



# Modular Arithmetic

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$$\sqrt{x} = 3, 14$$
$$3 \times 3 = 9$$



## MODULAR ARITHMETIC

Now is 4:00 pm, what time will be 100 hours from now?

4 days is equal to  $4 \times 24 = 96$  hours, so 100 hours is equal to 4 days and 4 hours. That means in 96 hours we will have the same time. After 4 hours it will be 8:00 pm.



**Modular arithmetic** is a type of arithmetic that deals with remainders of numbers.

In modular arithmetic, numbers "wrap around" upon reaching a given number (this given number is known as the **modulus**) to leave a remainder.

The value  $a \pmod{m}$  is a shorthand way of saying "the remainder when  $a$  is divided by  $m$ ", and is from 0 to  $m - 1$ .

We say that two numbers  $a$  and  $b$  are **congruent modulo  $m$**  if  $b - a$  is divisible by  $m$ . That is,

$$a \equiv b \pmod{m}$$

if and only if  $b - a = mk$  for some integer  $k$ .

38 is congruent to 14 (mod 12), because  $38 - 14 = 24$ , which is a multiple of 12, or, equivalently, because both 38 and 14 have the same remainder 2 when divided by 12. We use a triple equal sign to represent *congruence*:

$$38 \equiv 14 \pmod{12}.$$

The same rule holds for negative values:

$$-8 \equiv 7 \pmod{5};$$

$$2 \equiv -3 \pmod{5};$$

$$-3 \equiv -8 \pmod{5}.$$

In contests, you will often find modular arithmetic a useful tool for solving problems involving remainders or divisibility.

## ADDITION AND MULTIPLICATION THEOREMS FOR MODULAR ARITHMETIC

If  $a + b = c$ , then

$$a \pmod{n} + b \pmod{n} \equiv c \pmod{n}.$$

To find the **remainder of the sum** of  $a$  and  $b$ , we can instead **sum the remainders** of the two terms.

The proof is straightforward:

Let's write  $a$ ,  $b$  and  $c$  as  $q_a n + r_a$ ,  $q_b n + r_b$  and  $q_c n + r_c$  where  $q_a$ ,  $q_b$ , and  $q_c$  are *quotients* and  $r_a$ ,  $r_b$ , and  $r_c$  are *remainders* of division by  $n$ . Then  $a + b = q_a n + r_a + q_b n + r_b = (q_a + q_b)n + (r_a + r_b)$

That means we can ignore  $(q_a + q_b)n$  and  $q_c n$  when considering numbers *modulo*  $n$ . So we have

$$r_a + r_b \equiv r_c \pmod{n}.$$

Which we wanted to proof.

For example:

$$2021 \equiv 2000 + 20 + 1 \pmod{n}$$

If  $a \times b = c$ , then

$$a \pmod{n} \times b \pmod{n} \equiv c \pmod{n}.$$

To find the **remainder of the product** of  $a$  and  $b$ , we can instead **multiply the remainders** of the two factors.

Doing the same substitution as in proof of the *addition theorem*

$$\begin{aligned} a \times b &= (q_a n + r_a)(q_b n + r_b) = \\ &= q_a q_b n^2 + (q_a r_b + q_b r_a)n + r_a r_b. \end{aligned}$$

We can ignore  $q_a q_b n^2$ ,  $(q_a r_b + q_b r_a)n$ , and  $q_c n$  because all of them is divisible by  $n$ . So we have

$$r_a \times r_b \equiv r_c \pmod{n}.$$

For example:

$$2021 \times 2022 \equiv 21 \times 22 \equiv 462 \equiv 62 \pmod{100};$$

$$42 \times 24 \equiv 3 \times -2 \equiv -6 \equiv 7 \pmod{13}.$$

## DIVISIBILITY RULES FOR POWERS OF 2 AND 5

Is 721,456 divisible by 8?

8 is  $2 \times 2 \times 2$ . That means that we can multiply it by  $5 \times 5 \times 5$  and get  $10 \times 10 \times 10$ , so 1,000 must be divisible by 8.

