



Inclusion/Exclusion Principle

Graham Middle School Math Olympiad Team



$$\sqrt{x} = 3, 14$$
$$3 \times 3 = 9$$



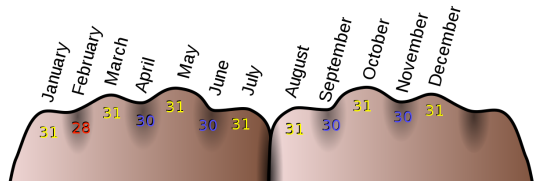
BASIC STATISTICAL TERMS

The **average (arithmetic mean)** of a set of N numbers is the sum of all N numbers divided by N .

The **mode** of a set of numbers is the number occurring *most frequently* in the set.

The **median** of an ordered set (from smallest to largest) of N numbers is the *middle number* if N is odd, or the *mean of the two middle numbers* if N is even.

The **range** of a set is a difference between the largest and the smallest elements.



Consider the set containing the number of days in each month of the year (non-leap year). What are the mean, mode, and median for this set?

The elements in this set are

$\{31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30, 31\}$.

To find mode and median it is useful to sort them

$\{28, 30, 30, 30, 30, 31, 31, 31, 31, 31, 31, 31\}$.

31 occurs most frequently, so it is the **mode**.

The median is the mean of the 6th and 7th elements of the set (both of which are **31**), so the **median** is **31**.

The **mean** is the sum of all numbers in the set **365** divided by **12**, which is $30\frac{5}{12}$.

The range is difference between **31** (the largest value) and **28** (the smallest), so the **range** is **3**.

SETS

In mathematics, a **set** is a *collection* of elements.

The elements that make up a set can be any kind of mathematical objects: numbers, symbols, points in space, lines, etc. Sets are usually denoted as capital Latin letters.

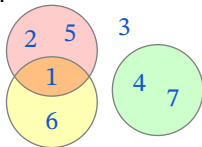
Sometimes it is helpful to visualize sets as circles with all their elements inside.

For example, this diagram represents 3 sets: the red set with elements 2, 5, and 1, the yellow set with elements 1 and 6, and the green with elements 4 and 7. 3 doesn't belong to any set. If we denote the sets as R , Y and G , we can write

$$R = \{1, 2, 5\}, \quad Y = \{1, 6\} \quad \text{and} \quad G = \{4, 7\}.$$

The **number** of elements (the set *cardinality*) in a set S is denoted as $|S|$.

$$|R| = 3, \quad |Y| = 2, \quad |G| = 2.$$



The **union** (denoted by \cup) of sets is the set of all elements in *any* of this set.

For example

$$R \cup Y = \{1, 2, 5, 6\}, \quad Y \cup G = \{1, 4, 6, 7\}.$$

The **intersection** (denoted by \cap) of sets is the set whose elements are in *all* sets.

For example

$$R \cap Y = \{1\}, \quad Y \cap G = \{\} = \emptyset.$$

The symbol \emptyset denotes the **empty set**, the set without elements.

The **complement** of a set S (denoted as S' or \bar{S}) is a set of all elements *not in* a set S .

For example if we consider only elements from 1 to 7 (it is an **universal set** of the problem)

$$R' = \{3, 4, 6, 7\}, \quad G' = \{1, 2, 3, 5, 6\}.$$

INCLUSION/EXCLUSION PRINCIPLE

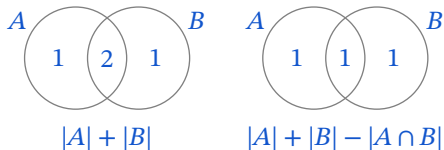
What the number of integers less than or equal to 100 that are factors of 4 or factors of 5?

We know there are 25 factors of 4 and 20 factors of 5, but the final answer is

$$25 + 20 - 5 = 40,$$

because we have to subtract the 5 factors of 20 that are being counted twice.

We can generalize this to the counting of elements in any two sets A and B .



The numbers in the diagram above indicate how many times the elements in each region are counted.

