BAMO-8 2023, Solutions

PROBLEM A

A. A *tangent line* to a circle is a line that intersects the circle in exactly one point. A *common tangent line* to two circles is a line that is tangent to both circles.

Given two distinct circles in the plane, let n be the number of common tangent lines that can be drawn to these two circles. What are all possible values of n? Your answer should include drawings with explanations.

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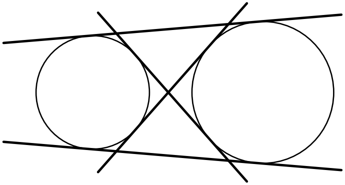
Case 1: two circles outside each other and not touching.



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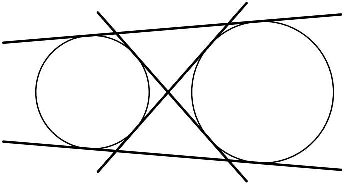

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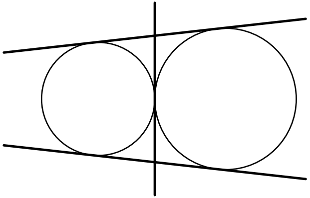
Case 2: two circles outside each other and touching.



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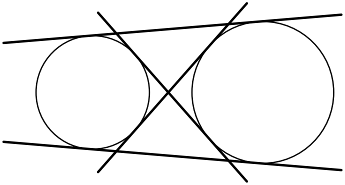

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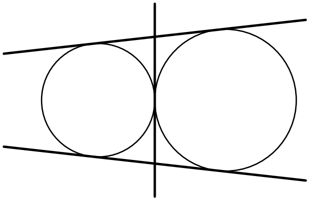
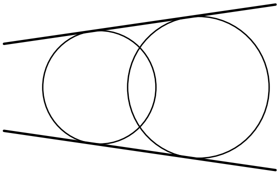
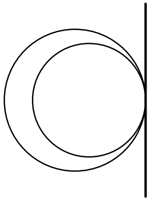


In this case we have $n = 3$.
Case 3: two circles intersect.



In this case we have $n = 2$.

Case 4: one circle is inside another and they are touching.



In this case we have $n = 1$.

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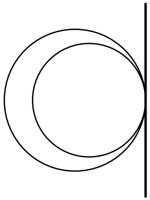

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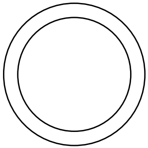
Case 5: one circle is inside another and they aren't touching.



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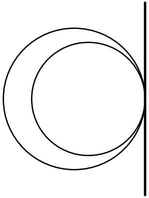

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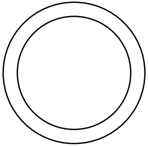

In this case we have $n = 0$. So n can has 5 values: 0, 1, 2, 3, and 4.

PROBLEM B

B. Ara and Bea play a game where they take turns putting numbers from 1 to 5 into the cells of the X-shaped diagram below. Each number can be played only once, and a cell cannot have more than one number placed in it. Ara's goal is for the two diagonals of the X diagram to have the same sum when the game is over; Bea's goal is for these two sums to be unequal.

(a) Show that Ara can always win if he goes first.

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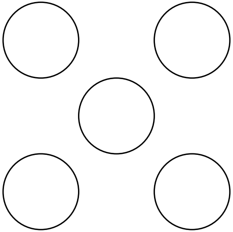

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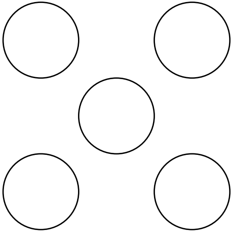

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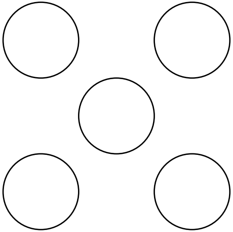

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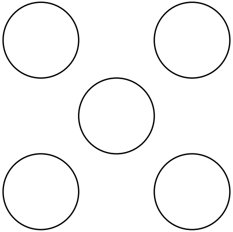

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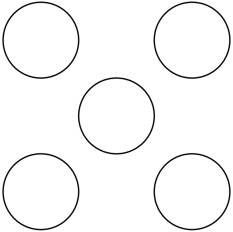

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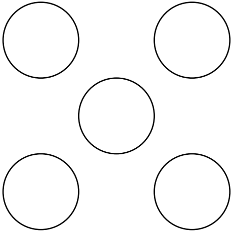

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For the second move, Bea has to place one number from one of these groups into a corner cell, so Ara places the remaining number from the group into the cell in the same diagonal.

