Number Theory 102

Graham Middle School Math Olympiad Team

TERMINAL ZEROES

The **factorial** $n!$ is the product of all positive integers less than or equal to n.

Count the number of trailing zeros in $7!$.

We may count $7!$ as

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040.$$

So the number of trailing zeros of $7!$ is 1.

But what to do for numbers much bigger?

Count the number of trailing zeros in $78!$.

Even the computer will have trouble counting $78!$, but we don't need the whole number. Instead, we need only trailing zeros. How do they appear? Let's look from another side. What does it mean a number has n trailing zeros? We may think of that as the number is n times divisible by 10. So the number is divisible by 10^n.

A number is divisible by 10 if its prime factorization has 2 and 5.

And to be divisible by 10^n it needs to have n times factors 2 and 5. So, let's count how many 2 has prime factorization of $78!$. Every even number in $78!$ give at least one 2, so we already have $78 \div 2 = 39$ twos. Also, every multiple of 4 gives us an additional 2, so we will have additional 19 twos. Multiples of 8: 9 twos, multiples of 16: 4 twos, and multiples of 32 and 64: 2 and 1 twos.

$$39 + 19 + 9 + 4 + 2 + 1 = 74$$

And this means $78!$ is divisible by 2^{74}.

What about 5^n. $78!$ has 15 multiples of 5 and 3 multiples of 25. So $78!$ is divisible by 5^{18}. As we see, the power of 5 is much *smaller* then the power of 2.

$n!$ factorial has the same number of trailing zeroes as **maximal power** of 5 it is *divisible*.

BASES

What is 6547 on *Planet 51*?

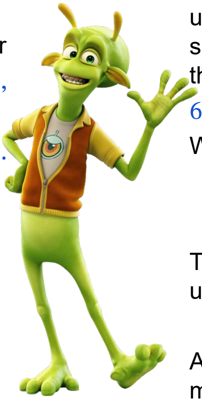
We, earthlings, use the positional numeral system, which was invented between the 1st and 4th centuries by Indian mathematicians.
In English, we read 6547 as *six thousand five hundred forty-seven*, or

$$6547 = 6\times 1000 + 5\times 100 + 4\times 10 + 7,$$

in short writing:

$$6547 = 6\times 10^3 + 5\times 10^2 + 4\times 10 + 7.$$

There
10 is the essential piece of how we understand our numbers. We called our system the **base-ten positional numeral system** or short **decimal**.
Moreover we have just 10 *digits*:
0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.



But 10 is just the number of fingers on our hands, some random result of evolution on the Earth. Does it mean that an alien from *Planet 51*, who has 8 fingers on their hands, can't count? But if they also use the positional system, aliens may use only 8 digits. We will translate them into our spelling as 0, 1, 2, 3, 4, 5, 6, and 7. As a result, they may treat the number like this:

$$6547 \text{ (on Planet 51)} = 6\times 8^3 + 5\times 8^2 + 4\times 8 + 7.$$

Which in our world would mean:

$$6547 \text{ (on Planet 51)} =$$
$$= 6\times 512 + 5\times 64 + 4\times 8 + 7 = 3431.$$

To distinguish ”our” numbers from ”their,” we will use subscript with the base of the system, so

$$6547_8 = 3431_{10}.$$

And now we can communicate with aliens without misunderstanding.

BASE 2

However, we don't need to fly to a faraway planet to find aliens using non 10 based numeral systems. Other creatures around us use another numeral system. We call them *computers*, and since they only can distinguish the presence of electricity or its absence, they may use only 2 digits - 0 and 1, where 0 - no electricity and 1 - there is electricity.

We call the numeral system with base 2 **binary**.

Present 6547_{10} in the binary system?

Let's write down the first powers of 2: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096 and so on. Since digits can be only 0 and 1, the number would be sum of some powers of 2:

$$6547_{10} = 4096+2048+256+128+16+2+1 =$$
$$= 2^{12} + 22^{11} + 2^{8} + 2^{7} + 2^{4} + 2^{1} + 2^{0} =$$
$$= 1100110010011_2$$

Instead of 4 digits, the binary system uses 12 to represent 6547, but they can sum and multiply numbers much faster. The reason is the binary addition and multiplication tables.

+	0	1
0	0	1
1	1	10

×	0	1
0	0	0
1	0	1

For example:

$$\begin{array}{r} 10011 \\ +\,1010 \\ \hline 11101 \end{array}$$

$$\begin{array}{r} 1011 \\ \times\,101 \\ \hline 1011 \\ 1011 \\ \hline 110111 \end{array}$$

In programming, we often use the **hexadecimal system** (base 16). Since we use only 10 digits, and base 16 system need 16, we use letters as digits $10 = A$, $11 = B$, $12 = C$, $13 = D$, $14 = E$, $15 = F$. For example

$$7A_{16} = 7 \times 16 + 10 = 122_{10}$$

CONVERTING BETWEEN BASES

Convert 6547_{10} to 9-base numeral system?

