Induction

Graham Middle School Math Olympiad Team

ARITHMETIC SEQUENSE

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So we can say that our formula stays for $k+1$ if it stays for k. And since it stays for 1, it also stays for 2, and so stays for 3, then for 4 and so on. That means we proved it for all possible n.

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$$=\frac{(k+1)(k(2k+1)+6(k+1))}{6}=$$
$$=\frac{(k+1)(2k^2+k+6k+6)}{6}=$$
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$$=\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}.$$

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So our formula holds for $k+1$ and that means for all counting numbers n. Q.E.D.

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Prove by induction has 4 steps:

1. Check a statement for starting values.
   It is called an ”**Induction Base**”.
2. Assume the statement is correct for some value k.
   It is called an ”**Induction Assumption**” or an ”**Induction Hypothesis**”.
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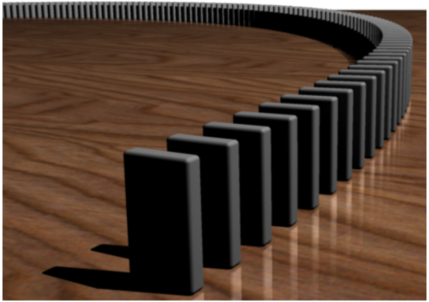

You may think about proof by induction as failing dominoes: for the whole set to be failed all you need is to push the first domino and make sure that each failing domino will cause the failure of the next domino.

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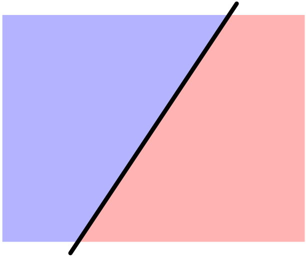

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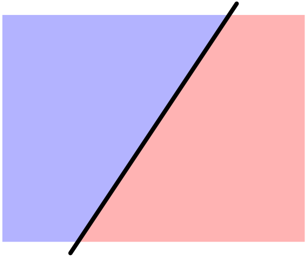

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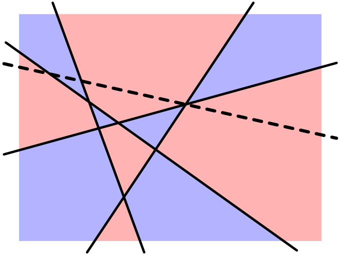

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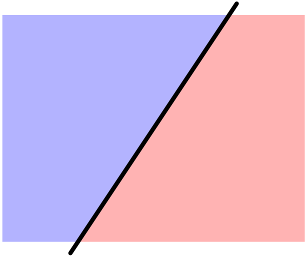

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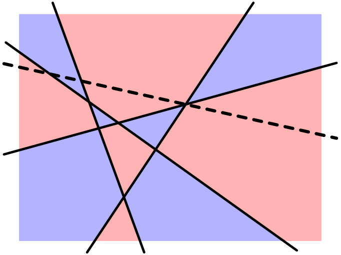
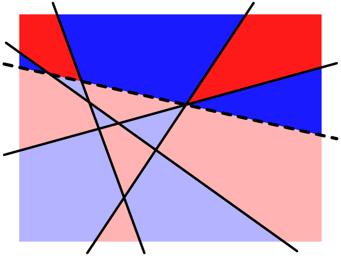


We will reverse the color of each region on one side of the line:



That means that if regions were connected by a previous line remain to be of different colors. And new region connections via the new line will also be of different colors, so it is possible to color a plane in two colors, it doesn't matter how many straight lines are drawn.

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$$\left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) = x^2 + 2 + \frac{1}{x^2},$$

so

$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x + \frac{1}{x}\right) - 2.$$

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Here we can assume that the statement is true for k, but this isn't enough, because we also have $\left(x^{k-1} + \frac{1}{x^{k-1}}\right)$ term. So we can do our induction step this way:

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Suppose the statement is true for all positive integers less or equal k and then we will prove for $k+1$.

This approach is called **complete (strong) Induction**.

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Step: Assume that any group of k horses has the same color. Now consider a group of $k + 1$ horses. Taking any k of them, the induction hypothesis states that they all have the same color, say, brown. The only issue is the color of the remaining "uncolored" horse. Consider, therefore, any other group of k of the $k + 1$ horses that contains the uncolored horse. Again, by the induction hypothesis, all of the horses in the new group have the same color. Then, because all of the colored horses in this group are brown, the uncolored horse must also be brown.