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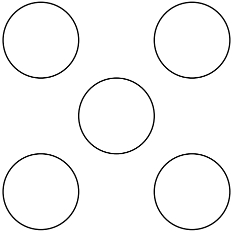

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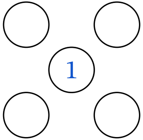

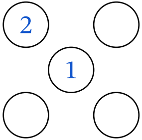

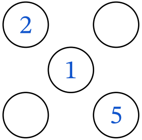

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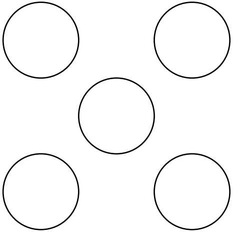

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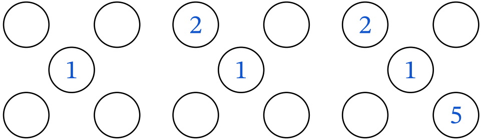

With this strategy, both diagonals will have the same sums. For example, when we start with 1 in the center, the diagonal sums will be

$2 + 1 + 5 = 8 = 3 + 1 + 4$.

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If Bea starts with 2, the remaining numbers are $1 + 3 + 4 + 5 = 13$, so it doesn't matter how they are distributed, one of the sums on the diagonals will be odd and one will be even.

PROBLEM C

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Mr. Murgatroyd sends his teaching assistant to post the ten messages in locations 1 to 10. Unfortunately, the assistant jumbles up the message cards at random before posting them. If the students begin at location 1 as planned and follow the directions at each location, show that they will still get to the pizza.

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PROBLEM D

D. Given a positive integer N (written in base 10), define its integer substrings to be integers that are equal to strings of one or more consecutive digits from N, including N itself. For example, the integer substrings of 3208 are 3, 2, 0, 8, 32, 20, 320, 208, 3208. (The substring 08 is omitted from this list because it is the same integer as the substring 8, which is already listed.)
What is the greatest integer N such that no integer substring of N is a multiple of 9? (Note: 0 is a multiple of 9.)

Let our number is $\overline{a_1 a_2 \dots a_n}$, where a_i are some digits.
By division by 9, no sum $a_i + a_{i+1} + \dots + a_{i+k}$ is divisible by 9.

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The largest possible digit of the N is 8, since 9 is divisible by 9, so we got $\boxed{N = 88,888,888}$.

PROBLEM D

D. Given a positive integer N (written in base 10), define its integer substrings to be integers that are equal to strings of one or more consecutive digits from N, including N itself. For example, the integer substrings of 3208 are 3, 2, 0, 8, 32, 20, 320, 208, 3208. (The substring 08 is omitted from this list because it is the same integer as the substring 8, which is already listed.)
What is the greatest integer N such that no integer substring of N is a multiple of 9? (Note: 0 is a multiple of 9.)

Let our number is $\overline{a_1 a_2 \dots a_n}$, where a_i are some digits.
By division by 9, no sum $a_i + a_{i+1} + \dots + a_{i+k}$ is divisible by 9.

Let's consider sums:

$$a_1,$$
$$a_1 + a_2,$$
$$a_1 + a_2 + a_3,$$
$$\dots,$$
$$a_1 + a_2 + \dots + a_n.$$

If any of these sums have the same reminder when divisible by 9, that means that their difference is divisible by 9, so we have some sequence $a_i + a_{i+1} + \dots + a_{i+k}$ divisible by 9, which contradicts.
Since we can have only 8 different reminders when divide by 9 (except 0), the maximal length of the N is 8.
The largest possible digit of the N is 8, since 9 is divisible by 9, so we got $\boxed{N = 88{,}888{,}888}$. It is easy to check that it doesn't have any substring divisible by 9.

PROBLEM E

E. In the following figure—not drawn to scale!—E is the midpoint of BC, triangle FEC has area 7, and quadrilateral $DBEG$ has area 27. Triangles ADG and GEF have the same area, x. Find x.

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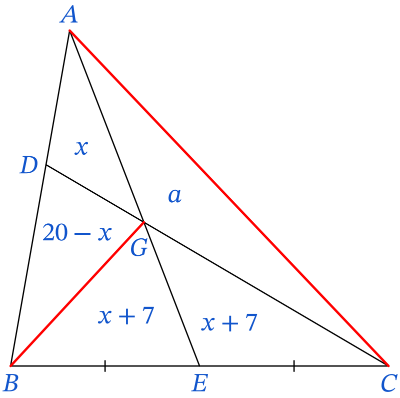

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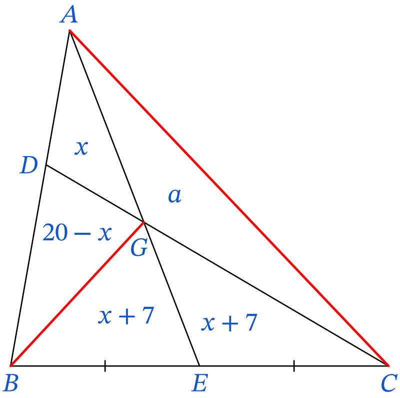

