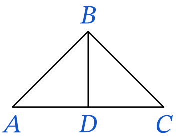
Right Triangles solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-4

E1. In $\triangle ABC$, $AB = BC = 29$, and $AC = 42$. What is the area of $\triangle ABC$?



Since $\triangle ABC$ is isosceles, BD is a median and a height. By the Pythagorean theorem $BD = \sqrt{29^2 - 21^2} = 20$, and $S(\triangle ABC) = \dfrac{42 \cdot 20}{20} = \boxed{420}$.

E2. A square-shaped floor is covered with congruent square tiles. If the total number of tiles that lie on the two diagonals i37, how many tiles cover the floor?

Since 37 is an odd number, the central square is shared with both diagonals, so there are $(37+1)/2 = 19$ squares on each of the diagonals. So we have 19×19 square, and it has $19 \cdot 19 = \boxed{361}$ tiles.

E3. Triangle ABC has sides of length 13 inches, 14 inches and 15 inches. What is the length of the altitude to the side of length 14 inches?

By Heron's formula the area of the triangle is $S = \sqrt{21 \cdot (21-13) \cdot (21-14) \cdot (21-15)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. From other side $S = \dfrac{14 \cdot h_{14}}{2} = 84$. So $h_{14} = \boxed{12}$.

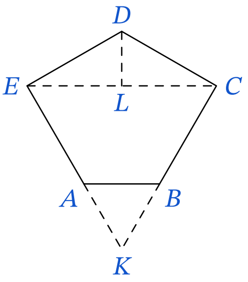
E4. The altitude of an equilateral triangle is $\sqrt{6}$ units. What is the area of the triangle, in square units?

Let the side of the triangle is a, so $a^2 = \dfrac{a^2}{4} + 6$, and $\dfrac{3}{4}a^2 = 6$. $a^2 = 8$, so $a = 2\sqrt{2}$.

$S = \dfrac{2\sqrt{2} \cdot \sqrt{6}}{2} = \boxed{2\sqrt{3}}$.

EXERCISES 5-6

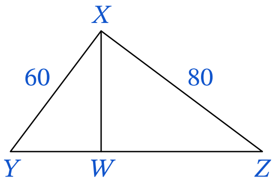
E5. In pentagon $ABCDE$, $\angle E$ and $\angle C$ are right angles and $m\angle D = 120°$. If $AB = 12$, $AE = BC = 18$ and $ED = DC$, what is ED?



When we continue EA and CB until the intersection in K, we will get equilateral $\triangle ABK$ with the side 12, so equilateral $\triangle ECK$ has a side 30. ED is hypotenuse in the $30-60-90$ triangle with the long side 15. So

$$ED = 15 \cdot \frac{2}{\sqrt{3}} = \boxed{10\sqrt{3}}.$$

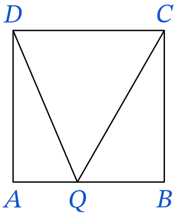
E6. In the diagram, $\triangle XYZ$ is right-angled at X, with $YX = 60$ and $XZ = 80$. The point W is on YZ so that WX is perpendicular to YZ. Determine the length of WZ.



$\triangle XYZ$ is similar to $3-4-5$ triangle, so $\triangle XWZ$ is similar to $3-4-5$ triangle. So $WZ : XZ = 4 : 5$ and

$$WZ = XZ \cdot 4/5 = 80 \cdot 4/5 = \boxed{64}.$$

E7. If point Q lies on side AB of square $ABCD$ such that $QC = \sqrt{10}$ units and $QD = \sqrt{13}$ units, what is the area of square $ABCD$?



