Probability

Graham Middle School Math Olympiad Team

PROBABILITY

An **outcome** is a possible result of an experiment.

We say ”the outcome of throwing the dice was 4.” The experiment was throwing the die and its result (outcome) was that the face with 4 pips on it came up.

We write each of the names of all 30 students in a math club on a card and we draw a card at random. If there are 14 sixth-graders, 9 seventh-graders, and 7 eighth-graders in this club, what is the probability of the name on the card being that of a sixth-grader?

The total number of *outcomes* is all students in the club. The desired *outcomes* are all sixth-graders. So *probability* is

$$P = \frac{14}{30} = \frac{7}{15}.$$

If the number of outcomes is countable and *if all outcomes have the same probability*, then the **probability** of the desired outcome is:

$$P = \frac{\text{(number of desired outcomes)}}{\text{(total number of events)}}.$$

Probability problems are either based on *countable* outcomes or on outcomes that are not countable. The later case is called *geometric probability* or *continuous probability*.

In the *geometric case*, the **probability** of a desired outcome is calculated as the ratio:

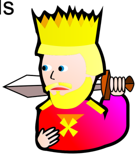
$$P = \frac{\text{(desired area)}}{\text{(total area)}}.$$

Probabilities, in any case, are always between 0 and 1, inclusive.
Probability is sometimes also given in percent:

$$P(\text{in percent}) = P(\text{as a decimal or fraction}) \times 100\%.$$

ADDITION RULE FOR PROBABILITY

What is the probability of choosing *an ace* **or** *a king* from a full deck of 52 cards?



A card that we got from the deck of cards may not be both an ace and a king. So we have 4 *outcomes* of getting an ace and 4 *outcomes* of getting a king. The resulting number of *desired outcomes* is 8. The *total number of possible outcomes* is 52. The total probability is

$$P = \frac{4+4}{52} = \frac{8}{52} = \frac{2}{13}.$$

We also may count the probability in another way. The *probability* of getting an ace is $4/52$ and the *probability* of getting a king is also $4/52$. Since both cases are good for us, the total probability is

$$P = \frac{4}{52} + \frac{4}{52} = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}.$$

If two events are *outcomes* of one experiment and they *may not happen together*, then *probability* of any of this event is **sum of their probabilities**

$$P(A \text{ or } B) = P(A) + P(B).$$

What is the probability of choosing *a multiple of* 4 or *a multiple of* 5 from numbers from 1 to 100?

The probability of getting a multiple of 4 is $25/100$, and the probability of getting a multiple of 5 is $20/100$. But there are multiples of 20, which we counted twice. To make up for this case, we need to subtract the probability of getting a multiple of 20, which is $5/100$. So total probability would be

$$P = \frac{25}{100} + \frac{20}{100} - \frac{5}{100} = \frac{25+20-5}{100} = \frac{40}{100} = \frac{2}{5}.$$

So the *probability of two events* of one experiment

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

MULTIPLICATION RULE FOR PROBABILITY

If the experiment *"throw a die and toss a coin"* is performed, what is the probability of the event *"a 4 and a tails"* to happen?

Let's first throw a die. If 4 didn't happen, we don't need to toss a coin, the desired event will not happen in any case. So we need to toss a coin only when the die rolls 4. The 4 has a probability of $1/6$ and, in this case only, we toss a coin and with a probability of $1/2$ we will get tails. So the overall probability would be

$$P = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

If two events are *independent* the **probability** of both of them happen is *a multiple of their probabilities*

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

The **conditional probability** of an event E that depends on another event F to happen is denoted

$$P(E \mid F)$$

and is a probability of E to happen if we know that F has already happened.

We used that rule to solve the problem *"throw a die and toss a coin"*.

If two events are *dependent* the **probability** of both of them happening is

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

What is the probability of choosing *an ace* **and then** *a king* from a full deck of 52 cards?

