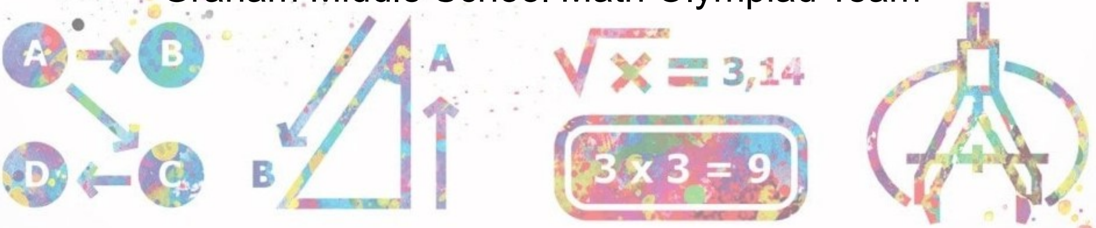
Binomial Expansions, Identities and More

Graham Middle School Math Olympiad Team



BINOMIAL IDENTITIES

Let's expand $(x+y)^2$. This is $(x+y)(x+y)$. Applying the distributive property by considering $(x+y)$ as a single term, we obtain: $(x+y)x+(x+y)y$. Using the distributive property again we obtain $x^2+xy+xy+y^2=x^2+2xy+y^2$. So

$$(x+y)^2=x^2+2xy+y^2.$$

Of course, we could replace y with $(-y)$ and see that:

$$(x-y)^2=x^2-2xy+y^2.$$

Knowing the result when you square a binomial comes up pretty frequently in middle school contests. Even more useful to know is the identity for the difference of perfect squares:

$$x2-y2=(x-y)(x+y).$$

Examples:

What is the value of 1002^2-998^2?

Without using a calculator, we employ the identity for the difference of perfect squares:

$$1002^2-998^2=(1002-998)(1002+998)=$$
$$=(4)(2000)=8000.$$

Given that $ab=10$, what is the value of $(a+b)^2-(a-b)^2$?

Squaring the binomials we have

$$(a+b)^2-(a-b)^2=$$
$$=a^2+b^2+2ab-(a^2+b^2-2ab)=$$
$$=4ab=40.$$

THE BINOMIAL THEOREM

The binomial theorem is an algebraic method of expanding a **binomial expression**. It shows what happens when you multiply a binomial by itself a positive integer number of times. For example, consider the expression $(4x+y)^7$. It would be tedious to multiply the binomial $(4x+y)$ out seven times. The binomial theorem provides a formula that yields the expanded form of this expression.

According to the theorem, it is possible to expand $(x+y)^n$ into a sum involving terms of the form $a_i x^b y^c$, where the exponents b and c are non-negative integers with $b+c=n$, and the coefficient a_i of each term is a specific positive integer depending on n and b. When an exponent is zero, the corresponding power is usually omitted from the term (so that $3x^2y^0$ would be written as $3x^2$).

The Binomial Theorem

It is possible to expand any power of $(x+y)^n$ as:

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y^1 +$$
$$+ C(n,2)x^{n-2}y^2 + \dots + C(n,n-1)x^1y^{n-1} + C(n,n)y^n$$

where the coefficients in the expansion can be calculated using combinatorics. For example, the coefficient of the third term is $C(n,2)$ or $\binom{n}{2}$ which is simply n choose 2.

Expand $(x+y)^3$.

Using the binomial theorem we have:

$$\begin{aligned} C(3,0)x^3 + C(3,1)x^2y^1 + C(3,2)x^1y^2 + C(3,3)y^3 &= \\ &= x^3 + 3x^2y^1 + 3x^1y^2 + y^3. \end{aligned}$$

What is the coefficient in front of x^5y^2 in the expansion of $(4x+y)^7$?

We start with $C(7,2)(4x)^5y^2$ (note we need to use $4x$ rather than x when applying the binomial theorem here). $C(7,2) = 21.\ 4^5 = 1024$. So the coefficient is $21{,}504$.

THE BINOMIAL THEOREM AND PASCAL'S TRIANGLE

The rows of Pascal's triangle are numbered, starting with row $n = 0$ at the top. The entries in each row are numbered from the left beginning with $k = 0$ and are usually staggered relative to the numbers in the adjacent rows. A simple construction of the triangle proceeds in the following manner. On row 0, write only the number 1. Then, to construct the elements of following rows, add the two above numbers to find the new value. If either of the above numbers is not present, substitute a zero in its place. For example, each number in row one is $0 + 1 = 1$.