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The mistake occurs in the last sentence, where it states that, "Then, because all the colored horses in this (second) group are brown, the uncolored horse must also be brown." How do you know that there is a colored horse in the second group? In fact, when the original group of $k+1$ horses consists of exactly 2 horses, the second group of k horses does not contain a colored horse. The entire difficulty is caused by the fact that the statement should have been verified for the initial integer $k = 2$, not $k = 1$. This, of course, you will not be able to do.

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Consider, therefore, a line of $k+1$ people in which the first is a woman and the last is a man. To relate $P(k+1)$ to $P(k)$, consider the second person in line. If that person is a man, then that man is standing behind the woman in the front of the line and so $P(k+1)$ is true. If, however, the second person in line is a woman, then consider the line from that second woman to the end. This line then consists of k people, the first of which is a woman and the last of which is a man. In this case, the induction hypothesis applies and so somewhere there is a man standing behind a woman and so $P(k+1)$ is true, thus completing the proof. Q.E.D.

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Step: Suppose such ordering exists for a tournament with k teams.
Get the tournament of $k+1$ teams and temporarily exclude a team with the number $(k+1)$, call it v. There is an order for k teams:

$$v_1 \to v_2 \to v_3 \to \cdots \to v_k$$

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In a tournament, everybody plays with everyone one game (round-robin tournament). Each game finished with a win for one team and a loss for another. Prove that we order teams that way, that the first wins the second, the second wins the third and so on.

Base: For two teams only one game is played and we put a winner in the first place and a loser in the second.

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Get the tournament of $k+1$ teams and temporarily exclude a team with the number $(k+1)$, call it v. There is an order for k teams:

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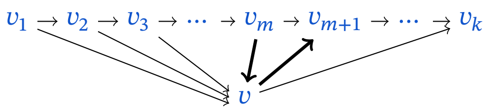
So the remaining case is when v is lost to v_1 and won v_k. But, by the previous problem, there are two teams v_m and v_{m+1} that are next to each other in the order, but v_m won v and v_{m+1} lost to v. We just put v between them and we are done.



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To relate $P(n+1)$ to $P(n)$, consider the first pair of candies dispensed. If this pair consists of one of each type, then $P(n+1)$ is true. Otherwise, this pair consists of two candies of the same type. In this case, the machine has $2n$ remaining candies still consisting of an odd number of caramel candies and an odd number of chocolate candies. Hence, the induction hypothesis applies and so the machine eventually dispenses a pair consisting of one of each type of candy. Thus, $P(n+1)$ is true and the proof is complete.

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Any positive integer x written on the board can be replaced with either $3x + 1$ or $\lfloor x/2 \rfloor$. Prove that if the number 1 is written initially, then by using these operations any natural number can be obtained.

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Let us draw some diagonals in a convex polygon in such a way that no two of them intersect (several diagonals may start from one vertex). Prove that there are at least two vertices of the polygon from which no diagonals are drawn.

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If one of the smaller polygon is a triangle, we take a vertex that is not M and N. Q.E.D.

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n bandits are dividing the loot. Each of them has their own opinion on the value of a particular share of the loot, and each of them wants to get no less than $1/n$ of the loot (from their point of view). Come up with a way to divide the loot among the bandits.

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Prove for positive numbers:

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$$\frac{x_1 + \cdots + x_{k-1} + \frac{x_1 + \cdots + x_{k-1}}{k-1}}{k} = \frac{x_1 + \cdots + x_{k-1}}{k-1}$$

So,

$$\frac{x_1 + \cdots + x_{k-1}}{k-1} \geq \sqrt[n]{x_1 \cdots x_{k-1} \cdot \frac{x_1 + \cdots + x_{k-1}}{k-1}}$$

$$\Rightarrow \left(\frac{x_1 + \cdots + x_{k-1}}{k-1}\right)^k \geq x_1 \cdots x_{k-1} \cdot \frac{x_1 + \cdots + x_{k-1}}{k-1}$$

$$\Rightarrow \left(\frac{x_1 \cdots + x_{k-1}}{k-1}\right)^{k-1} \geq x_1 \cdots x_{k-1}$$

$$\Rightarrow \frac{x_1 + \cdots + x_{k-1}}{k-1} \geq \sqrt[k-1]{x_1 \cdots x_{k-1}}$$

Q.E.D.