Inclusion/Exclusion Principle solutions

Graham Middle School Math Olympiad Team

EXCERCISES 1-3

E1. Four students take an exam. Three of their scores are $70, 80,$ and 90. If the average of their four scores is 70, then what is the remaining score?

We can call the remaining score r. We also know that the average, 70, is equal to $\frac{70+80+90+r}{4} = 70$. So $r = \boxed{40}$.

E2. How many subsets of two elements can be removed from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ so that the mean (average) of the remaining numbers is 6?

Since there will be 9 elements after removal, and their mean is 6, we know their sum is 54. We also know that the sum of the set pre-removal is 66. Thus, the sum of the 2 elements removed is $66 - 54 = 12$. There are only $\boxed{5}$ subsets of 2 elements that sum to 12: $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, $\{5, 7\}$.

E3. Jefferson Middle School has the same number of boys and girls. $\frac{3}{4}$ of the girls and $\frac{2}{3}$ of the boys went on a field trip. What fraction of the students on the field trip were girls?

Let there be b boys and g girls in the school. We see $g = b$, which means $\frac{3}{4}b + \frac{2}{3}b = \frac{17}{12}b$ kids went on the trip and $\frac{3}{4}b$ kids are girls. So, the answer is $\frac{\frac{3}{4}b}{\frac{17}{12}b} = \frac{9}{17}$, which is $\boxed{\frac{9}{17}}$

EXERCISES 4-6

E4. The mean, median, and unique mode of the positive integers 3, 4, 5, 6, 6, 7, and x are all equal. What is the value of x?

Notice that the mean of this set of numbers, in terms of x, is:

$$\frac{3+4+5+6+6+7+x}{7} = \frac{31+x}{7}$$

Because we know that the mode must be 6 (it can't be any of the numbers already listed, as shown above, and no matter what x is, either 6 or a new number, it will not affect 6 being the mode), and we know that the mode must equal the mean, we can set the expression for the mean and 6 equal:

$$\frac{31+x}{7} = 6 \quad 31+x = 42 \quad x = \boxed{11}$$

E5. How many ways to distribute 10 candies among 4 kids?

Using stars and bars we need to distribute 3 bars across 13 stars and bars:

$$\binom{13}{3} = \frac{13\cdot 12\cdot 11}{1\cdot 2\cdot 3} = \boxed{286}.$$

E6. How many ways to distribute 10 candies among 4 kids, so each kid should get at least one candy?

We just give every kid one candy and distribute the remaining 6 candies. So 3 bars and 6 stars:

$$\binom{9}{3} = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = \boxed{84}.$$

EXERCISES 7-8

E7. If 20 girls are on my school's soccer team, 25 girls are on my school's hockey team, and 11 girls play both sports, then how many girls play soccer and hockey?

$20 - 11 = 9$ girls plays only soccer, $25 - 11 = 14$ plays only hockey. So the total number of girls who play soccer and hockey is $9 + 14 + 11 = \boxed{34}$.

Using PIE: $25 + 20 - 11 = \boxed{34}$.

E8. At one hospital, there are 100 patients, all of whom have at least one of the following ailments: a cold, the flu, or an earache. 38 have a cold, 40 have the flu, and some number have earaches. If 17 have both colds and the flu, 10 have colds and earaches, 23 have the flu and earaches, and 7 have all three, how many have an earache?

Let x is the number of patients with earache.
Using PIE:

$$x + 38 + 40 - 17 - 10 - 23 + 7 = 100,$$
$$x + 35 = 100,$$
$$x = \boxed{65}.$$

CHALLENGE PROBLEMS 1-2

CP1. One day the Beverage Barn sold 252 cans of soda to 100 customers, and every customer bought at least one can of soda. What is the maximum possible median number of cans of soda bought per customer on that day?

In order to maximize the median, we need to make the first half of the numbers as small as possible. Since there are 100 people, the median will be the average of the 50th and 51st largest amount of cans per person. To minimize the first 49, they would each have one can. Subtracting these 49 cans from the 252 cans gives us 203 cans left to divide among 51 people. Taking $\frac{203}{51}$ gives us 3 and a remainder of 50. Seeing this, the largest number of cans the 50th person could have is 3, which leaves 4 to the rest of the people. The average of 3 and 4 is 3.5. Thus our answer is $\boxed{3.5}$

CP2. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2022. On how many days during the next year did she not receive a phone call from any of her grandchildren?

We use PIE. The first grandchild called $\left\lfloor \frac{365}{3} \right\rfloor = 121$ days, the second: $\left\lfloor \frac{365}{4} \right\rfloor = 91$, and the third: $\left\lfloor \frac{365}{5} \right\rfloor = 73$. The first and the second called: $\left\lfloor \frac{365}{12} \right\rfloor = 30$, the first and the second called $\left\lfloor \frac{365}{15} \right\rfloor = 24$ and the second and the third called $\left\lfloor \frac{365}{20} \right\rfloor = 18$ days. All three called $\left\lfloor \frac{365}{60} \right\rfloor = 6$ days. As a result she will have $121 + 91 + 73 - 30 - 24 - 18 + 6 = 219$ days with calls and $365 - 219 = \boxed{149}$ days without calls.

CHALLENGE PROBLEMS 3-4

CP3. A list of 2022 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

To minimize the number of distinct values, we want to maximize the number of times they appear. So, we could have 223 numbers appear 9 times, 1 number appear five times, and the mode appear 10 times, giving us a total of $223 + 1 + 1 = \boxed{225}$.

CP4. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Using stars and bars:
Six to stars for the cookies in and two bars to divide the cookies by type. Let the number of chocolate chip cookies be the number of stars to the left of the first divider, the number of oatmeal cookies be the number of stars between the two dividers, and the number of peanut butter cookies be the number of stars to the right of the second divider. There are $\binom{8}{2} = \boxed{28}$ ways to place the two dividers, so there are 28 ways to select the six cookies.