Let's write down 721,456 as  $721,000 + 456$ , as we know from the *addition theorem* the remainder of the division of 721,456 is equal to the sum of remainders of each of summands. Since 21,000 is multiple of 1,000 its remainder modulo 8 is 0. So

$$721,456 \equiv 456 \pmod{8}.$$

That means to check whether 721,456 is divisible by 8, we need to check, whether 456 is divisible by 8. Indeed  $456 = 8 \times 57$ , so 721,456 is also divisible by 8.

To know whether a number is divisible by  $2^n$ , we need to check whether the **last  $n$  digits** of the number is divisible by  $2^n$ .

Is 6385 divisible by 25?

25 is  $5 \times 5$ , so multiplying it by  $2 \times 2$  we will get that 25 is multiple of 100. Doing the same stuff as with divisibility by 2, we write 6385 as  $6300 + 85$ . 6300 is a multiple of 100 therefore it is also multiple of 25. That means we only need to check whether 85 is divisible by 25.  $85 = 75 + 10$ , so 6385 is not divisible by 25.

To know whether a number is divisible by  $5^n$ , we need to check whether the **last  $n$  digits** of the number is divisible by  $5^n$ .

Both rules are true, because  $10^n$  is divisible by  $2^n$  and  $5^n$ . Using this fact allows us to exclude from consideration all numbers bigger than  $10^n$  and that left us with the last  $n$  digits of a number.



## DIVISIBILITY RULES FOR 3 AND 9

Is 5479 divisible by 9?

Let's write a number 5479 as a sum

$$5472 = 5 \cdot 1000 + 4 \cdot 100 + 7 \cdot 10 + 2.$$

And then write 1000 as a sum  $999 + 1$ ,

$100 = 99 + 1$  and  $10 = 9 + 1$ . So

$$\begin{aligned} 5472 &= 5 \cdot (999 + 1) + 4 \cdot (99 + 1) + 7 \cdot (9 + 1) + 9 = \\ &= 5 \cdot 999 + 5 + 4 \cdot 99 + 4 + 7 \cdot 9 + 7 + 2. \end{aligned}$$

And finally we can realize that 999, 99, and 9 are divisible by 9. That means that divisibility of 5479 by 9 depends on divisibility of sum  $5 + 4 + 7 + 2$ .

A number is divisible by 9, if the **sum of its digits** is divisible by 9.

And since 9, 99, 999, 9999 and so on is also divisible by 3, we get

A number is divisible by 3, if the **sum of its digits** is divisible by 3.

$\overline{abcd}$  is a mathematical notation that a number is written with digits  $a$ ,  $b$ ,  $c$ , and  $d$ . So

$$\overline{abcd} = a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d.$$

This allows us to write the rules as

$$\overline{abcd} \equiv a + b + c + d \pmod{9 \text{ or } 3}.$$

The divisibility rule by 9 allows us quickly check if our arithmetic operations are correct.

$$\begin{array}{r} 384 \\ \times 56 \\ \hline 2304 \\ 1820 \phantom{0} \\ \hline 20504 \end{array}$$

Lets see remainders of factors and product

$$384 \equiv 6 \pmod{9}, \quad 56 \equiv 2 \pmod{9},$$

$$20504 \equiv 2 \pmod{9}.$$

But  $6 \times 2 \not\equiv 2 \pmod{9}$ , so there is an error.

## DIVISIBILITY RULES FOR 11

The divisibility rules for 9 and 3 works because 10, 100, 1,000, and so on are all divisible by 9 with remainder 1. But what about 11?

$$1 \underbrace{00 \dots 0}_{\text{even zeros}} - 1 = \underbrace{99 \dots 9}_{\text{even nines}}$$

is divisible by 11 because it is sum of

$$99 \ 99 \dots 99 = 99 \cdot 10^n + \dots + 99$$

which all are divisible by 11.

From other hand

$$1 \underbrace{00 \dots 0}_{\text{odd zeros}} + 1 = \underbrace{99 \dots 9}_{\text{even nines}} 0 + 11$$

is also divisible by 11.

