Angles and Triangles solutions

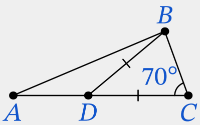
Graham Middle School Math Olympiad Team

PROBLEMS 1-3

E1. If the degree measures of the angles of a triangle are in the ratio $3 : 3 : 4$, what is the degree measure of the largest angle of the triangle?

Let the angles are $3x$, $3x$, and $4x$, so $3x + 3x + 4x = 10x = 180°$. So $x = 18°$ and $4x = \boxed{72°}$.

E2. In $\triangle ABC$, D is a point on side $\overline{AC}$ such that $BD = DC$ and $\angle BCD$ measures $70°$. What is the degree measure of $\angle ADB$?



$\angle ADB = \angle DCB + \angle DBC$, and since $\triangle BDC$ is isosceles, $\angle DBC = \angle DCB = 70°$, so $\angle ADB = 70° + 70° = \boxed{140°}$.

E3. How many scalene triangles have all sides of integral lengths and perimeter less than 15?

Let's look at the longest side of a triangle. Then iterate on the length of the middle side.

Case 0: 3 or smaller. The only option $3 - 2 - 1$ is degenerated triangle.

Case 1: 4, the only one scalene triangle possible: $4 - 3 - 2$.

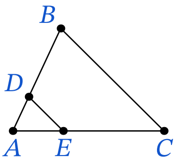
Case 2: 5: $5 - 4 - 2$ and $5 - 4 - 3$.

Case 3: 6: $6 - 4 - 3$, $6 - 5 - 2$, and $6 - 5 - 3$.

Case 4: 7: the sum two other sides will be at least 8.

As a result we have $\boxed{6}$ scalene triangles.

E4. On sides AB and AC of $\triangle ABC$, we pick points D and E, respectively, so that $DE \parallel BC$. If $AB = 3AD$ and $DE = 6$, find BC.



Since $\triangle ADE \sim \triangle ABC$ via **AA** similarity, $DE : BC = AD : AB$, so $BC = DE \times AB/AD = 6 \times 3 = \boxed{18}$.

E5. Let $\angle ABC = 24°$ and $\angle ABD = 20°$. What is the smallest possible degree measure for $\angle CBD$?

$\angle ABD$ and $\angle ABC$ share ray AB. In order to minimize the value of $\angle CBD$, D should be located between A and C. $\angle ABC = \angle ABD + \angle CBD$, so $\angle CBD = 4$. The answer is $\boxed{4}$.

E6. A polygon has N sides and q obtuse interior angles. Each of its obtuse interior angles has a measure $150°$ and each of its acute interior angles has measure $80°$. How many sides does the polygon have?

The total sum of angles in the polygon is $180 \cdot (N-2)$ and from another side, the sum of angles is $150 \cdot q + 80 \cdot (N-q)$.

$$
\begin{aligned}
180(N-2) &= 150q + 80(N-q), \\
180N - 360 &= 150q + 80N - 80q, \\
100N - 360 &= 70q, \\
100N &= 360 + 70q, \\
10N &= 36 + 7q.
\end{aligned}
$$

So $7q$ should be ended up with 4. This is possible when $q = 2, 12, 22$ etc. For $q = 2$, $\boxed{N = 5}$, for $q = 12$, $\boxed{N = 12}$, but for each $q > 12$, $N < q$, so the there are no more solutions.

E7. In how many ways can we form a nondegenerate triangle by choosing three distinct numbers from the set $\{1, 2, 3, 4, 5\}$ as the sides?

Let's look at the longest side:
Case 1: The longest side is 5. The sum of two other sides should be at least 6, so the only options are $2-4-5$ and $3-4-5$.
Case 2: The longest side is 4. The sum of two other sides should be at least 5, so the only option is $2-3-4$.
Case 3: The longest side is 3. The rest two sides are 1 and 2 and we can got only degenerated triangle.
So we have only three cases $2-4-5$, $3-4-5$, and $2-3-4$.

