Quadratic Equations solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-4

E1. Use factoring to find the roots of $x^2 - 22x - 48 = 0$.

Let $x^2 - 22x - 48 = (x-a)(x-b)$, so $a+b = 22$ and $ab = -48$. Looking into divisors of -48 we may find that 24 and -2 has a sum of 22, so the root of the equation are $\boxed{-2}$ and $\boxed{24}$.

E2. Complete the square to find the possible values of x for which $x^2 + 4x + 3 = 0$?

To complete square for $x^2 + 4x$ we need to add 4, since $(x+2)^2 = x^2 + 4x + 4$. So, in our equation we have $x^2 + 4x + 3 + 1 = 1$ or $(x+2)^2 = 1$. This gives us $x + 2 = \pm 1$ and $x = \boxed{-3}$ or $\boxed{-1}$.

E3. What is the value of i^3?

$i^3 = (i)^2 \times i = -1 \times i = -i$.

E4. For what value(s) of x does the fraction of 3 raised to the power x^2 over 3 raised to the power $3x$ equal one-ninth?
(*Hint*: If all exponents have the same base, then we can solve the problem by equating the exponents.

We need to solve this equation:

$$\frac{3^{x^2}}{3^{3x}} = \frac{1}{9} = \frac{1}{3^2}.$$

Which can be rewritten as $3^{x^2} \times 3^{-3x} = 3^{-2}$, or, using our hint, $x^2 - 3x = -2$, $x^2 - 3x + 2 = (x-1)(x-2) = 0$ and our equation has two roots $x = \boxed{1}$ and $\boxed{2}$. Plugging them back into our original equation we can check that they are works.

EXERCISES 5-8

E5. Find the roots of $x = \frac{28}{x-3}$.

$x(x-3) = 28$ or $x^2 - 3x - 28 = (x-7)(x+4) = 0$ and two roots $x = \boxed{-4}$ and $\boxed{7}$. Plugging them back into our original equation we can check that they are works.

E6. If b and c are both rational numbers and one of the roots of $x^2 + bx + c = 0$ is $3 + \sqrt{2}$, find b and c.

The roots are $\frac{-b}{2} \pm \sqrt{\frac{b^2 - 4c}{4}}$, so we may see that rational and irrational parts of both roots should be the same and the second root is $3 - \sqrt{2}$. $x^2 + bx + c = (x - (3 + \sqrt{2}))(x - (3 - \sqrt{2})) = x^2 - 6x + 7 = 0$. So $\boxed{b = -6}$ and $\boxed{c = 7}$.

E7. For how many different integer values of b are both roots of $x^2 + bx - 16 = 0$ integers?

Let $x^2 + bx - 16 = (x+p)(x+q) = 0$, so $pq = -16$ and $b = p + q$. Since $-16 = 1 \cdot -16 = 2 \cdot -8 = 4 \cdot -4 = 8 \cdot -2 = 16 \cdot -1 = -1 \cdot 16 = -2 \cdot 8 = -4 \cdot 4 = -8 \cdot 2 = -16 \cdot 1$. We got $\boxed{5}$ different values for b: -15, -6, 0, 6, and 15.

E8. Let m and n be roots of: $x^2 - 60x + 864 = 0$. Find a polynomial with roots $m+1$ and $n+1$.

$(x - (m+1))(x - (n+1)) = x^2 - (m+n+2)x + (mn + m + n + 1) = 0$ has roots $m+1$ and $n+1$, since $m + n = 60$ and $mn = 864$, we got $\boxed{x^2 - 62x + 925 = 0}$.

CHALLENGE PROBLEMS 1 - 2

CP1. Find the minimum possible value of the absolute value of $(m-n)$, where m and n are integers satisfying $m+n=mn-2021$.
(*Hint*: could completing the square be useful here if the variables were all grouped on one side of the equation?)

$mn-m-n-2021=(m-1)(n-1)-2022$ or $(m-1)(n-1)=2022$, for $|m-n|$ to be as minimal as possible we need m and n as close as possible. The closest factors of 2022 are 6 and 337, so answer is $337-6=\boxed{331}$.

CP2. (For fun) In the novel, "The Curious Incident of the Dog in the Nighttime," a student in England taking his A-level college entrance exam in maths was given the following question:
Prove that a triangle with sides that can written in the form n^2+1, n^2-1 and $2n$ (where $n>1$) is right-angled.

Since $(n^2+1)^2=n^4+2n^2+1=n^4-2n^2+1+4n^2=(n^2-1)^2+(2n)^2$, and using *Converse of Pythagoras Theorem* we got that triangle with sides n^2+1, n^2-1 and $2n$ (where $n>1$) is right-angled.

CHALLENGE PROBLEMS 3 - 4

CP3. Let m and n be roots of the polynomial $x^2 - 60x + 899 = 0$. What is $m^2 + n^2$?
(*Hint*: think about how $m^2 + n^2$ can be rewritten in terms of the sum and product of the roots m and n).

$m^2 + n^2 = m^2 + 2mn + n^2 - 2mn = (m+n)^2 - 2mn = (60)^2 - 2 \cdot 899 = 3600 - 1798 = \boxed{1802}$.

CP4. Find all real values of n such that $2^{2n} + 2^n + 1 = 73$.
(*Hint*: What substitution would turn this into a quadratic?)

Let $y = 2^n$, then our equation turns into $y^2 + y + 1 = 73$ or $y^2 + y - 72 = (y-8)(y+9) = 0$. So $y = 8$ or -9. If $2^n = 8$, $n = \boxed{3}$, if $2^n = -9$ we don't have solutions since $2^n > 0$ for any real n.