overline{TY}$ and $\overline{OP}$ meet at point K such that $TK = 2$, $KY = 16$, and $KP = 2 \cdot KO$. Find OP.

Let $KO = x$, than $KP = 2x$ and $OP = 3x$. Since $KP \cdot KO = TK \cdot KY$, $2x^2 = 32$ and $x = 4$. So $OP = \boxed{12}$.



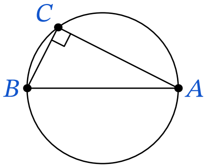
E7. Points A, B, C, and D are on a circle in that order and AB intersects DC in point P outside the circle. We have $BP = 8$, $AB = 10$, $CD = 7$, and $\angle APC = 60°$. Find the radius of the circle.



$PC \cdot PD = PA \cdot PB$, so if $PC = x$, $x(x+7) = 8 \cdot 18$ and $x = 9$ or -16. The later root doesn't make sense, so $PC = 9$. So $\triangle PCA$ is $30-60-90$ triangle and $\angle ACD = 90°$. So DA is the diameter of the circle and $DA = \sqrt{DC^2 + CA^2} = \sqrt{DC^2 + (\sqrt{3}PC)^2} = \sqrt{49 + 3 \cdot 81} = \sqrt{292} = 2 \cdot \sqrt{73}$. So radius is $\boxed{\sqrt{73}}$.

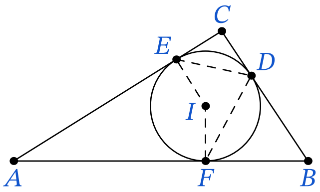
E8. Diameter AB of a circle has length 25. Point C is chosen along the circumference such that AC has length 24. What is the length of BC?

Since AB is diameter, $\angle C = 90°$. And using the Pythagorean theorem $BC^2 = AB^2 - AC^2$. So $BC = \sqrt{25^2 - 24^2} = \sqrt{(25+24)(25-24)} = \sqrt{49} = \boxed{7}$.



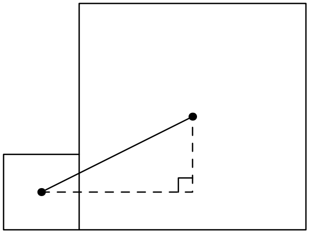
CHALLENGE PROBLEMS 1-2

CP1. Let the incircle of triangle ABC be tangent to sides BC, AC, and AB at points D, E, and F, respectively. Given that $\angle A = 32°$, find $\angle EDF$.



$\angle IEA = \angle IFA = 90°$, so $\angle EIF = 360° - 90° - 90° - 32° = 148°$. And $\angle EDF = \frac{1}{2}\angle EIF = \boxed{74°}$.

CP2. The areas of two adjacent squares are 256 square inches and 16 square inches, respectively, and their bases lie on the same line. What is the number of inches in the length of the segment that joins the centers of the two inscribed circles?

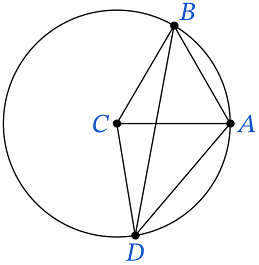


The sides of both squares are $\sqrt{16} = 4$ and $\sqrt{256} = 16$. So horizontal distance between centers is $4/2 + 16/2 = 10$ inches, and vertical distance is $16/2 - 4/2 = 6$ inches. So distance between the points is $\sqrt{10^2 + 6^2} = \sqrt{136} = \boxed{2\sqrt{34}}$.

CHALLENGE PROBLEMS 3-4

CP3. We are given points A, B, C, and D in the plane such that $AD = 13$ while $AB = BC = AC = CD = 10$. Find $\angle ADB$.

Since A, B, and C all equidistance from C, points A, B, and D are all on the circle with the center C. Moreover, $\triangle ABC$ is equilateral. So $\angle ADB = \frac{1}{2}\angle BCA = \boxed{30^\circ}$.



CP4. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

Let $BX = q$, $CX = p$, and AC meets the circle at Y and Z, with Y on AC. Then $AZ = AY = 86$. Using the Power of a Point, we get that $p(p+q) = 11(183) = 11 \cdot 3 \cdot 61$. We know that $p+q > p$, so p is either 3, 11, or 33. We also know that $p > 11$ by the triangle inequality on $\triangle ACX$. Thus, p is 33 so we get that $BC = p + q = \boxed{61}$.