Probability solutions

Graham Middle School Math Olympiad Team

EXERCISES 1-3

E1. When a pair of 6-sided dice are rolled, what is the probability that the numbers rolled sum to 8?

We can get 8 in 5 cases: $2-6$, $3-5$, $4-4$, $5-3$, $6-2$. The total number of possible cases is $6 \times 6 = 36$, so the probability is $\boxed{\frac{5}{36}}$.

E2. If you flip a fair coin 5 times, what is the probability that you will flip a total of 3 heads and 2 tails?

We can choose 2 coins that flip tails in $\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10$. The total number of possible combinations is $2^5 = 32$. So the probability is $\frac{10}{32} = \boxed{\frac{5}{16}}$.

E3. Each of two boxes contains three chips numbered 1, 2, 3. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?

Let's do complimentary counting: the product is odd if both multiplicands are odd. The probability that a chip from the first box odd is $\frac{2}{3}$, the same for the second box. So the probability that two chips are odd is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. The probability that the product is even is $1 - \frac{4}{9} = \boxed{\frac{5}{9}}$.

EXERCISES 4-5

E4. A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

There are $\binom{5}{3}$ possible groups of cards that can be selected. If 4 is the largest card selected, then the other two cards must be either 1, 2, or 3, for a total $\binom{3}{2}$ groups of cards. Then the probability is just $\frac{\binom{3}{2}}{\binom{5}{3}} = \boxed{\frac{3}{10}}$.

E5. Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown.

X X X
X X X

If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

We can ignore the 4 other classmates because they aren't relevant. We can treat Abby and Bridget as a pair, so there are $\binom{6}{2} = 15$ total ways to seat them. If they sit in the same row, there are $2 \cdot 2 = 4$ ways to seat them. If they sit in the same column, there are 3 ways to seat them. Thus our answer is $\frac{4+3}{15} = \boxed{\frac{7}{15}}$.

EXERCISES 6-7

E6. Two different numbers are randomly selected from the set $\{-2, -1, 0, 3, 4, 5\}$ and multiplied together. What is the probability that the product is 0?

The product can only be 0 if one of the numbers is 0. Once we chose 0, there are 5 ways we can choose the second number, or $6-1$. There are $\binom{6}{2}$ ways we can choose 2 numbers randomly, and that is 15. So, $\frac{5}{15} = \frac{1}{3}$ so the answer is $\boxed{\frac{1}{3}}$.

E7. On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

The number of people wearing caps and sunglasses is $\frac{2}{5} \cdot 35 = 14$. So then 14 people out of the 50 people wearing sunglasses also have caps. $\frac{14}{50} = \boxed{\frac{7}{25}}$

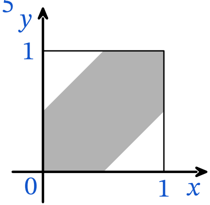
EXERCISE 8

E8. A top hat contains 3 red chips and 2 green chips. Chips are drawn randomly, one at a time without replacement, until all 3 of the reds are drawn or until both green chips are drawn. What is the probability that the 3 reds are drawn?

Assume that after you draw the three red chips in a row without drawing both green chips, you continue drawing for the next turn. The last/fifth chip that is drawn must be a green chip because if both green chips were drawn before, we would've already completed the game. So technically, the problem is asking for the probability that the ”fifth draw” is a green chip. This probability is symmetric to the probability that the first chip drawn is green, which is $\frac{2}{5}$. So the probability is $\boxed{\frac{2}{5}}$.

C1. Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that their difference is less than 0.5?

Let's draw our points on the Cartesian plane and color all points whose coordinates difference is less that 0.5. They are limited by lines $y = x + 0.5$ and $y = x - 0.5$



The ratio of the shaded region area to the total area of the square is $\boxed{\frac{3}{4}}$.

C2. A radio program has a quiz consisting of 3 multiple-choice questions, each with 3 choices. A contestant wins if he or she gets 2 or more of the questions right. The contestant answers randomly to each question. What is the probability of winning?

There are two ways the contestant can win.
Case 1: The contestant guesses all three right.
This can only happen $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$ of the time.
Case 2: The contestant guesses only two right.
We pick one of the questions to get wrong, 3, and this can happen $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}$ of the time. Thus, $\frac{2}{27} \cdot 3 = \frac{6}{27}$.
So, in total the two cases combined equals $\frac{1}{27} + \frac{6}{27} = \boxed{\frac{7}{27}}$.

CHALLENGE PROBLEMS 3-4

C3. When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as $\frac{n}{6^7}$, where n is a positive integer. What is n?

The minimum number that can be shown on the face of a die is 1, so the least possible sum of the top faces of the 7 dice is 7.
In order for the sum to be exactly 10, 1 to 3 dices numbers on the top face must be increased by a total of 3.
There are 3 ways to do so: 3, $2+1$, and $1+1+1$
There are 7 for Case 1, $7 \cdot 6 = 42$ for Case 2, and $\frac{7 \cdot 6 \cdot 5}{3!} = 35$ for Case 3.
Therefore, the answer is $7 + 42 + 35 = \boxed{84}$

C4. A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \dots, 2018\}$. What is the probability that m^n has a units digit of 1?

When a number's unit's digit is 1, then any power to this number will also end in 1 (since 1^n for any n is always 1), so we have 20 choices for 11.
When a number's unit's digit is 3, then 3^{4n} for any n will produce a number ending with 1. So, $20 \div 4 = 5$ choices for 13.
5^n always ends in 5, so there are 0 possibilities for 15.
When a number's unit's digit is 7, then this is also the same thing with 3, so we have 5 choices.
When a number's unit's digit is 9, then 9^{2n} will produce a number ending in 1, so we have $20 \div 2 = 10$ possibilities.
Hence, we have a total of $5 \cdot 20 = 100$ ways, so the probability is $\frac{20+5+0+5+10}{100} = \frac{40}{100} = \boxed{\frac{2}{5}}$.