Geometric sequencies

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GEOMETRIC SERIES AND THE INVENTION OF CHESS

This story happened (*or probably not*) in the Persian Empire (*or probably in India*). The Shah (king) was so delighted by the game of chess that he invited its creator to his court and asked him to make any wish as a reward for creating the game. The clever inventor said, “Oh generous Shah, all I ask is that you place 1 grain of rice (*or probably wheat*) on the first square, 2 grains on the second square, 4 grains on the 3rd square, and continue doubling until you reach the 64th square.” “Is that all you ask for?” laughed the Shah. “I would have happily given you a fortune in gold. But if that is your request, I promise you shall have it.” With this promise made, an imperceptible grin spread over the inventor’s face as he bowed and took his leave of the Shah.

The court mathematician was in attendance and overhead this conversation. “Oh, powerful Shah,” began the alarmed mathematician, “we cannot meet his request. It would bankrupt the kingdom.” After the Shah heard the explanation, he summoned his guards to bring him the inventor and had him beheaded for his insolence (*or hire him as a minister*). How many grains of rice would have been required to satisfy the request?

This request was asking the king to sum the terms of a geometric series

$$1 + 2 + 4 + 8 + \ldots + 2^{63}.$$

The total amount of grains was

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

which is enough to feed *100 tons* of rice to every single human on Planet Earth.

GEOMETRIC SEQUENCES

Binary fission is the process by which bacterium divide into two identical daughter cells.

After 3 minutes of life, a bacterium divides into two new bacteria. A new bacterium also split into two bacteria after 3 minutes, and so on. One bacterium gets into a cup with a nutrient solution. How many bacteria will be in a cup after one hour?

After 3 minutes, it would be 2 bacteria. After 6 minutes, it would be 4 bacteria and so on. Every 3 minutes amount of bacteria doubled. Since an hour contains 20 3 minutes intervals, the number of bacteria will be doubled 20 times. So total amount of bacteria after one hour would be

$$1 \times 2^{20} = 1{,}048{,}576 \approx 1 \text{ million bacteria.}$$

The sequence $1, 2, 4, 8, \ldots, 2^{20}$ is called a geometric sequence.

A **geometric sequence** is a sequence of numbers in which the ratio between *consecutive terms* is the same.
The *ratio* between consecutive terms is called the **common ratio** of the sequence.

The common ratio is usually denoted as r.
Some examples

geometric sequence	a_1	r
$1, 2, 4, 8, 16, 32, \ldots$	1	2
$3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$	3	$\frac{1}{3}$
$1, -1, 1, -1, 1, -1, 1, \ldots$	1	-1
$2, 2, 2, 2, 2, \ldots$	2	1

A sequence can also have *infinitely* many terms.

A general term (n^{th} term) of a geometric sequence may be found using formula

$$a^n = a_1 \times r^{n-1}.$$

GEOMETRIC SERIES

The *sum* of the terms of a geometric sequence is called a **geometric series**.

Find the sum $1 + 3 + 9 + 27 + 81 + 243 + 729$.

Direct summation gives us 1093. But what to do if series would be longer?
Let's multiply this sum (denoted as S) by the common ratio

$$3S = 3 + 9 + 27 + 81 + 243 + 729 + 2187.$$

It looks exactly the same as the original sum except 1 has been replaced with 2187. So

$$3S = S - 1 + 2187.$$

Solving for S

$$S = \frac{2187 - 1}{3 - 1} = \frac{2186}{2} = 1093.$$

Let's get general formula for a geometric series. The initial term is a, the ratio is r, and we have a total of n terms. Hence we have the following series

$$S = a + a \times r + a \times r^2 + \ldots + a \times r^{n-1}.$$

Multiply it by r

$$S \times r = a \times r + a \times r^2 + \ldots + a \times r^{n-1} + a \times r^n.$$

Subtract the initial sum

$$S \times r - S = a \times r^n - a.$$

The sum of geometric sequence is

$$S = \frac{a \times r^n - a}{r - 1}$$

Note, if $r = 2$, the formula become $S = a \times 2^n - a$, which gives us the total amount of grains the Shah owes to the inventor of chess.

INFINITE GEOMETRIC SERIES

The series goes on forever in an infinite geometric series, with a constant ratio between successive terms. What is the sum of the elements of an infinite geometric series?

If $r \geq 1$, the sum is infinite since we have an infinite number of non-zero terms that are either staying constant ($r = 1$) or growing ($r > 1$).

If $r < 1$, however, the series is said to *converge*. It is possible to calculate the value of the sum of the elements in the series.

Find the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$.

Let S is the sum of the elements in the series.

$S = 1/2 + 1/4 + \ldots$ Now multiply both sides of the equation by 2.

$2S = 1 + 1/2 + 1/4 + \ldots$ Wait, isn't the right side of the equation just $1 + S$?

So $2S = 1 + S$ (now subtract S from both sides)

$S = 1$. So, the sum of $1/2 + 1/4 + \ldots = 1$.