Where A and B overlap, elements are counted twice.

Therefore to find the number of the elements in A and B , we must subtract the number of the elements present in both sets.

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

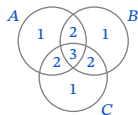
There are 12 cookies with chocolate chips, and 10 cookies with nuts. If there are 4 cookies with chips and nuts, how many cookies are there?

To find the total number of cookies, we sum the cookies from each category ($12 + 10$), and then subtract the cookies that overlap both categories (4) to find the total is 18.

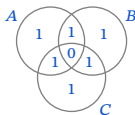
INCLUSION/EXCLUSION PRINCIPLE EXTENDED

This counting principle for overlapping sets can be extended to 3 sets.

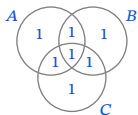
The diagram below shows 3 overlapping sets and the numbers labeling each region indicate how many times that region is counted in the operation shown beneath it. To find the total number of elements, we must count the elements in each set, then subtract the elements for the 3 pairwise intersections, and finally add back the elements found in all 3 sets.



$$|A| + |B| + |C|$$



$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$$



$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Again, the numbers in the diagrams above indicate how many times the elements in each region are counted.

There are 12 cookies with chocolate chips, 18 cookies with nuts, and 8 cookies with raisins. If there are 8 cookies with chips and nuts, 2 cookies with chips and raisins, 8 cookies with raisins and nuts, and 2 cookies with chips, nuts, and raisins, how many cookies are there in total?

Using the overlapping counting principle:

$$\text{chips} + \text{nuts} + \text{raisins} = 38,$$

$$\text{chips/nuts} + \text{chips/raisins} + \text{nuts/raisins} = 18,$$

$$\text{chips/nuts/raisins} = 2,$$

$$\text{Total number of cookies} = 38 - 18 + 2 = 22.$$



CASEWORK

Casework is a counting method where you divide a problem into *several groups* based on some property (known as cases), count these cases individually, and then sum the totals from each case. Casework is a very general problem solving approach, and is used a tremendous amount in middle school math contests. Be careful of potential case overlap.

If the same solution occurs in two of your cases, don't count it twice.

How many pairs of positive integers (x, y) satisfy the equation $x^4 + y < 70$?

Case 1: $x = 1$. $1 + y < 70$. So $0 < y < 69$, and there are 68 possible pairs $(1, 1), (1, 2), \dots, (1, 68)$.

Case 2: $x = 2$. $16 + y < 70$. So $0 < y < 54$, and there are 53 possible pairs $(2, 1), (2, 2), \dots, (2, 53)$.

Note $3^4 = 81$, so there are only two cases to consider ($x = 1$ or 2) with no overlap between them.

Total pairs = $68 + 53 = 121$.

There are a 3 distinct urns (containers A, B, C) and 4 identical balls. I need to put all of the balls into the urns, and I am free to place 0–4 balls into any urn. How many distinct ways are there to put the balls into the urns?

Case 1: All 4 balls in a single urn – 3 ways.

Case 2: 3 in one urn, 1 in a different urn. There are 3 choices for the urn with 3 balls, and then 2 choices remaining for the urn with the single ball, so there are $3 \times 2 = 6$ ways.

Case 3: 2 balls in one urn, and 2 balls in a different urn. It is 6 arrangements (3 ways to pick the first urn, and then 2 ways to pick the second urn), and because 2 balls first into urn A and then 2 balls into urn B is the same if we swap urns A and B , we need to divide by 2. So the answer is $\frac{3 \times 2}{2} = 3$ ways.

Case 4: 2 balls in one urn, and one ball in each of the other urns. There are 3 choices for the urn with 2 balls. There is no other flexibility to choose, since one ball must occupy each of the other 2 urns. So 3 ways for this case.

Therefore there are $3 + 6 + 3 + 3 = 15$ ways to arrange the balls into urns.

