Arithmetic sequencies

Graham Middle School Math Olympiad Team

GAUSSIAN SUMS

To keep his 3rd grade students busy, a teacher in 18th century Germany asked to find the sum of the numbers from 1 to 100. But one of the students in the class was 10-year-old Carl Friedrich Gauss. He instantly answered 5050. How did he do that? To sum the digits

$$1 + 2 + 3 + \ldots + 98 + 99 + 100$$

$$101$$

$$101$$

$$101$$

we pair terms as shown. So total, we have 50 pairs with a value 101. Then multiplying them together, we will get 5050.

What is the sum of first n numbers?

If n is even, there are $\dfrac{n}{2}$ pairs, each equal to $n+1$, and the sum is $\dfrac{n(n+1)}{2}$.

If n is odd, the middle term, which is equal to $\dfrac{n+1}{2}$, is unpaired, and there are $\dfrac{n-1}{2}$ paired terms that each sum to $n+1$. So in the odd case the total sum is $\dfrac{n+1}{2} + \dfrac{(n-1)(n+1)}{2}$ which also equals $\dfrac{n(n+1)}{2}$.

The sum of the counting numbers from 1 to n is equal to $\dfrac{n(n+1)}{2}$.

This formula is easy to remember as n is the number of terms, and $(n+1)/2$ is the average or arithmetic mean value of the terms in the series.

ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence of numbers in which *consecutive terms* differ by the **same amount**.

For example

$$5,\ 11,\ 17,\ 23,\ 29,\ 35,\ 41.$$

An order of elements in a sequence matters.The sequence 1, 2, 3, 4 is not the same as the sequence 4, 3, 2, 1.

Arithmetic sequences are sometimes referred to as **arithmetic progressions**, and terms that form an arithmetic sequence are said to be ”in arithmetic progression.” For example 7, 9, 11, 13 are in arithmetic progression.

Sequence can also have *infinitely* many terms. For example *counting numbers* are *infinite arithmetic sequence*

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ \ldots$$

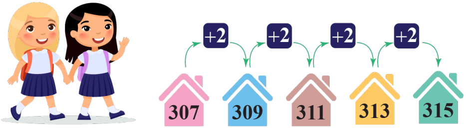
In an arithmetic progression, the *first number* in the series is called the **initial (first) term**.

The value by which *consecutive terms increase or decrease* is called the **common difference**.

Usually initial term denoted as a or a_1 and common difference as d.

Some examples

arithmetic sequence	a_1	d
$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \ldots$	1	1
$5.5,\ 7.8,\ 10.1,\ 12.4,\ \ldots$	5.5	2.3
$4,\ 2,\ 0,\ -2,\ -4,\ -6,\ \ldots$	4	-2
$\frac{2}{3},\ \frac{5}{6},\ 1,\ \frac{7}{6},\ \frac{4}{3},\ \ldots$	$\frac{2}{3}$	$\frac{1}{6}$
$x,\ 2x,\ 3x,\ 4x,\ 5x,\ \ldots$	x	x



GENERAL TERM OF ARITHMETIC SEQUENCE

On February 1st, Alice starts reading a book and ends on page 10 (she read the forward and introduction). Then she reads 3 pages per day. How many pages will she read by February 15th?

She will read 3 pages for 14 days (from February 2nd till February 15th), so she will read total of

$$3 \times 14 = 42$$

pages after February 1st. So total she will read

$$42 + 10 = 52$$

pages.

The number of pages she has read by the end of the day from an arithmetic sequence

$$10,\ 13,\ 16,\ 19,\ 21,\ \ldots$$

And 15th term of this sequence is 52.

An initial term of an arithmetic series is a, and a common difference is d. Find a term on n^{th} place?

The second term is the initial term and the common difference, the third term is the second term plus the common difference, and so on.

$$\text{Term} = \text{Initial Term} + \\ + (\text{Common Difference} \times \\ \times \text{Number of steps from the initial term}).$$

Or written as a formula for a general term of an arithmetic sequence:

$$a_n = a_1 + d \times (n - 1).$$

It is also worth mentioning, each term in the arithmetic series is an average of its neighbors.

ARITHMETIC SERIES

The *sum* of the terms of an arithmetic sequence is called an **arithmetic series**.

Find the sum $7 + 11 + 15 + 19 + \ldots + 83 + 87$?

Let's do the trick done by Gauss in 3rd grade. We pair numbers: 7 with 87, 11 with 83, 15 with 79, and so on. Each pair has the sum 94, and we just need to find a total number of pairs.

How to know how many terms we have?
To get from 7 to 87 we need to add 4 20 times, which means that we have a total of 21 terms. That means we have 10 pairs and one element in the middle of the sequence. The element is on 11th place, so it should be equal to

$$7 + 4 \times (11-1) = 47.$$

The sum of the arithmetic series is

$$94 \times 10 + 47 = 987.$$

The **number of terms** in the arithmetic series is

$$\frac{\text{last term} - \text{first term}}{\text{increment}} + 1.$$

Instead of using elements in pairs, we can replace both terms with 2 average values. This will not change the sum, but all terms will become the same.