If binomial x+y is raised to a positive integer power:

$$(x+y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n.$$

Where the coefficients a_i in this expansion are precisely the numbers on row n of Pascal's triangle.

Row 0:							1						
Row 1:						1		1					
Row 2:					1		2		1				
Row 3:				1		3		3		1			
Row 4:			1		4		6		4		1		
Row 5:		1		5		10		10		5		1	
Row 6:	1		6		15		20		15		6		1

Using Pascal's triangle, expand $(x+y)^4$.

Looking at the triangle above, we see the coefficients are:

$$1 \quad 4 \quad 6 \quad 4 \quad 1.$$

The expansion is:

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

SPECIAL FACTORIZATIONS

Slick use of factorizations can make many intimidating algebra problems easier than they first seem. For middle school contests, you really need to know the first two factorizations below. You can verify they are true using the distributive property, which we did earlier in this unit.

Difference of squares:

$$a^2 - b^2 = (a-b)(a+b).$$

Sum of squares:

$$a^2 + b^2 = (a+b)^2 - 2ab.$$

The cubic identities below may show up in very hard middle school contests, and will be useful in high school competitions:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Factor completely: $x^2 + 2mn - m^2 - n^2$.

As a first step, we recognize the last 3 terms come from $-(m-n)^2$. Substituting this into the expression, we get

$$x^2 - (m-n)^2.$$

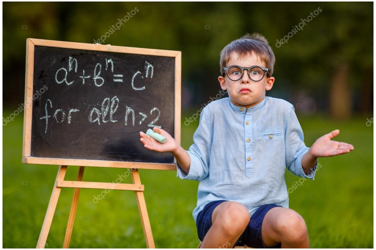
This is a difference of squares equal to

$$[x+(m-n)][x-(m-n)] = (x+m-n)(x-m+n).$$

MANIPULATIONS

There are some problems in which you will need to manipulate the equations or terms in order to use the given information to solve the problem. These manipulations are often combined with the identities from the previous slide.

These manipulations will take time for you to become comfortable with. At first, you may feel like the student in the photo below, but as you move into high school contests, you will learn the techniques needed to master these problems.



Find $1/a + 1/b$ if $a + b = 6$ and $ab = 3$.

The first step here is to add the fractions by creating the common denominator ab. Then we have

$$b/ab + a/ab = (a + b)/ab = 6/3 = 2.$$

Find $x^6 + 1/x^6$ if $x + 1/x = 3$.

We are given an equation with x raised to the first power and we need to get to an equation with x to the sixth power. First, we square both sides of the equation: $(x + 1/x)^2 = x^2 + 1/x^2 + 2 = 9$.
So $x^2 + 1/x^2 = 7$. Cubing squares gives us sixth powers, so we cube both sides:
$(x^2 + 1/x^2)^3 = 343$.
Writing out the binomial expansion, we obtain

$$x^6 + 1/x^6 + 3(x^2 + 1/x^2) = 343.$$

But $x^2 + 1/x^2 = 7$, so $x^6 + 1/x^6 + 3(7) = 343$.
Hence $x^6 + 1/x^6 = 322$.

EXERCISES

1. The *harmonic mean* of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1, 2, and 4?

2. If $\dfrac{x+1}{x} = 2$, what is the value of $\dfrac{x^2+1}{x^2}$?

3. What is the greatest integer value of x such that $\dfrac{x^2+2x+5}{x-3}$ is an integer?

4. What is the value of the product
$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

5. What is the value of
$$\frac{2022^3 - 2\cdot 2022^2\cdot 2023 + 3\cdot 2022\cdot 2023^2 - 2023^3 + 1}{2022\cdot 2023}?$$

6. If $\left(\dfrac{1}{x+y}\right)\left(\dfrac{1}{x}+\dfrac{1}{y}\right) = \dfrac{1}{13}$, what is the value of the product of x and y?

7. If $a\clubsuit b = \dfrac{ab}{a+b}$ and $a\clubsuit 4 = 3$, what is the value of a?

8. Compute the sum of the distinct prime factors of $2^{15} + 2^8 - 2^7 - 1$.

CHALLENGE PROBLEMS

1. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?
2. Let a, b, and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c)=0$?
3. If $y+4=(x-2)^2$, $x+4=(y-2)^2$, and $x \neq y$, what is the value of x^2+y^2?
4. For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?