The probability to choose an ace is $4/52$. Then we select a king from the deck without one card, so the probability to choose a king is $4/51$. The total probability is

$$P(A \text{ and then } K) = P(A) \cdot P(K \mid A) = \frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}.$$

USING THE NEGATIVE CASE

All possible outcomes of an experiment must equal the entire experiment. That means that the probability for all outcomes is

$$P(\text{All outcomes}) = 1.$$

What is the probability of throwing the dice and not to get 4.

There are 5 outcomes that are desired: 1, 2, 3, 5, and 6. So total probability would be

$$P(\text{not throwing } 4) = 5 \cdot \frac{1}{6} = \frac{5}{6}.$$

On the other hand, there is only one event that is not desired, so we can calculate the same probability as

$$P(\text{not throwing } 4) =$$
$$= P(\text{all possible results}) - P(\text{throwing } 4) =$$
$$= 1 - \frac{1}{6} = \frac{5}{6}.$$

The **complement event** of a *desired event* E, denoted as $\overline{E}$, is the event that E does not occur. Since for any event, we know whether it is desired or not

$$P(\overline{E}) = 1 - P(E).$$

Sometimes that simplifies the solution.

We are drawing 4 cards from a full deck of cards. What is the probability to get *at least one ace*?

If we try to solve the problem using the *conditional probability* we find out that the number of cases is pretty big. But we may count the probability to *not get any ace*.

$$P(\text{no ace}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} = \frac{38916}{54145} \approx 72\%.$$

And probability to get *an ace with* 4 *cards* is

$$P(\text{at least one ace}) = 1 - \frac{38916}{54145} = \frac{15229}{54145} \approx 28\%.$$

THE BIRTHDAY PROBLEM

How many people need to attend a party until there is a 50% chance that at least two guests share a birthday?

It is easier to calculate the probability that *no two people share a birthday*.
Let's start with the first guest. He doesn't share a birthday with anyone. So the probability is 1.
The second guest may share a birthday with the first guest with a probability $364/365$.
The third one doesn't share a birthday with previous guests with a probability $363/365$ and so on.
For n guests

$$P(\text{no shared birthday}) =$$

$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-n+1}{365} =$$

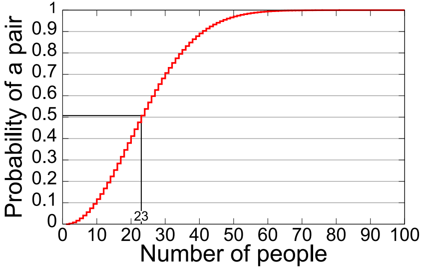
$$= \frac{365!}{(365-n)!} \cdot \frac{1}{365^n}.$$

Using formula

$$P(\text{there is a shared birthday}) =$$

$$= 1 - P(\text{no shared birthday})$$

we can calculate, using a computer, the probability for any number of guests



So we may see that group of 23 people has the probability of 50.7% to have a shared birthday, and a group of 70 people has a probability of 99.9% to share a birthday.

SAMPLE PROBLEM 1

Mala and Joslyn number the cards in a deck of playing cards from 1 to 52. Mala and Joslyn each draw a card and show them to each other. What is the probability that the sum of the numbers on the cards is *odd*? Express your answer as a common fraction.

Solution 1: There are two ways for the sum to be *odd*: Mala draws an odd card and Joslyn draws an even, or vice versa. If both draw even or odd cards, the resulting sum is even.

