Number Theory 102 solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-4

E1. Convert to the base 7 the number 532_8?

$532_8 = 5\times 8^2 + 3\times 8 + 2 = 320 + 24 + 2 = 346_{10}$.

$$\begin{array}{rrl}
 & 0 & \text{R } \mathbf{1} \\
7 \,) & 1 & \text{R } \mathbf{0} \\
7 \,) & 7 & \text{R } \mathbf{0} \\
7 \,) & 49 & \text{R } \mathbf{3} \\
7 \,) & 346 &
\end{array}$$

$532_8 = 346_{10} = \boxed{1003_7}$.

E2. What is the last digit in 7^{149} ?

$7^4 \equiv 1 \pmod{10}$, and $149 \equiv 1 \pmod{4}$, so $7^{149} \equiv 7^1 \equiv \boxed{7} \pmod{10}$.

E3. How many trailing zeros in $144!$?

There are 28 multiples of 5 less or equal than 144, there are 5 multiples of $5^2 = 25$ less or equal than 144, and there is one multiple of $5^3 = 125$ less of equal than 144.

So $144!$ has $28 + 5 + 1 = \boxed{34}$ trailing zeros.

E4. $R + RR = BOW$. What is the last digit of $F \times A \times I \times N \times T \times I \times N \times G$?

The only way for $R + RR$ to be bigger than 100 is $R = 9$ and $BOW = 108$. So for 6 other digits F, A, I, N, T, and G we have only 6 other digits, so $F \times A \times I \times N \times T \times G = 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7$, and it is ended up with 0. So doesn't matter which digits are N and I the last digit of

$F \times A \times I \times N \times T \times I \times N \times G$ is $\boxed{0}$.

E5. What is the largest power of 2 that is a divisor of $13^4 - 11^4$?

First, we need to factor $13^4 - 11^4$. We will use formula

$$a^2 - b^2 = (a-b)(a+b).$$

$13^4 - 11^4 = (13^2)^2 - (11^2)^2 = (13^2 - 11^2)(13^2 + 11^2) = (13-11)(13+11)(13^2+11^2) = 2 \times 24 \times (169+121) = 2 \times 24 \times 290$. $2 = 2^1$, 24 contains 2^3, and 290 contains 2^1, so result is $1+3+1 = 5$.

E6. What is the smallest positive integer greater than 1 that leaves a remainder of 1 when divided by 4, 5, and 6?

If we subtract 1 from this number, the resulting number should be divisible by 4, 5, and 6. The smallest positive integer greater that 0 is $\operatorname{lcm}(4,5,6) = 60$, so answer for our problem is $60 + 1 = \boxed{61}$.

E7. What is the largest integer n for which 5^n is a factor of the sum $98! + 99! + 100!$?

$98! + 99! + 100! = 98! \times (1 + 99 + 99 \cdot 100) = 98! \cdot 100 \cdot 100$. $98!$ contains 19 multiples of 5 and 3 multiples of 25. $100 = 2^2 \cdot 5^2$. So $n = 19 + 3 + 2 + 2 = \boxed{26}$.

E8. Starting with some gold coins and some empty treasure chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest, but then I had 3 gold coins left over. How many gold coins did I have?

We can represent the amount of gold with g and the amount of chests with c. We can use the problem to make the following equations:

$$9c - 18 = g, \quad 6c + 3 = g$$

Therefore, $6c + 3 = 9c - 18$. This implies that $c = 7$. We therefore have $g = \boxed{45}$.

CHALLENGE PROBLEMS 1-2

C1. The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

We see that since QRS is divisible by 5, S must equal either 0 or 5, but it cannot equal 0, so $S = 5$. We notice that since PQR must be even, R must be either 2 or 4. However, when $R = 2$, we see that $T \equiv 2 \pmod 3$, which cannot happen because 2 and 5 are already used up; so $R = 4$. This gives $T \equiv 3 \pmod 4$, meaning $T = 3$. Now, we see that Q could be either 1 or 2, but 14 is not divisible by 4, but 24 is. This means that $Q = 2$ and $P = \boxed{1}$.

C2. What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2022?

Observe that $2022_{10} = 5616_7$. To maximize the sum of the digits, we want as many 6s as possible (since 6 is the highest value in base 7), and this will occur with either of the numbers 4666_7 or 5566_7. Thus, the answer is

$4 + 6 + 6 + 6 = 5 + 5 + 6 + 6 = \boxed{22}$.

C3. The base-ten representation for $19!$ is $121{,}6T5{,}100{,}40M{,}832{,}H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

We can figure out $H=0$ by noticing that $19!$ will end with 3 zeroes, as there are three factors of 5 in its prime factorization, so there would be 3 powers of 10 meaning it will end in 3 zeros.
Next, we use the fact that $19!$ is a multiple of both 11 and 9. Their divisibility rules tell us that $T+M \equiv 3 \pmod 9$ and that $T-M \equiv 7 \pmod{11}$.
Case 1: $T+M=3$, $T-M=7$ no solutions.
Case 2: $T+M=3$, $T-M=-4$ no solutions.
Case 3: $T+M=12$, $T-M=7$ no solutions.
Case 4: $T+M=12$, $T-M=-4$ one solution.
We got that $T=4$, $M=8$ is a valid solution.
Therefore the answer is $4+8+0=\boxed{12}$.

C4. There are 6 boxes of apples in a store, weighting 15, 16, 18, 19, 20, and 31 pounds. Two customers purchased 5 boxes, and one customer purchased twice as many apples as the second customer. Which box has been left?

The sum of purchased boxes should be divisible by 3, since if first customer purchased n pounds, they together purchased $3n$. Since remainder of sum of all 6 boxes is $0+1+0+1+2+1 \equiv 2 \pmod 3$, the box that left should weight $\boxed{20}$.
The purchase should be split 33 to 66 pounds, so first customer purchased boxes 15 and 18 and the second purchased 16, 19, and 31.