# 15 Strong Law of Large Numbers

We show Etemadi's version of the strong law of large numbers for identically distributed, pairwise independent random variables. There is a zoo of strong laws of large numbers, each of which varies in the exact assumptions it makes on the underlying sequence of random variables. For example, the assumption that the random variables be identically distributed can be waived if other assumptions are introduced such as bounded variances. We do not strive for completeness but show only a few of the statements.

In order to illustrate the method of the proof of Etemadi's theorem, we first present (and prove) a strong law of large numbers under stronger assumptions.

**Theorem 15.1.** Let $X_1, X_2, \ldots \in \mathcal{L}^2(\mathbf{P})$ be i.i.d. real random variables. Then $(X_n)_{n\in\mathbb{N}}$ fulfills the strong law of large numbers.

*Proof.* *Exercise.* □

We can weaken the condition in Theorem 15.1 by requiring integrability only instead of square integrability of the random variables.

**Theorem 15.2** (Etemadi's strong law of large numbers (1981))**.** Let $X_1, X_2, \ldots \in \mathcal{L}^1(\mathbf{P})$ be i.i.d. real random variables. Then $(X_n)_{n\in\mathbb{N}}$ fulfills the strong law of large numbers.

**Lemma 15.3.** For $n \in \mathbb{N}$, define $Y_n := X_n \mathbf{1}_{|X_n|\leq n}$ and $T_n = Y_1 + \ldots + Y_n$. The sequence $(X_n)_{n\in\mathbb{N}}$ fulfills the strong law of large numbers if $T_n/n \xrightarrow{\text{a.s.}} \mu := \mathbf{E}[X_1]$.

*Proof.* *Exercise.* □

**Lemma 15.4.** $2x \sum_{n>x} n^{-2} \leq 4$ for all $x \geq 0$.

*Proof.* *Exercise.* □

**Lemma 15.5.** $\sum_{n=1}^{+\infty} \frac{\mathbf{E}[Y_n^2]}{n^2} \leq \mathbf{E}[|X_1|]$.

*Proof.* *Exercise.* □

(*Proof of Theorem 15.2.*) *Exercise.* □

**Example 15.6** (Monte Carlo Integration)**.** □

**Definition 15.7** (Empirical distribution function)**.** Let $X_1, X_2, \ldots$ be real random variables. The map $F_n : \mathbb{R} \to [0,1]$, $x \mapsto \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{]-\infty,x]}(X_i)$ is called the **empirical distribution function** of $X_1, X_2, \ldots$. □

**Theorem 15.8** (Glivenko–Cantelli)**.** Let $X_1, X_2, \ldots$ be i.i.d. real random variables with distribution function $F$, and let $F_n$, $n \in \mathbb{N}$, be the empirical distribution functions. Then

$$\limsup_{n\to+\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0 \quad \text{a.s.}$$

*Proof.* *Exercise.* $\square$

**Example 15.9** (Shannon's theorem). $\square$

**Definition 15.10** (Entropy). Let $p = (p_e)_{e \in E}$ be a probability distribution on the countable set $E$. For $b > 0$, define

$$H_b(p) := -\sum_{e \in E} p_e \log_b(p_e)$$

with the convention $0 \log_b(0) := 0$. We call $H_b(p) := H_e(p)$, where $e$ is the Euler's number, the **entropy** and $H_2(p)$ the **binary entropy** of $p$. $\square$

**Lemma 15.11** (Entropy inequality). Let $b$ and $p$ as above. Further, let $q$ be a sub-probability distribution, i.e., $q_e \geq 0$ for all $e \in E$ and $\sum_{e \in E} q_e \leq 1$. Then

$$H_b(p) \leq -\sum_{e \in E} p_e \log_b(q_e)$$

with equality if and only if $H_b(p) = +\infty$ or $q = p$.

*Proof.* *Exercise.* $\square$

**Theorem 15.12** (Source coding theorem). Let $p = (p_e)_{e \in E}$ be a probability distribution on the finite alphabet $E$. For any binary prefix code $C = (c(e), e \in E)$, we have $L_p(C) \geq H_2(p)$. Furthermore, there is a binary prefix code $C$ with $L_p(C) \leq H_2(p) + 1$.

*Proof.* *Exercise.* $\square$