Using the *addition rule for probability*, we add the probabilities from the two scenarios that lead to an odd sum. Namely,

$$P(\text{odd sum}) = P\begin{pmatrix}\text{Mala odd}\\ \text{Joslyn even}\end{pmatrix} + P\begin{pmatrix}\text{Mala even}\\ \text{Joslyn odd}\end{pmatrix}.$$

Now we use the *product rule* to find the probabilities for each of those scenarios occurring. Note that after Mala draws her card, there are only 51 cards remaining in the deck, and 26 of them are the opposite flavor of even/odd

$$P(\text{odd sum}) = \frac{1}{2}\cdot\frac{26}{51} + \frac{1}{2}\cdot\frac{26}{51} = \frac{26}{51}.$$

Solution 2: Use *combinations*. The total number of ways to draw 2 cards is C_2^{52}, which equals $\dfrac{52\times 51}{2}$. The total number of combinations drawing 1 even and 1 odd is $\dfrac{52\times 26}{2}$. This is because we can select any card first, and then have 26 cards remaining in the deck that make the sum odd. We divide by 2 because the order of drawing doesn't matter in determining the number of combinations.

$$P(\text{odd sum}) = $$

$$= \frac{\text{Total ways to draw } 1 \text{ even and } 1 \text{ odd}}{\text{Total ways to draw } 2 \text{ cards}} =$$

$$= \frac{52\times 26}{52\times 51} = \frac{26}{51}$$

The initial intuition is that probability should be $1/2$. But this is not the case, because once you draw the first card, the probability to draw odd/even card has changed.

SAMPLE PROBLEM 2

You have a bin containing 20 *red* socks and 40 *blue* socks. You reach in and randomly pick out two socks. What is the probability, expressed as a ratio a/b, that you managed to pick out a matching pair (*red/red* or *blue/blue*)?

Solution 1: In this solution, we use the *addition* and *multiplication rules* for probability. Since there is no overlap between the *red/red* and *blue/blue* events,

$$P(\text{matching pair}) = P(\text{red/red}) + P(\text{blue/blue}).$$

We use the *multiplication rule* next, multiplying the probability of pulling out the sock of the first color by the probability of next pulling out a sock of the same color.

$$P(\text{red/red}) = \frac{20}{60} \cdot \frac{19}{59}, \qquad P(\text{blue/blue}) = \frac{40}{60} \cdot \frac{39}{59},$$

$$P(\text{matching pair}) =$$

$$= \frac{20 \times 19 + 40 \times 39}{60 \times 59} = \frac{19 + 78}{3 \times 59} = \frac{97}{177}.$$

Solution 2:
Use *combinations*. There are $C_2^{20} = \frac{20\times19}{2}$ ways to pick 2 *red* socks out of 20, and $C_2^{40} = \frac{40\times39}{2}$ ways to pick 2 *blue* socks out of 40. The total number of ways to pick 2 socks out of 60 is $C_2^{60} = \frac{60\times59}{2}$.

$$P(\text{red/red}) = \frac{C_2^{20}}{C_2^{60}}, \qquad P(\text{blue/blue}) = \frac{C_2^{40}}{C_2^{60}}.$$

Since there is no overlap between *red/red* and *blue/blue* events,

$$P(\text{matching}) = \frac{20 \times 19 + 40 \times 39}{60 \times 59} = \frac{97}{177}.$$

Armed with your knowledge of probability, you will never be frightened of sock drawer problems again.

EXERCISES

1. When a pair of 6-sided dice are rolled, what is the probability that the numbers rolled sum to 8?
2. If you flip a fair coin 5 times, what is the probability that you will flip a total of 3 heads and 2 tails?
3. Each of two boxes contains three chips numbered 1, 2, 3. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?
4. A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?
5. Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown.

   X X X
   X X X

   If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?
6. Two different numbers are randomly selected from the set $\{-2, -1, 0, 3, 4, 5\}$ and multiplied together. What is the probability that the product is 0?
7. On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?
8. A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

CHALLENGE PROBLEMS

1. Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that their difference is less than 0.5?
2. A radio program has a quiz consisting of 3 multiple-choice questions, each with 3 choices. A contestant wins if he or she gets 2 or more of the questions right. The contestant answers randomly to each question. What is the probability of winning?
3. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where n is a positive integer. What is n?
4. A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \ldots, 2018\}$. What is the probability that m^n has a units digit of 1?