



Number Theory 101 solutions

Graham Middle School Math Olympiad Team



$$\sqrt{x} = 3, 14$$
$$3 \times 3 = 9$$



PROBLEMS 1-4

E1. What is the remainder when 301×349 is divided by 9?

The remainder when 301 is divisible by 9 is $3 + 0 + 1 = 4$ and the remainder when 349 is divisible by 9 is $(3 + 4 + 9) - 9 = 7$. The remainder of $4 \times 7 = 28$ is divisible by 9 is $(2 + 8) - 9 = \boxed{1}$.

E2. Find the GCD and LCM of 42 and 98.

Using prime factorization:

$42 = 2 \times 3 \times 7$ and $98 = 2 \times 7^2$, so

$\gcd(42, 98) = 2 \times 7 = 14$, and

$\text{lcm}(42, 98) = 2 \times 3 \times 7^2 = 294$.

Using Euclidean algorithm:

$$98 = 42 \times 2 + 14,$$

$$42 = \boxed{14} \times 3 + 0.$$

and

$$\text{lcm}(42, 98) = \frac{42 \times 98}{\gcd(42, 98)} = \frac{42 \times 98}{14} = \boxed{294}.$$

E3. The GCD of two numbers A and B is 7. What are the possible values of GCD of $15 \cdot A$ and $35 \cdot B$?

The possible values are shown in the table:

	$7^2 \nmid A$	$7^2 \mid A$
$3 \nmid B$	<div>35</div>	<div>245</div>
$3 \mid B$	<div>105</div>	<div>735</div>

$x|y$ means x divides y , and $x \nmid y$ means x not divides y .

E4. The number 6545 can be written as the product of a pair of positive two-digit integers. What are these two integers?

The prime factorization of 6545 is $5 \times 7 \times 11 \times 17$, since $5 \times 7 \times 11 > 100$ and $7 \times 17 > 100$, the only possible way to form a pair of two-digit integers is $5 \times 17 = \boxed{85}$ and $7 \times 11 = \boxed{77}$.

PROBLEMS 5-8

E5. What is the smallest prime factor of $11^7 + 7^5$?

The smallest prime number is 2, and it is a factor of $11^7 + 7^5$, because both 11^7 and 7^5 are odd and their sum is even.

E6. The four-digit number $A55B$ is divisible by 36. What is the sum of A and B ?

$36 = 9 \times 4$, so $A55B$ should be divisible by 9 and 4. If $A55B$ is divisible by 9, so $A + 5 + 5 + B$ is divisible by 9, so $A + B$ is either 8 or 17. And since $5B$ is divisible by 4, B can be either 2 or 6. For both cases the sum $A + B$ can't be 17, so the only option for $A + B$ is 8.

E7. What is the sum of the digits of $\frac{10^{25} + 8}{9}$?

$$10^{25} = \underbrace{99 \cdot 9}_{25 \text{ nines}} + 1, \text{ so}$$

$$\frac{10^{25} + 8}{9} = \frac{\underbrace{99 \cdot 9}_{25 \text{ nines}} + 9}{9} = \underbrace{11 \cdot 1}_{25 \text{ ones}} + 1 = \underbrace{11 \cdot 1}_{24 \text{ nines}} 2. \text{ And}$$

the sum of the digits is $24 + 2 = \span style="border: 1px solid black; padding: 0 5px;">26.$

E8. Find the GCD of $2n + 13$ and $n + 7$ by Euclid's algorithm.

$$2n + 13 = (n + 7) \times 2 - 1,$$

$$n + 7 = \span style="border: 1px solid black; padding: 0 5px;">1 \times (n + 7) + 0.$$

CHALLENGE PROBLEMS 1-3

C1. The positive integers A , B , $A - B$, and $A + B$ are all prime numbers. What is the sum of these four primes?

Since A , B , and $A + B$ are all prime, that means two of them should be odd, so B should be 2. One of the numbers $A - 2$, A , and $A + 2$ should be divisible by 3 since all of them have different remainders when divisible by 3. So we got $A - B = 3$, $A = 5$, and $A + B = 7$ and the sum is $2 + 3 + 5 + 7 = \boxed{17}$.

C2. Show that every prime greater than 3 must be of the form $6n + 1$ or $6n - 1$ for a positive integer n .

$6n$ and $6n + 3$ can't be primes because they are divisible by 3;

$6n + 2$ and $6n + 4$ can't be primes because they are divisible by 2.

So the only way for primes is to have the form $6n + 1$ and $6n - 1$.

C3. If p , q and r are prime numbers such that their product is 19 times their sum, find $p^2 + q^2 + r^2$.

Since $pqr = 19(p + q + r)$, one of the numbers should be 19. So $pq = p + q + 19$.

$pq - p - q + 1 = 20$, so

$$(p - 1)(q - 1) = 20.$$

Let's take a look at different factorization of 20:

Case 1: $(p - 1)(q - 1) = 1 \times 20$, $p = 2$, $q = 21$.

Case 2: $(p - 1)(q - 1) = 2 \times 10$, $p = 3$, $q = 11$.

Case 3: $(p - 1)(q - 1) = 4 \times 5$, $p = 5$, $q = 6$.

Only in case 2 p and q are primes, so 3, 11, and 19 is the only possible options for p , q , and r in some order.

$$p^2 + q^2 + r^2 = 3^2 + 11^2 + 19^2 = \boxed{491}.$$

CHALLENGE PROBLEM 4

C4. Let a , b , c , and d be positive integers such that $\gcd(a, b) = 24$, $\gcd(b, c) = 36$, $\gcd(c, d) = 54$, and $70 < \gcd(d, a) < 100$. Which of the following must be a divisor of a : 5, 7, 11, 13, or 17?

Notice that

$$\begin{aligned}\gcd(a, b, c, d) &= \\ &= \gcd(\gcd(a, b), \gcd(b, c), \gcd(c, d)) = \\ &= \gcd(24, 36, 54) = 6,\end{aligned}$$

so $\gcd(d, a)$ must be a multiple of 6.

If $\gcd(d, a)$ is multiple of 2^2 , then $\gcd(c, d) \neq 54$, since d and c divisible by 4.

If $\gcd(d, a)$ is multiple of 3^2 , then $\gcd(a, b) \neq 24$, since a and b both divisible by 9.

The only answer choice that gives a value between 70 and 100 when multiplied by 6 is 13.