E8. The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

We let the measures be $5x$ and $4x$ giving us the ratio of $5 : 4$. We know $90-4x > 90-5x$ since this inequality gives $x > 0$, which is true since the measures of angles are never negative. We also know the bigger complement is twice the smaller, so

$$90-4x = 2(90-5x),$$
$$90-4x = 180-10x,$$
$$6x = 90,$$
$$x = 15.$$

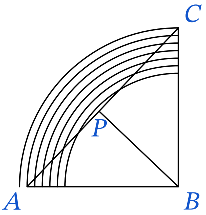
Therefore, the angles are 75 and 60, which sum to $\boxed{135}$.

C1. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

The triangle inequality generalizes to all polygons, so $x < 3+7+15$ and $15 < x+3+7$ yields $5 < x < 25$. Now, we know that there are 19 numbers between 5 and 25 exclusive, but we must subtract 2 to account for the 2 lengths already used that are between those numbers, which gives $19 - 2 = \boxed{17}$.

C2. Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

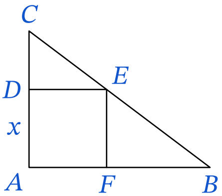
$AC = \sqrt{20^2 + 21^2} = 29$ and $\triangle ABC \sim \triangle APB$, so $BP = BC \times AB/AC = 21 \times 20/29 \approx 14.5$.

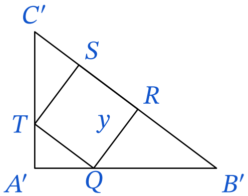


It follows that we can draw circles of radii $15, 16, 17, 18, 19,$ and $20,$ that each contribute two integer lengths (since these circles intersect the hypotenuse twice) from B to $\overline{AC}$ and one circle of radius 21 that contributes only one such segment. Our answer is then $6 \cdot 2 + 1 = \boxed{13}$.

CHALLENGE PROBLEM 3

C3. A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?





Analyze the first right triangle. Note that $\triangle ABC$ and $\triangle FBE$ are similar, so $\frac{BF}{FE} = \frac{AB}{AC}$. This can be written as $\frac{4-x}{x} = \frac{4}{3}$. Solving, $x = \frac{12}{7}$.

Now we analyze the second triangle.

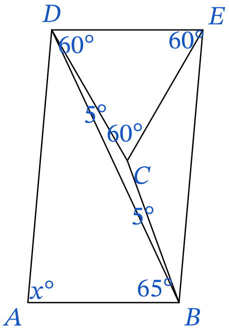
Similarly, $\triangle A'B'C'$ and $\triangle RB'Q$ are similar, so $RB' = \frac{4}{3}y$, and $C'S = \frac{3}{4}y$. Thus,

$$C'B' = C'S + SR + RB' = \frac{4}{3}y + y + \frac{3}{4}y = 5.$$

Solving for y, we get $y = \frac{60}{37}$. Thus, $\frac{x}{y} = \boxed{\frac{37}{35}}$.

CHALLENGE PROBLEM 4

C4. Quadrilateral $ABCD$ has $AB = BC = CD$, angle $ABC = 70°$ and angle $BCD = 170°$. What is the measure of angle BAD?



First, connect the diagonal DB, then, draw line DE such that it is congruent to DC and is parallel to AB. Because triangle DCB is isosceles and angle DCB is $170°$, the angles CDB and CBD are both $\dfrac{180-170}{2} = 5°$. Because angle ABC is $70°$, we get angle ABD is $65°$. Next, noticing parallel lines AB and DE and transversal DB, we see that angle BDE is also $65°$, and subtracting off angle CDB gives that angle EDC is $60°$.

Now, because we drew $ED = DC$, triangle DEC is equilateral. We can also conclude that $EC = DC = CB$ meaning that triangle ECB is isosceles, and angles CBE and CEB are equal. Finally, we can set up our equation. Denote angle BAD as $x°$. Then, because $ABED$ is a parallelogram, the angle DEB is also $x°$. Then, CEB is $(x-60)°$. Again because $ABED$ is a parallelogram, angle ABE is $(180-x)°$. Subtracting angle ABC gives that angle CBE equals $(110-x)°$. Because angle CBE equals angle CEB, we get

$$x - 60 = 110 - x,$$

solving into $x = \boxed{85°}$.