STARS AND BARS

Going through the casework for the simple case of 4 balls and 3 urns was time-consuming, and would become impossible to do without a computer as the numbers grow larger. There is, however, a technique that can be used to solve such problems. We will represent each ball by a star ★, and the boundary between urns by a vertical bar |. Any star to the left of the first bar is in urn *A*, any star between the pair of bars is in urn *B*, and any stars to the right of second bar is in urn *C*. So ★ | ★★ | ★ can be read as 1 ball in urn *A*, 2 balls in urn *B* and 1 ball in urn *C*. The 15 arrangements from the 4 cases on the previous slide are:

Case 1:	★★★★	★★★★	★★★★
Case 2:	★★★ ★	★★★★ ★	★★★★ ★
	★ ★★★★	★ ★★★★	★ ★★★★
Case 3:	★★ ★★	★★ ★★	★★ ★★
Case 4:	★★ ★ ★	★ ★★ ★	★ ★ ★★

If we have k balls and n urns, we can represent them with a row of k stars and $n - 1$ bars ($n - 1$ bars separate the stars into regions to represent the n urns).

Using our combinatorics knowledge, we can figure out how many distinct ways there are to arrange a row of identical star and bar symbols. There are

$$\frac{(n + k - 1)!}{k! \times (n - 1)!}$$

distinct arrangements. This fraction is equivalent to C_k^{n+k-1} . We divide the $(n + k - 1)!$ permutations by $k!$ because there are k indistinguishable stars to arrange, and by $(n - 1)!$ because there are $n - 1$ identical bars to arrange. So for the 4 balls and 3 urns, there should be $C_4^6 = 15$ arrangements.

STARS AND BARS EXAMPLE

This was Problem 25 (the final and allegedly hardest) question on the AMC 8 in 2019. It is not really possible to solve this problem with casework due to time constraints on the exam, but seeing how we can apply "Stars and Bars" makes it computationally simple.

Alice has 24 apples. In how many ways can she share them with Ben and Chris so that each of the three people has at least two apples?

The 24 apples are identical in terms of counting, so they can be thought of as the "Stars". Each person is the destination for the apple, so they are the 3 urns (meaning we have 2 "Bars"). There is an added complication, in that each person must have at least 2 apples. No problem. Just subtract these 6 apples from the 24, and Alice must now split 18 apples ($k = 18$) into 3 groups ($n = 3$). This is $C_{18}^{20} = 190$.



Transforming a counting problem into a combinations calculation is possible here because we see that it's equivalent to arranging a pattern consisting of only two types of symbols: stars and bars.

EXERCISES

1. Four students take an exam. Three of their scores are 70, 80, and 90. If the average of their four scores is 70, then what is the remaining score?
2. How many subsets of two elements can be removed from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ so that the mean (average) of the remaining numbers is 6?
3. Jefferson Middle School has the same number of boys and girls. $\frac{3}{4}$ of the girls and $\frac{2}{3}$ of the boys went on a field trip. What fraction of the students on the field trip were girls?
4. The mean, median, and unique mode of the positive integers 3, 4, 5, 6, 6, 7, and x are all equal. What is the value of x ?
5. How many ways to distribute 10 candies among 4 kids?
6. How many ways to distribute 10 candies among 4 kids, so each kid should get at least one candy?
7. If 20 girls are on my school's soccer team, 25 girls are on my school's hockey team, and 11 girls play both sports, then how many girls play soccer and hockey?
8. At one hospital, there are 100 patients, all of whom have at least one of the following ailments: a cold, the flu, or an earache. 38 have a cold, 40 have the flu, and some number have earaches. If 17 have both colds and the flu, 10 have colds and earaches, 23 have the flu and earaches, and 7 have all three, how many have an earache?

CHALLENGE PROBLEMS

1. One day the Beverage Barn sold 252 cans of soda to 100 customers, and every customer bought at least one can of soda. What is the maximum possible median number of cans of soda bought per customer on that day?
2. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2022. On how many days during the next year did she not receive a phone call from any of her grandchildren?
3. A list of 2022 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?
4. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?