Geometric Sequences solutions

Graham Middle School Math Olympiad Team

EXERCISES 1 - 4

E1. Find the third term of the geometric sequence $3, 4, \ldots$

The ratio is $r = \dfrac{a_2}{a_1} = \dfrac{4}{3}$, so the third term is

$a_3 = a_2 \times r = 4 \cdot \dfrac{4}{3} = \boxed{\dfrac{16}{3}}$.

E2. The first term of a geometric sequence is 1, the third term is 9. Find the second term. Is your answer the only possible one?

$a_3 = a_2 \times r = a_1 \times r^2$, so $r^2 = 9$ and $r = \pm 3$. That means $a_2 = a_1 \times r = \boxed{\pm 3}$, and there are *two possible solutions*.

E3. Express the repeating decimal $0.3636363\ldots$ as a fraction.

$0.36363636\ldots = 0.36 + 0.0036 + 0.000036 + \ldots = \dfrac{0.36}{1 - 0.01} = \dfrac{36}{99} = \boxed{\dfrac{4}{11}}$.

E4. Express the repeating decimal $0.428571428571428\ldots$ as a fraction.

$\dfrac{1}{7} = 0.\overline{142857}$ and this sequence of digit is good to know, since every decimal representation of a fraction $\dfrac{n}{7}$ has the same sequence of digits but starts from different one. In this case it is $\boxed{\dfrac{3}{7}}$.

EXERCISES 5-8

E5. For what value of x does the infinite geometric series $1 + x + x^2 + x^3 + \ldots = 5$?

Using formula for geometric sequence $1 + x + x^2 + \ldots = \frac{1}{1-x} = 5$, so $1 = 5 - 5x$ or $5x = 4$ and $x = \boxed{\frac{4}{5}}$.

E6. If we subtract the geometric series $1 + \frac{1}{9} + \frac{1}{81} + \ldots$ from the infinite geometric series $1 + \frac{1}{3} + \frac{1}{9} + \ldots$, what is the sum of the resulting infinite geometric series?

The remaining terms will be:

$$\frac{1}{3} + \frac{1}{27} + \frac{1}{243} + \ldots = \frac{\frac{1}{3}}{1 - \frac{1}{9}} = \frac{\frac{1}{3}}{\frac{8}{9}} = \frac{9}{8 \cdot 3} = \boxed{\frac{3}{8}}.$$

E7. Find a digit d, so $0.d25d25d25\ldots = \frac{n}{810}$ for some positive integer n.

$\frac{\overline{d25}}{999} = \frac{n}{810}$, $(100 \times d + 25) \times 810 = n \times 999$ and $(100d + 25) \cdot 30 = n \cdot 37$, that means that $(100d + 25)$ must be divisible by 37 and since it is ended by 25 also must by divisible by 25. So $(100d + 25) = k \cdot 925$. The only possible k is 1 and $d = \boxed{9}$.

E8. Find a geometric sequence in which 8, 18, and 27 are terms (not necessary adjacent).

$r^a = 18/8 = 3^2/2^2$ and $r^b = 27/8 = 3^3/2^3$. Just by looking to the numbers we can find out that $a = 2$ and $b = 3$, so $r = 3/2$ and our geometric sequence is 8, 12, 18, 27, $81/2$ and so on.

CHALLENGE PROBLEMS 1-2

CP1. The fraction

$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \ldots + b_{n-1}$?

$$\frac{1}{99^2} = \frac{1}{99}\cdot\frac{1}{99} = \frac{0.\overline{01}}{99} = 0.\overline{00\,01\,02\,03\ldots 97\,99}.$$

It is basically all numbers from 1 til 99 except 98. So, the answer is

$0+0+0+1+0+2+0+3+\ldots+9+7+9+9 = 2\cdot 10\cdot\frac{9\cdot 10}{2} - (9+8)$ or $\boxed{883}$.

CP2. Give the base ten, common fraction representation for $0.\overline{123}_4$ (base 4).

$$0.\overline{123}_4 = \frac{1}{4}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{1}{4^4}+\frac{2}{4^5}+\frac{3}{4^6}+\frac{1}{4^6}+\ldots.$$

Groping them into three series we will get

$$\frac{\frac{1}{4}}{1-\frac{1}{64}}+\frac{\frac{2}{16}}{1-\frac{1}{64}}+\frac{\frac{3}{64}}{1-\frac{1}{64}} = \frac{16}{63}+\frac{8}{63}+\frac{3}{63} = \boxed{\frac{3}{7}}.$$

CP3. Two geometric sequences $a_1,\ a_2,\ a_3,\ \ldots$ and $b_1,\ b_2,\ b_3,\ \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_5.

Let the common ratio r. Now since the n^{th} term of a geometric sequence with first term x and common ratio y is xy^{n-1}, we see that

$a_1\cdot r^{14} = b_1\cdot r^{10}$ so $r^4 = \frac{99}{27} = \frac{11}{3}$. But a_5 equals

$$a_1\cdot r^4 = 27\cdot\frac{11}{3} = \boxed{99}.$$

CP4. The very hungry caterpillar lives on the number line. For each non-zero integer i, a fruit sits on the point with coordinate i. The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?

On the n^{th} straight path, the caterpillar travels n units before hitting food and his weight is $n-1$. Then his speed is 2^{1-n}. Then right before he turns around for the n^{th} time, he has traveled a total time of $\sum_{i=1}^{n} \frac{1}{2^{1-i}} = \frac{1}{2}\sum_{i=1}^{n} i \cdot 2^i$. We want to know how many days the catepillar moves before his weight is 10, so we want to take $n = 10$ so that his last straight path was taken at weight 9.

Hence we want to evaluate $S = \frac{1}{2}\sum_{i=1}^{10} i \cdot 2^i$. Note that $2S = \frac{1}{2}\sum_{i=2}^{11} (i-1) \cdot 2^i$, so

$$S = 2S - S = \frac{1}{2}\left(11 \cdot 2^{11} - \sum_{i=1}^{10} 2^i\right) = \frac{1}{2}(10 \cdot 2^{11} - 2^{11} + 2) = \boxed{9217}.$$