The natural approach is to find out all needed powers of 9: 9, 81, 729, 6561, and so on. Then find the biggest power of 9 lower than our number and divide it with reminder:

$$\begin{aligned} 6547 &= \mathbf{8} \times 729 + 715, \\ 715 &= \mathbf{8} \times 81 + 67, \\ 67 &= \mathbf{7} \times 9 + 4, \\ 4 &= \mathbf{4} \times 1 + 0, \end{aligned}$$

and we get $6547 = 8 \times 9^3 + 8 \times 9^2 + 7 \times 9 + 4$. So

$$6547_{10} = 8874_9$$

Convert 6547_9 to *decimal* numeral system?

$$\begin{aligned} 6547_9 &= 6 \times 9^3 + 5 \times 9^2 + 4 \times 9 + 7 = \\ &= 6 \times 729 + 5 \times 81 + 4 \times 9 + 7 = 4822. \end{aligned}$$

But there is another approach: lets rewrite

$$6547_{10} = 8 \times 9^3 + 8 \times 9^2 + 7 \times 9 + 4.$$

as

$$6547_{10} = ((8 \times 9 + 8) \times 9 + 7) \times 9 + 4.$$

and it gives us second method:

$$\begin{aligned} 6547 &= 727 * 9 + \mathbf{4}, \\ 727 &= 80 * 9 + \mathbf{7}, \\ 80 &= 8 * 9 + \mathbf{8}, \\ 8 &= \mathbf{8}. \end{aligned}$$

and we get our number in reverse order.
It also may be written this way (from bottom to top)

$$\begin{array}{r l} 0 & \text{R} \;\; \mathbf{8} \\ 9\,\overline{)\,8} & \text{R} \;\; \mathbf{8} \\ 9\,\overline{)\,8\,0} & \text{R} \;\; \mathbf{7} \\ 9\,\overline{)\,7\,2\,7} & \text{R} \;\; \mathbf{4} \\ 9\,\overline{)\,6\,5\,4\,7} & \end{array}$$

THE LAST DIGIT

Find the last digit of $743 + 24$?

Let's write our numbers as $74 \times 10 + 3$ and $2 \times 10 + 4$ and find the sum:

$743{+}\mathbf{24} = (74{\times}10{+}\mathbf{3}){+}(2{\times}10{+}\mathbf{4}) = (76{\times}10){+}\mathbf{7}.$

As we see, every multiple of 10 does not contribute to the last digit because all they can apply is tenth and higher.

The same is true for multiplication.

Find the last digit of 743×24?

$$743 \times 24 = (74 \times 10 + \mathbf{3})(2 \times 10 + \mathbf{4}) =$$
$$= 74 \times 2 \times 100 + 74 \times 4 \times 10 +$$
$$+ 3 \times 2 \times 10 + \mathbf{3} \times \mathbf{4} =$$
$$= \text{something}{\times}10{+}\mathbf{12} = (\text{something}{+}1){\times}10{+}\mathbf{2}.$$

The **last digit** of a sum, difference, or product *depends only* on the **last digit** of terms.

Find the last digit of 7^{42}?

As usual for the problems with big numbers, it is sometimes fruitful to start with small numbers and look for a pattern.

$7^1 = 7$, $7^2 = ...9$, $7^3 = ...3$, $7^4 = ...1$, $7^5 = ...7$, $7^6 = ...9$, $7^7 = ...3$. And the last digits start repeating. Indeed, as soon as we get a digit we already met, the next digit would be produced by the same operation since we always multiply by 7.

As a result, the numbers will form a repeated pattern with period 4.

To find the last digit of 7^{42}, we need to find a reminder of 42 divided by 4. $42 = 10 \times 4 + 2$, so the last digit of 7^{42} will be the same as of 7^2. That means that the last digit of 7^{42} is 9.

EXERCISES

1. What is the last digit in $5432 \times 234 \times 747$?
2. What is the last digit in 7^{49}?
3. Write 2021 in base 7 notation.
4. What is base 10 representation of 2021_8?
5. In base 6 notation, what is the sum of $2021_6 + 2022_6 + 2023_6$?
6. A binary number consists of 5 digits, all of which are ones. When that number is doubled, how many digits in the resulting number are now 1's?
7. What is the largest base 10 number that can be expressed as a 2-digit base 5 number?
8. How would you represent $531{,}441$ in base 9?

CHALLENGE PROBLEMS

1. Find the units digit of $3^{1986} - 2^{1986}$.
2. In base 16 (also known as *hexadecimal* notation), the digits for $10 - 15$ are given by the letters $A - F$ respectively. What is the base 10 value of the number $ACED_{16}$?
3. How many terminal zeros are there when $24^6 \times 9^3$ is written in base 6 notation?

4*. When the number n is written in base b its representation is the two-digit number AB where $A = b - 2$ and $B = 2$. What is the representation of n in base $(b-1)$?

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* very hard