In an arithmetic series, the **average value** is

$$\frac{\text{first term} + \text{last term}}{2}.$$

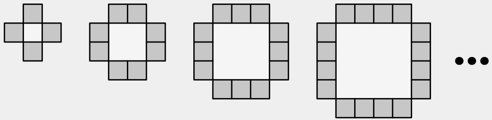
This will leads us to formula for a sum of arithmetic series.

In an arithmetic series, the sum is

$$\text{average value} \times \text{number of terms}.$$

REPEATING GEOMETRIC PATTERN

A popular variety of question in middle school math contests involves series of geometric patterns. Consider this question from MOEMS in December 2020:



Each figure in the sequence shown is composed entirely of 1×1 shaded squares. If the pattern is continued, how many 1×1 shaded squares will there be altogether in the first 21 figures?

There are 4 squares in the first figure, 8 squares in the second figure, 12 squares in the third figure and so on.

With each successive figure, they are adding one square to each side, and there are 4 sides, so the number of squares for the n^{th} figure in the series is $4n$. So we have figured out the pattern, and now have the general expression for the value of any term in this arithmetic series.

The total number of squares in the first 21 figures

$$4 + 8 + 12 + \ldots + 84 = 4\,(1 + 2 + 3 + \ldots + 21)\,.$$

Using Gauss' rule for summing integers from 1 to n, the sum of the integers 1 to 21 is:

$$\frac{22 \times 21}{2} = 231.$$

The total number of squares is $4 \times 231 = 924.$

ARITHMETIC SERIES SHORTCUTS

You don't always need to use a formula to calculate the sum of arithmetic series.

Find the sum of the sequence: $7 + 8 + \ldots + 106$?

When we replace 101 with 1, 102 with 2, and so on until 106 with 6 we will get the sum of the first 100 consecutive integers. The result must be the sum of the numbers from 1 to 100 ($= 5050$) plus 600 for a total of 5650. The 600 is added because the first 6 terms of the series have been reduced to get the example that Gauss summed.

Consider the arithmetic series

$$7,\ 14,\ 21,\ \ldots,\ 693,\ 700.$$

What is the sum of the elements in the series?

We may see that each element in this series is seven times larger than the numbers $1 - 100$, so this sum must be $7\times$ larger than 5050. Well,

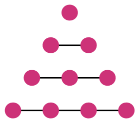
$$7 \times 5050 = 35350.$$

The sums of arithmetic series are usually denoted as S_n, where n is the number of terms.
It is also helpful to know some sums that are often used in olympiad problems.

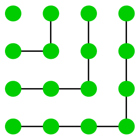
Sums of arithmetic series of *counting numbers* 1, 2, 3, 4, $\ldots$ are called the **triangular numbers**.

$$T_n = \frac{n(n+1)}{2},$$

$T_1 = \mathbf{1}$, $T_2 = \mathbf{3}$, $T_3 = \mathbf{6}$, $T_4 = \mathbf{10}$,
$T_5 = \mathbf{15}$, $T_{10} = \mathbf{55}$, $T_{100} = \mathbf{5050}$.



Sums of arithmetic series of *odd number* 1, 3, 5, 7, $\ldots$ are called the **square numbers**.



$S_n = n^2$, $S_1 = \mathbf{1}$, $S_2 = \mathbf{4}$, $S_3 = \mathbf{9}$,
$S_4 = \mathbf{16}$, $S_5 = \mathbf{25}$, $S_{10} = \mathbf{100}$, $\ldots$.

The general name for that kind of series is the **figurate numbers**.

EXERCISES

1. What is the average value of the terms in the arithmetic series $5, 13, 21, \ldots 93, 101$?
2. Find the first term of the arithmetic sequence which 12th term is equal to 43 and the difference is equal to 4.
3. In an arithmetic sequence the 6th term is equal to 26 and 7th term is equal to 30. Find the common difference and the initial term of the sequence.
4. How many terms in the arithmetic sequence $2, 7, 12, 17, \ldots$ are three-digit numbers?
5. What is the sum of all the integers between -20 and 50 inclusive?
6. If we subtract the arithmetic series $1 + 5 + 9 + \ldots + 97 + 101$ from the arithmetic series $1 + 3 + 5 + \ldots + 99 + 101$, what is the sum of the remaining terms?
7. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

8. Figures 0, 1, 2, and 3 consist of 1, 5, 13, and 25 non-overlapping unit squares, respectively. If the pattern were continued, how many non-overlapping unit squares would there be in figure 100?

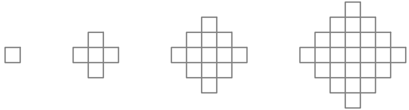


Figure 0 Figure 1 Figure 2 Figure 3

CHALLENGE PROBLEMS

1. Sum of 3^{rd} and 7^{th} terms of the arithmetic sequence is equal to 12. Second term is two times smaller than 5^{th} term. Find the initial term and the common difference of the sequence.
2. Find the sum of the first 15^{th} terms of the arithmetic series, if the sum of 4^{th}, 5^{th}, 7^{th}, and 16^{th} is equal to 32.
3. The pages of a book are numbered from 1 through n. When the page numbers of the book are added, one of the page numbers was mistakenly added twice, resulting in the incorrect sum of 2021. What was the page that was added twice?
4. In an attempt to copy down a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers 113, 137, 149, 155, 173, accidentally omitting one. He later discovered that he also miscopied one of them. What is the original sequence?