Binomials solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-4

E1. The *harmonic mean* of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1, 2, and 4?

The sum of the reciprocals is $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}=\frac{7}{4}$. Their average is $\frac{7}{12}$. Taking the reciprocal of this gives $\boxed{\frac{12}{7}}$

E2. If $\frac{x+1}{x}=2$, what is the value of $\frac{x^2+1}{x^2}$?

$\frac{x+1}{x}=1+\frac{1}{x}=2$, so $\frac{1}{x}=1$ or $x=1$.

$\frac{x^2+1}{x^2}=\frac{1^2+1}{1^2}=\boxed{2}$.

E3. What is the greatest integer value of x such that $\frac{x^2+2x+5}{x-3}$ is an integer?

$\frac{x^2+2x+5}{x-3}=\frac{(x-3)x+5x+5}{x-3}=\frac{(x-3)x+5(x-3)+20}{x-3}=x+5+\frac{20}{x-3}$. Since $x+5$ is an integer for any integer x, $\frac{20}{x-3}$ should be integer. So $x=\boxed{23}$ is the greatest possible integer, for all $x>23$ the nominator is smaller than the denominator.

E4. What is the value of the product

$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

$$\left(\frac{1\cdot \cancel{3}}{2\cdot \cancel{2}}\right)\left(\frac{\cancel{2}\cdot \cancel{4}}{\cancel{3}\cdot \cancel{3}}\right)\left(\frac{\cancel{3}\cdot \cancel{5}}{\cancel{4}\cdot \cancel{4}}\right)\cdots\left(\frac{\cancel{98}\cdot 100}{\cancel{99}\cdot 99}\right)=\frac{1\cdot 100}{2\cdot 99}=\boxed{\frac{50}{99}}$$

EXERCISES 5-8

E5. What is the value of $\frac{2022^3-2\cdot 2022^2\cdot 2023+3\cdot 2022\cdot 2023^2-2023^3+1}{2022\cdot 2023}$?

The first four terms in nominator are very close to $(a-b)^3$.

$$\frac{2022^3-2\cdot 2022^2\cdot 2023+3\cdot 2022\cdot 2023^2-2023^3+1}{2022\cdot 2023} =$$

$$= \frac{[2022-(2023)]^3+2022^2\cdot 2023+1}{2022\cdot 2023} =$$

$$= \frac{-1+2022^2\cdot 2023+1}{2022\cdot 2023} = \boxed{2022}.$$

E6. If $\left(\frac{1}{x+y}\right)\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{13}$, what is the value of the product of x and y?

$\left(\frac{1}{x+y}\right)\left(\frac{1}{x}+\frac{1}{y}\right)=\left(\frac{1}{x+y}\right)\left(\frac{y+x}{xy}\right)=\frac{1}{xy}=\frac{1}{13}$, so $xy=\boxed{13}$.

E7. If $a\clubsuit b=\frac{ab}{a+b}$ and $a\clubsuit 4=3$, what is the value of a?

$\frac{a\cdot 4}{a+4}=3$, so $4a=3a+12$ or $a=\boxed{12}$.

E8. Compute the sum of the distinct prime factors of $2^{15}+2^8-2^7-1$.

$$2^{15}+2^8-2^7-1=(2^7+1)(2^8-1)=$$

$$=(2^7+1)(2^4+1)(2^4-1)=$$

$$=(2^7+1)(2^4+1)(2^2+1)(2^2-1)=$$

$$=(2^7+1)(2^4+1)(2^2+1)(2^1+1)(2^1-1)=$$

$$=129\cdot 17\cdot 5\cdot 3\cdot 1=$$

$$=43\cdot 3\cdot 17\cdot 5\cdot 3\cdot 1.$$

So answer is $43+17+5+3=\boxed{68}$.

CP1. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

We compute $2^8 + 1 = 257$. We're all familiar with what 6^3 is, namely 216, which is too small. The smallest cube greater than it is $7^3 = 343$.
$2^{18} + 1$ is too large to calculate, but we notice that $2^{18} = (2^6)^3 = 64^3$, which therefore clearly will be the largest cube less than $2^{18} + 1$.
So, the required number of cubes is
$64 - 7 + 1 = \boxed{58}$

CP2. Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c) = 0$?

Factoring out $(x - b)$ from the equation yields

$$(x-b)(2x-(a+c)) = 0 \Rightarrow (x-b)\left(x - \frac{a+c}{2}\right) = 0.$$

Therefore the roots are b and $\frac{a+c}{2}$. Because b must be the larger root to maximize the sum of the roots, letting $a, b,$ and c be $8, 9,$ and 7 respectively yields the sum $9 + \frac{8+7}{2} = 9 + 7.5 = \boxed{16.5}$.

CP3. If $y+4=(x-2)^2$, $x+4=(y-2)^2$, and $x \neq y$, what is the value of x^2+y^2?

Subtract the two equations to get

$$(x-y)=(x+y-4)(y-x) \iff x+y=3.$$

Plugging back into any of the original equations yields

$$(3-x)+4=(x-2)^2 \iff x^2-3x=3.$$

However, we know $x^2+y^2=x^2+(3-x)^2=2x^2-6x+9=2(x^2-3x)+9=2\cdot 3+9=\boxed{15}$

CP4. For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

All the desired terms are in the form $a^x b^y c^z d^w 1^t$, where $x+y+z+w+t=N$ (the 1^t part is necessary to make stars and bars work better.) Since x, y, z, and w must be at least 1 (t can be 0), let $x'=x-1$, $y'=y-1$, $z'=z-1$, and $w'=w-1$, so $x'+y'+z'+w'+t=N-4$. Now, we use stars and bars to see that there are $\binom{(N-4)+4}{4}$ or $\binom{N}{4}$ solutions to this equation. We notice that $1001=7\cdot 11\cdot 13$, which leads us to guess that N is around these numbers. This suspicion proves to be correct, as we see that $\binom{14}{4}=1001$, giving us our answer of $\boxed{14}$.