School of Mathematics
Faculty of Science
National University of Engineering

Final Exam

Subject: Probability Theory **Period: 2016.1**

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1. [4 pts.] Prove that if $X \in \mathcal{L}^1(\mathbf{P})$ and has density f, then
$$\mathbf{E}[X] = \int x f(x) \lambda(dx).$$

2. [4 pts.] Let $\theta > 0$ and let $X \sim \exp_\theta$. Show that $\mathbf{E}[X] = \theta^{-1}$ and $\mathbf{Var}[X] = \theta^{-2}$.

3. [4 pts.] Let $\Omega =]0,1[$, $\mathcal{A}$ be the class of Borel sets and $\mathbf{P}$ be the Lebesgue measure. If $X_n(\omega) = \sin(2\pi n \omega)$, $n = 1, 2, \ldots$, then prove that $X_1, X_2, \ldots$ are uncorrelated but not independent.

4. [4 pts.] Let $X_1, X_2, \ldots$ be i.i.d. real random variables with
$$\frac{1}{n}(X_1 + \ldots + X_n) \xrightarrow{\text{a.s.}} Y.$$
Show that $X_1 \in \mathcal{L}^1(\mathbf{P})$ and $Y = \mathbf{E}[X_1]$ a.s. (Hint: first show that
$$\mathbf{P}[|X_n| > n \text{ for infinitely many } n] = 0 \quad \Leftrightarrow \quad X_1 \in \mathcal{L}^1(\mathbf{P}).)$$

5. [4 pts.] Let $X_1, X_2, \ldots \geq 0$ be i.i.d. By virtue of Borel–Cantelli lemma, show that
$$\limsup_{n\to+\infty} \frac{1}{n} X_n = \begin{cases} 0 \text{ a.s.} & \text{if } \mathbf{E}[X_1] < +\infty, \\ +\infty \text{ a.s.} & \text{if } \mathbf{E}[X_1] = +\infty. \end{cases}$$

Monday, July 18