For positive numbers $r < 1$, as n becomes arbitrarily large, r^n *tends to zero*. And, taking the *limit of the sequence*, we can consider r^n is equal to 0 when n goes to infinity. Let's apply this to the formula for the sum of a geometric sequence when $-1 < r < 1$ and for infinite sequence.

$$S = \frac{a \times r^n - a}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

The sum of an infinite geometric series with a common ratio less than 1

$$S = \frac{a}{1 - r}.$$

ACHILLES AND THE TORTOISE

Zeno of Elea was a Greek philosopher born around 490 BC. He wrote a book featuring 40 paradoxes, of which *Achilles and the Tortoise* is perhaps the most famous.
Achilles, the ancient Greek hero, is chasing after a tortoise. Achilles is much faster than the tortoise, and so he must eventually catch up with it.
Zeno argued that on his way to catching the tortoise, Achilles must first reach the tortoise's starting point. By the time he has reached this however, the tortoise will also have moved further forward to a new point.
By the time Achilles reaches this new point, the tortoise will have moved a bit further forward and so on. Every time Achilles reaches the point where the tortoise was, the tortoise has moved forwards to a new point. Hence Achilles **will never catch up**.

Let's say Achilles run 10 *meters per second* and the tortoise's speed is 0.1 *meters per second* and initial distance between them was 100 meters.
After 10 seconds Achilles run 100 meters and the tortoise crawled only 1 meters. The next 0.1 seconds Achilles run 1 meter and the tortoise crawled 0.01 meters and so on.
The distance and time will form infinite geometric series:
Distance (m): $100 + 1 + 0.01 + 0.0001 + \ldots$
Time (s): $10 + 0.1 + 0.001 + 0.00001 + \ldots$
Let's find the sums

$$\text{distance} = \frac{100}{1 - 0.01} = \frac{100}{0.99} \approx 101.0101 \text{ m},$$

$$\text{time} = \frac{10}{1 - 0.01} = \frac{10}{0.99} \approx 10.101 \text{ s}.$$

As we see, "never" will happen 10.1 seconds after the race begins, and Achilles catches the tortoise almost immediately when he gets to its starting point.

REPEATING DECIMAL

We instantly recognize the fractional representations for some repeating decimals. For example, $0.333\ldots = \frac{1}{3}$, $0.111\ldots = \frac{1}{9}$.

A repeating decimal can be represented as an infinite geometric series, and the sum of that series can be used to calculate the fractional representation for that repeating decimal.

Represent $0.4444\ldots$ as a common fraction.

This is the sum of the infinite geometric series

$$0.4 + 0.4 \times 0.1 + 0.4 \times 0.1^2 + 0.4 \times 0.1^3 + \ldots$$

$$S = 0.4 + 0.4 \times 0.1 + 0.4 \times 0.1^2 + \ldots$$

$$10 \times S = 4 + 0.4 + 0.4 \times 0.1 + 0.4 \times 0.1^2 + \ldots$$

$$10 \times S = 4 + S$$

$$9 \times S = 4$$

$$S = \frac{4}{9}.$$

What if there is more than one repeating digit in the decimal, such as $0.81818181\ldots$?

We can once again represent this as the sum of a geometric series, but now the r is 0.01.

$$S = 0.81 + 0.81 \times 0.01 + 0.81 \times 0.01^2 + \ldots$$

$$100 \times S = 81 + 0.81 + 0.81 \times 0.01 + \ldots$$

$$100 \times S = 81 + S$$

$$99 \times S = 81$$

$$S = \frac{81}{99}.$$

In general every repeating decimal fraction with period of length n may be represented as a common fraction

$$0.a_1a_2\ldots a_na_1\ldots = \frac{0.a_1a_2\ldots a_n}{1 - 0.\underbrace{0\ldots0\,1}_{n\text{ times}}} = \frac{a_1a_2\ldots a_n}{\underbrace{999\ldots9}_{n\text{ times}}}.$$

EXERCISES

1. Find the third term of the geometric sequence $3, 4, \ldots$
2. The first term of a geometric sequence is 1, the third term is 9. Find the second term. Is your answer the only possible one?
3. Express the repeating decimal $0.3636363\ldots$ as a fraction.
4. Express the repeating decimal $0.428571428571428\ldots$ as a fraction.
5. For what value of x does the infinite geometric series $1 + x + x^2 + x^3 + \ldots = 5$?
6. If we subtract the geometric series $1 + \frac{1}{9} + \frac{1}{81} + \ldots$ from the infinite geometric series $1 + \frac{1}{3} + \frac{1}{9} + \ldots$, what is the sum of the resulting infinite geometric series?
7. Find a digit d, so $0.d25d25d25\ldots = \frac{n}{810}$ for some positive integer n.
8. Find a geometric sequence in which 8, 18, and 27 are terms (not necessary adjacent).

CHALLENGE PROBLEMS

1. The fraction
$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\dots b_2b_1b_0},$$
where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \dots + b_{n-1}$?
2. Give the base ten, common fraction representation for $0.\overline{123}_4$ (base 4).
3. Two geometric sequences $a_1,\ a_2,\ a_3,\ \dots$ and $b_1,\ b_2,\ b_3,\ \dots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_5.
4. The very hungry caterpillar lives on the number line. For each non-zero integer i, a fruit sits on the point with coordinate i. The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?