Probability Theory
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List 3

Topic: Kolmogorov's 0-1 law **Period: 2016.1**

Throughout this list $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set) and I is an index set.

1. Let $X_1, X_2, \ldots$ be independent real random variables with $X_n \sim \mathcal{U}_{\{1,\ldots,n\}}$. Compute $\mathbf{P}[X_n = 5 \quad \text{i.o.}]$.

2. Let $X_1, X_2, \ldots$ be i.i.d real random variables with $X_1 \sim \text{Rad}_{1/2}$, i.e., $\mathbf{P}[X_1 = -1] = \mathbf{P}[X_1 = 1] = 1/2$, and let $S_n = X_1 + \ldots + X_n$ for any $n \in \mathbb{N}$. Show that $\mathbf{P}[S^* = \infty] = 1$.

3. Let $A_1, A_2, \ldots$ be events such that
   (a) $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are independents whenever $i_{j+1} \geq i_j + 2$ for $j = 1, \ldots, k-1$, and
   (b) $\sum_{n=1}^{+\infty} \mathbf{P}[A_n] = +\infty$.

   Prove that $\mathbf{P}[A^*] = 1$.

4. Consider infinite, independent, fair coin tossing, and let H_n be the event that the nth coin is heads. Determine the following probabilities.
   (a) $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+9} \quad \text{i.o.}]$.
   (b) $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{2^n} \quad \text{i.o.}]$.
   (c) $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor 2\log_2 n \rfloor} \quad \text{i.o.}]$ must equal 0 or 1.
   (d) Prove that $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+\lfloor \log_2 n \rfloor} \quad \text{i.o.}]$.

5. Let $A_1, A_2, \ldots$ be independent events, let $x \in \mathbb{R}$, and let

$$S_x = \left[\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{A_i} \leq x\right].$$

   Prove that $\mathbf{P}[S_x]$ must equal 0 or 1.

6. Let $A_1, A_2, \ldots$ be independent events. Let Y be a real random variable which is measurable with respect to $\sigma(A_n, A_{n+1} \ldots,)$ for each $n \in \mathbb{N}$. Prove that Y is deterministic.