Probability Theory
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List 1 of Exercises

Topic: Independence of events, classes of events and random variables **Ciclo: 2016.1**

Throughout this list $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set) and I is an index set.

1. Let $A_1, A_2, \ldots, A_n$ be independent events. Show that the probability that none of the $A_1, A_2 \ldots, A_n$ occur is less that or equal to $\exp(-\sum_{i=1}^{n} \mathbf{P}[A_i])$.

2. Toss a coin with $\mathbf{P}[\text{Heads}] = p$ repeatedly. Let A_k be the event that k or more consecutive heads occurs amongst the tosses numbered $2^k, 2^k+1, \ldots, 2^{k+1}-1$. Show that
$$\mathbf{P}[A_k \quad \text{i.o.}] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2}. \end{cases}$$

3. Let $(\mathcal{E}_i)_{i \in I}$ be a collection of σ-algebras over Ω with $\mathcal{E}_i \subset \mathcal{A}$ for every i. Let $\mathcal{E}$ be a σ-algebra over Ω with $\mathcal{E} \subset \mathcal{A}$. Prove that if for every $J \subset I$ finite $\mathcal{E}$ and $\sigma(\mathcal{E}_i\,;i \in J)$ are independent, then $\mathcal{E}$ and $\sigma(\mathcal{E}_i\,;i \in I)$ are independent.

4. Let $X_1, X_2, \ldots$ be a sequence of random variables. Prove that if for every $k \geq 1$ $\sigma(X_{k+1})$ and $\sigma(X_1, X_2, \ldots, X_k)$ are independent, then $(X_n)_{n \in \mathbb{N}}$ is independent.

5. Let F be the distribution function of the (real) random variable X. Prove that X has symmetric distribution iff $F(a-) = 1 - F(a)$ for all $a \in \mathbb{R}$.

6. Let X, Y be (real) random variables i.i.d. Prove that $X - Y$ has symmetric distribution.

7. Let $X_1, X_2, \ldots$ be a sequence of (real) random variables i.i.d. with $X_n \sim \exp(1)$ for every $n \in \mathbb{N}$. Show that
$$\mathbf{P}\left[\frac{X_n}{\ln(n)} > 1 \quad \text{i.o.}\right] = 1$$
and
$$\mathbf{P}\left[\frac{X_n}{\ln(n)} > 2 \quad \text{i.o.}\right] = 0.$$

8. Let X, Y be independent (real) random variables, both with distribution $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Show that X, Y and $Z := XY$ are not independent, but are 2-2 independent.

9. Let X be a (real) random variable. We say that X is **deterministic** if there exists $c \in \mathbb{R}$ such that $\mathbf{P}[X = c] = 1$. Prove that X and X are independent iff X is deterministic.

10. Let X, Y be independent (real) random variables. Prove that if for some $c \in \mathbb{R}$, $\mathbf{P}[X + Y = c] = 1$, then X and Y are constant random variables.

11. Let $X_0, X_1, \ldots$ be a sequence of independent (real) random variables with $\mathbf{P}[X_n = 1] = \mathbf{P}[X_n = -1] = \frac{1}{2}$ for $n = 0, 1, \ldots$ Let $Y_n = \prod_{i=0}^{n} X_i$ for $n = 1, 2, \ldots$ Show that $(Y_n)_{n \in \mathbb{N}}$ is independent.