Let the side of the square is a and $AQ = b$. So $a^2 + b^2 = 10$ and $a^2 + (a-b)^2 = 13$. $(a-b)^2 - b^2 = a^2 - 2ab = 3$, $b = \frac{a^2-3}{2a}$. Plugin back into the first equation, we got:

$$a^2 + \left(\frac{a^2-3}{2a}\right)^2 = 10,$$

$$a^2 + \frac{a^4 - 6a^2 + 9}{4a^2} = 10,$$

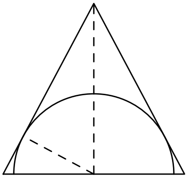
$$4a^4 + a^4 - 6a^2 + 9 = 40a^2,$$

$$5a^4 - 46a^2 + 9 = 0,$$

$$(a^2-9)(5a^2-1) = 0, \quad \Rightarrow \quad a^2 = \boxed{9}.$$

Another root is too small.

E8. A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



The radius of the semicircle is an altitude in the half of the triangle. The area of the half of the triangle is $\frac{8 \cdot 15}{2} = \boxed{60}$. The hypotenuse of the half of the triangle is $\sqrt{8^2 + 15^2} = 17$. Since $\frac{17 \cdot r}{2} = 60$, $r = 60 \cdot 2/17 = \boxed{120/17}$.

CHALLENGE PROBLEMS 1-2

CP1. Points $A(11,9)$ and $B(2,-3)$ are vertices of $\triangle ABC$ with $AB = AC$. The altitude from A meets the opposite side at $D(-1,3)$. What are the coordinates of point C?

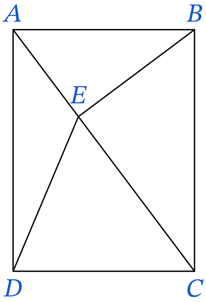
Since $\triangle ABC$ is isosceles, D is a median of BC. So $x_D = \frac{x_B + x_C}{2}$ and $y_D = \frac{y_B + y_C}{2}$. So $C(x_C, y_C) = \boxed{(-4,9)}$.

CP2. In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

Without loss of generality, let AB be a horizontal segment of length 10. Now realize that C has to lie on one of the lines parallel to AB and vertically 20 units away from it. But $10 + 20 + 20$ is already 50, and this doesn't form a triangle. So there are no such points, the answer is $\boxed{0}$.

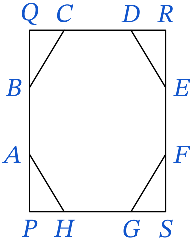
CHALLENGE PROBLEMS 3-4

CP3. Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$?



$S(\triangle AED) = S(\triangle AEB)$ since they share the base and altitudes have the same length because of symmetry. $\triangle ABE$ similar to $3-4-5$ triangle, so $AE : AB = 3 : 5$ and $AE = 9/5$, $EB : AB = 4 : 5$, so $EB = 12/5$. $S(\triangle AED) = S(\triangle AEB) = \dfrac{\frac{9}{5}\cdot\frac{12}{5}}{2} = \dfrac{12\cdot 9}{2\cdot 5\cdot 5} = \boxed{\dfrac{54}{25}}$.

CP4. In rectangle $PQRS$, $PQ = 8$ and $QR = 6$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, points E and F lie on $\overline{RS}$, and points G and H lie on $\overline{SP}$ so that $AP = BQ < 4$ and the convex octagon $ABCDEFGH$ is equilateral. Find the length of a side of this octagon.



Let $AP = BQ = x$. Then $AB = 8 - 2x$. Now notice that since $CD = 8 - 2x$ we have $QC = DR = x - 1$. Thus by the Pythagorean Theorem we have $x^2 + (x-1)^2 = (8-2x)^2$ which becomes

$$2x^2 - 30x + 63 = 0 \implies x = \frac{15 - 3\sqrt{11}}{2}.$$

Our answer is $8 - (15 - 3\sqrt{11}) = \boxed{3\sqrt{11} - 7}$.

EMPTY

TA1. $\frac{9}{37}$ is changed to a decimal. What digit lies in the 2022$^{\text{th}}$ place to the right of the decimal point?

$\frac{9}{37} = 0.243\,243\,243\ldots = 0.\overline{243}$ so every position divisible by 3 has digit 3. So on 2022$^{\text{th}}$ there is the digit $\boxed{3}$.