Is 3432 divisible by 11?

$$\begin{aligned} 3432 &= 3 \cdot 1000 + 4 \cdot 100 + 3 \cdot 10 + 2 = \\ &= 3 \cdot (1001 - 1) + 4 \cdot (99 + 1) + 3 \cdot (11 - 1) + 2 \equiv \end{aligned}$$

when we get rid of all multiples of 11

$$\equiv -3 + 4 - 3 + 2 = 0 \pmod{11}.$$

As we see our final computation is the digits of the number with alternate signs.

A number is divisible by 11 if the **difference** between the **sum of the odd-place digits** and the **sum of the even-place digits** is 0 or a multiple of 11.

The combinations of the divisibility rules may also create addition divisibility rules.

For example to check if number is divisible by 6 we may check if it divisible by 2 and 3. This will work when the divisor may be split in product of

**coprime<sup>a</sup>** factors.

But the rule will not work for example for  $27 = 9 \times 3$ , since testing for 9 we already testing for 3.



<sup>a</sup>coprime numbers are numbers that don't have common divisors.

## SUMMARY OF THE DIVISIBILITY RULES

An integer number  $n$  is divisible by

2. if the *last digit of  $n$  is divisible by 2*, e.g. is equal to 0, 2, 4, 6, or 8;
3. if the *sum of digits of  $n$  is divisible by 3*;
4. if the *last two digits of  $n$  is divisible by 4*.
5. if the *last digit of  $n$  is divisible by 5*, e.g. is equal to 0 or 5;
6. if  $n$  is divisible by both 2 and 3;
7. if *subtracting twice the last digit of  $n$  from the remaining digits* gives a multiple of 7;
8. if the *last three digits of  $n$  is divisible by 8*;
9. if the *sum of digits of  $n$  is divisible by 9*;
10. if  $n$  is divisible by both 2 and 5;
11. if the *difference of the alternating sum of digits of  $n$*  is a multiple of 11;
12. if  $n$  is divisible by both 3 and 4.

You might notice the strange divisibility rule for 7. The proof of this rule is not straightforward as for all other. Let's write  $n = a \cdot 10 + b$ , where  $b$  is the last digit and  $a$  is the number build with remaining digits. Then suppose  $n$  is divisible by 7

$$a \cdot 10 + b \equiv 0 \pmod{7}, \text{ multiply all by } 2$$

$$a \cdot 20 + b \cdot 2 \equiv 0 \pmod{7}, \text{ subtract } 21a$$

$$a \cdot (-1) + b \cdot 2 \equiv 0 \pmod{7}, \text{ multiply by } -1$$

$$a - b \cdot 2 \equiv 0 \pmod{7}.$$

Doing the same operations in reverse order, we can get the divisibility rule for 7. This is possible because 2 and  $-1$  are *coprime* with 7.

The rule may be repeated several times to get a number small enough for the direct check.

Similar divisibility rules may be obtained for other primes

13. if *adding 4 times the last digit of  $n$  to the remaining digits* gives a multiple of 13;
17. if *subtracting 5 times the last digit of  $n$  from the remaining digits* gives a multiple of 17.

But these rules aren't simpler than divide by the number.

## EXERCISES

1. What is the largest integer less than 100 which is congruent to  $3 \pmod{5}$ ?
2. How many integers are there between 50 and 250 inclusive which are congruent to  $1 \pmod{7}$ ?
3. Find the value of the digit represented by  $A$  in the 5-digit number  $12A3B$  if that number is divisible by both 4 and 9 and  $A$  does not equal  $B$ .
4. In how many ways can a debt of \$69 be paid exactly using only \$5 and \$2 bills?
5. When  $n$  is divided by 5 the remainder is 1. What is the remainder when  $3n$  is divided by 5?
6. The 4-digit number  $4\_89$  is divisible by 11. What digit goes in the blank?
7. If the 5-digit number  $3367\_$  is divisible by 15, what digit must go in the blank?

## CHALLENGE PROBLEMS

1. The Fibonacci sequence is the sequence 1, 1, 2, 3, 5, 8, 13, ..., where every term after the second is equal to the sum of the two preceding terms. Using the addition theorem for modular arithmetic, what is the remainder when the 100<sup>th</sup> term of the Fibonacci sequence is divided by 8? (Hint: *look for a repeating pattern*).
2. Prove that the square of any integer is congruent to either 0, 1, or  $4 \pmod{8}$ .