1. Find all sequences of positive integers $a_1,\ a_2,\ \ldots$ such that

$$a_i + a_{i+1} + 2022 = a_{i+2} + a_{i+3}$$

for all positive integers i.

2. Let I be the incenter of a triangle ABC with $\angle BAC = 90^\circ$. Let M and N be the midpoints of AB and BI, respectively. Prove that CI is tangent to the circumcircle of BMN.

3. Let $q > 1$ be an odd positive integer. Show that q is prime if and only if for any set of $\dfrac{q+1}{2}$ distinct positive integers, there exist two integers x and y in the set such that $\dfrac{x+y}{\gcd(x,\ y)} \geq q$.

4. We say that two strings $x,\ y \in \{0,\ 1\}^n$ are similar if it's possible to delete one bit from each to obtain the same length $n-1$ string. (For instance, if you delete the 2nd bit from 110010, you get the string 10010.)

   - Show that there exist at least $\dfrac{2n}{n+1}$ binary strings of length n such that no two strings are similar.
   - Show that for all sets of more than $\dfrac{2n+1}{n+2}$ binary strings of length n, at least two strings are similar.

   For both parts a and b, write up the best possible bounds you can find. We are interested in any bounds you can come up with, even if they are weaker than what we asked; you may even get better bounds!