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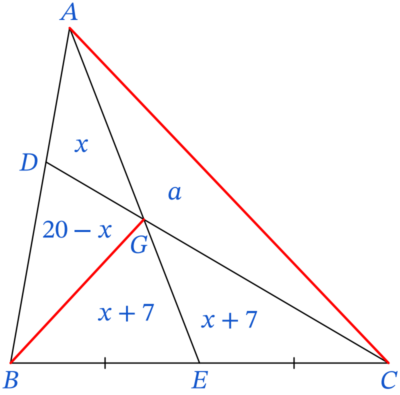

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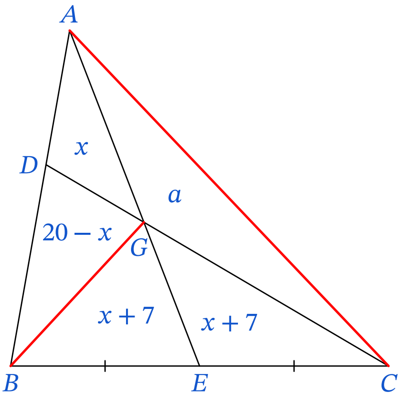

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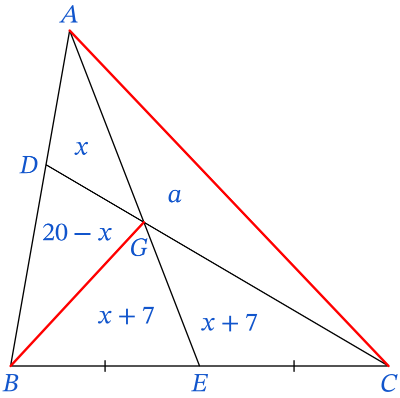

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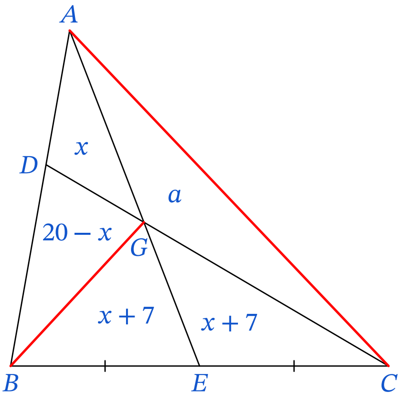

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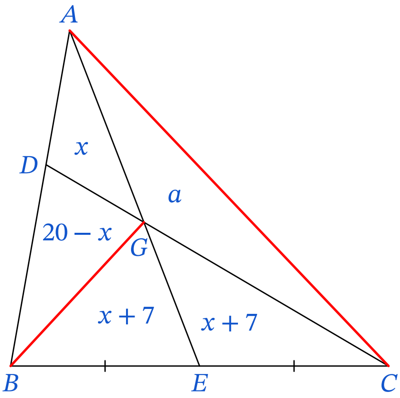

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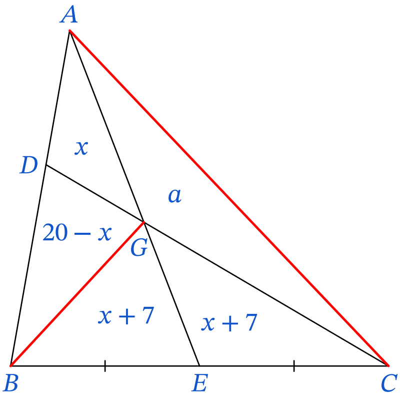

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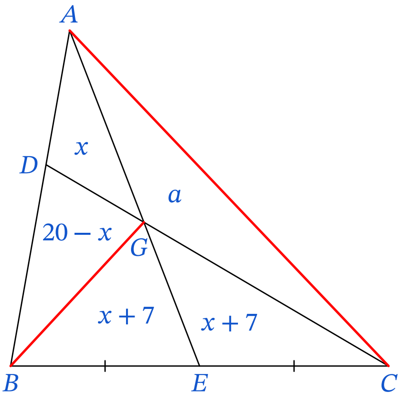

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Or $2x^2+14x = 400-20x$ or $x^2+17x-200 = 0$ or $(x-8)(x+25) = 0$. So $\boxed{x = 8}$, since other root is negative.

PROBLEM 4

4. Zaineb makes a large necklace from beads labeled $290, 291, \ldots, 2023$. She uses each bead exactly once, arranging the beads in the necklace any order she likes. Prove that no matter how the beads are arranged, there must be three beads in a row whose labels are the side lengths of a triangle.

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Contradiction! So there are three beads in a row whose labels are the side lengths of a triangle.

PROBLEM 5

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