TITLE

TEAM ATTACK 4 SOLUTIONS, PROBLEMS 1-2

TA1. It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to her home along the same route. What is her average speed, in km/hr, for the round trip?

Since she walked 1 km to school and 1 km back home, her total distance is $1 + 1 = 2$ km.
Since she spent 30 minutes walking to school and 10 minutes walking back home, her total time is
$30 + 10 = 40$ minutes = $\frac{40}{60} = \frac{2}{3}$ hours.

Therefore her average speed in km/hr is $\frac{2}{\frac{2}{3}} = \boxed{3}$.

TA2. Members of the Rockham Soccer League buy socks and T-shirts. Socks cost \$4 per pair and each T-shirt costs \$5 more than a pair of socks. Each member needs one pair of socks and a shirt for home games and another pair of socks and a shirt for away games. If the total cost is \$2366, how many members are in the League?

Since T-shirts cost 5 dollars more than a pair of socks, T-shirts cost $5 + 4 = 9$ dollars.
Since each member needs 2 pairs of socks and 2 T-shirts, the total cost for 1 member is
$2(4 + 9) = 26$ dollars.
Since 2366 dollars was the cost for the club, and 26 was the cost per member, the number of members in the League is $2366 \div 26 = \boxed{91}$.

TA3. How many non-congruent triangles with perimeter 7 have integer side lengths?

By the triangle inequality, no side may have a length greater than the semiperimeter, which is $\frac{1}{2} \cdot 7 = 3.5$.
Since all sides must be integers, the largest possible length of a side is 3. Therefore, all such triangles must have all sides of length 1, 2, or 3. Since $2+2+2=6<7$, at least one side must have a length of 3. Thus, the remaining two sides have a combined length of $7-3=4$. So, the remaining sides must be either 3 and 1 or 2 and 2. Therefore, the number of triangles is $\boxed{2}$.

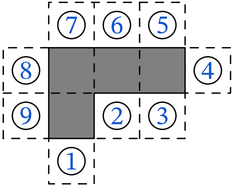
TA4. What is the probability that a randomly drawn positive factor of 60 is less than 7?

Notice that
$1 \cdot 60 = 2 \cdot 30 = 3 \cdot 20 = 4 \cdot 15 = 5 \cdot 12 = 6 \cdot 10 = 60$.
Hence, 60 has 12 factors, of which 6 are less than 7. Thus, the answer is $\frac{6}{12} = \boxed{\frac{1}{2}}$.

TA6. The sum of the two 5-digit numbers $AMC10$ and $AMC12$ is 123422. What is $A+M+C$?

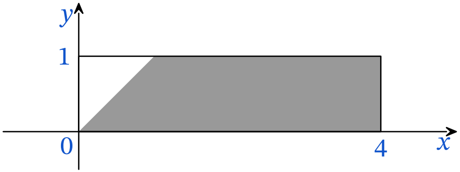
We know that $AMC12$ is 2 more than $AMC10$. We set up $AMC10 = x$ and $AMC12 = x+2$. We have $x+x+2 = 123422$. Solving for x, we get $x = 61710$. Therefore, the sum $A+M+C = \boxed{14}$.

TA5. The polygon enclosed by the solid lines in the figure consists of 4 congruent squares joined edge-to-edge. One more congruent square is attached to an edge at one of the nine positions indicated. How many of the nine resulting polygons can be folded to form a cube with one face missing?



A cube missing one face has 5 of its 6 faces. Since the shape has 4 faces already, we need another face. The only way to add another face is if the added square does not overlap any of the others. 1, 2, and 3 overlap, while squares 4 to 9 do not. The answer is $\boxed{6}$.

TA7. A point (x, y) is randomly picked from inside the rectangle with vertices $(0, 0)$, $(4, 0)$, $(4, 1)$, and $(0, 1)$. What is the probability that $x < y$?



The area of this rectangle is $4 \cdot 1 = 4$.
The line $x = y$ intersects the rectangle at $(0,0)$ and $(1,1)$.

The area of this triangle is $\frac{1}{2} \cdot 1^2 = \frac{1}{2}$

Therefore, the probability that $x < y$ is $\frac{\frac{1}{2}}{4} = \boxed{\frac{1}{8}}$.