TA2. Emily has 21 dimes. She placed then in three piles, with an odd number of dimes in each pile. In how many different ways can she accomplish this? [*Consider piles of 1, 1, 19 dimes, for example, to be equivalent to piles 1, 19, and 1 dimes.*]

Start with the biggest pile and count down:
Case 19: $19-1-1$; Case 17: $17-3-1$;
Case 15: $15-5-1$ and $15-3-3$;
Case 13: $13-7-1$, $13-5-3$;
Case 11: $11-9-1$, $11-7-3$, and $11-5-5$;
Case 9: $9-9-3$, $9-7-5$.
Case 7: $7-7-7$. Total: $\boxed{12}$ cases.

TA3. Suppose $\frac{2}{N}$, $\frac{3}{N}$, and $\frac{5}{N}$ are three fractions in lowest terms. Find a sum of all the possible composite whole number values for N between 20 and 80?

N should not have prime factors of 2, 3 or 5. The only options for composite numbers is $7\cdot 7 = 49$ and $7\cdot 11 = 77$. $7\cdot 13 > 80$ and $11\cdot 11 > 80$. So answer is $49 + 77 = \boxed{126}$.

TA4. The Mathematical Olympiad began in the prime year 1979. Find the product of the fractions below in a simplest form:

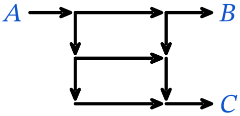
$$\left(1-\frac{1}{1980}\right)\times\left(1-\frac{1}{1981}\right)\times\ldots\times\left(1-\frac{1}{2022}\right).$$

$$\left(1-\frac{1}{1980}\right)\times\left(1-\frac{1}{1981}\right)\times\ldots\times\left(1-\frac{1}{2022}\right) =$$

$$= \frac{1979}{1980}\cdot\frac{1980}{1981}\cdot\ldots\cdot\frac{2020}{2021}\cdot\frac{2021}{2022} = \boxed{\frac{1979}{2022}}.$$

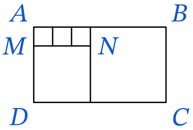
TEAM ATTACK PROBLEMS 5-6

TA5. In this street map, all traffic enters at A and exits at either B or C. All traffic flows either south or east. At each intersection where there is a choice of direction, 70% of the traffic goes east and 30% goes south. What percent of the traffic exists at C?



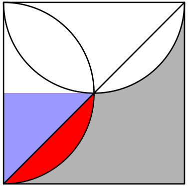
The probability to get point B is $0.7 \cdot 0.7 = 0.49$. So $1 - 0.49 = 0.51 = \boxed{51\%}$ of traffic will exists at C.

TA6. Rectangle $ABCD$ is partitioned into five squares as shown. The length, in centimeters, of $\overline{AM}$ is a whole number. The area of rectangle $ABCD$ is greater than 100 sq cm. Find the smallest possible area of rectangle $ABCD$, in sq cm.



Let $AM = n$ cm, then $AN = 3n$ cm, $AD = 4n$ cm and $AB = 7n$ cm, so $S(ABCD) = 4n \cdot 7n = 28n^2$ sq cm. If $n = 1$, $28 < 100$, if $n = 2$, $28 \cdot 4 = \boxed{112}$ sq cm.

TA7. Two semicircles are inscribed in a square with side 8 meters as shown. Approximate the area of the shaded region to the nearest tenth of a square meter. Use the approximation 3.14 for π.



The gray area is a half of the big square minus two red sections.
To calculate area of the red section, we need to get area of quarter of the circle minus area of the blue triangle.

$S(\text{quarter of the circle}) = \dfrac{\pi r^2}{4} = \dfrac{\pi 4^2}{4} = 4\pi \approx 4 \cdot 3.14 = 12.56.$

$S(\text{blue triangle}) = \dfrac{4 \cdot 4}{2} = 8.$

$S(\text{red area}) = 12.56 - 8 = 4.56.$

$S(\text{grey area}) = \dfrac{8 \cdot 8}{2} - 2 \cdot 4.56 = 32 - 9.12 = 22.88 \approx \boxed{22.9}$ sq meters.