

Week 10: The Central limit theorem (CLT)

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The space $\mathcal{L}^p(\mu)$

Definition

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space $(\Omega \text{ is a nonempty set})$ and $\overline{\mathbb{R}} := [-\infty, +\infty]$. For measurable $f: \Omega \to \overline{\mathbb{R}}$, define

$$|f|_{
ho}:=egin{cases} \left(\int|f|^{
ho}d\mu
ight)^{1/
ho} & ext{if }
ho\in[1,+\infty[] \ \inf\{K\geq0\;;\;\mu(|f|>K)=0\} \end{cases} & ext{if }
ho=+\infty.$$

Further, for any $p \in [1, +\infty]$, define the vector space

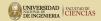
$$\mathcal{L}^p(\mu) := \{ f : \Omega \to \overline{\mathbb{R}} : f \text{ is measurable and } |f|_p < +\infty \}.$$



The space $\mathcal{L}^p(\mu)$

Property

 $(\mathcal{L}^p(\mu), ||_p)$ is a seminormed vector space.



Moments

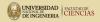
Definition

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space $(\Omega \text{ is a nonempty set})$ and let $X : \Omega \to \mathbb{R}$ be a random variable.

1. If $X \in \mathcal{L}^1(\mathbf{P})$, then X is called **integrable** and we call

$$\mathbf{E}[X] := \int X d\mathbf{P}$$

the expectation or mean of X. If $\mathbf{E}[X] = \mathbf{0}$, then X is called centered.



Moments

Definition

2. If $n \in \mathbb{N}$ and $X \in \mathcal{L}^n(\mathbf{P})$, then the quantities

$$m_k := \mathbf{E}[X^k]$$
, for any $k = 1, \ldots, n$,

are called the kth moments of X.

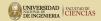
3. If $X \in \mathcal{L}^2(\mathbf{P})$, then X is called square integrable and

$$\operatorname{Var}[X] := \operatorname{E}[X^2] - \operatorname{E}[X]^2$$

is the **variance** of X. The number

$$\sigma := \sqrt{\operatorname{Var}[X]}$$

is called the **standard deviation** of X.



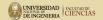
Polish spaces

Definition

A topological space (\mathcal{E}, τ) is called a **Polish space** if it is separable and if there exists a complete metric that induces the topology τ .

Example

Examples of Polish spaces are the Euclidean spaces \mathbb{R}^n and the space C([0,1]) of continuous functions $[0,1] \to \mathbb{R}$, equipped with the supremum norm. In practice, all spaces that are of importance in probability theory are Polish spaces.



Spaces of measures on E

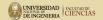
Definition

In the following, let (E, τ) be a Polish space with Borel σ -algebra $\mathcal{E} = \mathcal{B}(E) := \sigma(\tau)$. We introduce the following spaces of measures on E:

$$\mathcal{M}_f(E) := \{ \text{finite measures on } (E, \mathcal{E}) \},$$

$$\mathcal{M}_{\leq 1}(E) := \{ \mu \in \mathcal{M}_f(E) \; ; \; \mu(E) \leq 1 \}.$$

The elements of $\mathcal{M}_{\leq 1}(E)$ are called **sub-probability measures** on E.



Spaces of continuous functions

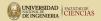
Definition

We agree on the following notation for spaces of continuous functions:

$$C(E) := \{ f : E \to \mathbb{R} \text{ is continuous} \},$$

$$C_b(E) := \{ f \in C(E) \text{ is bounded} \}.$$

Unless otherwise stated, the vector spaces C(E) and $C_b(E)$ are equipped with the supremum norm.



Weak convergence

Definition

Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_f(E)$. We say that $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to μ , and we write $\mu = w - \lim_{n \to +\infty} \mu_n$, if for all $f \in C_b(E)$:

$$\int f \, d\mu = \lim_{n \to +\infty} \int f \, d\mu_n.$$



Portmanteau theorem

Theorem

Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_{\leq 1}(E)$. The following are equivalent:

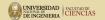
1.
$$\mu = \mathbf{w} - \lim_{n \to +\infty} \mu_n$$
.

2.
$$\mu(A) = \lim_{n \to +\infty} \mu_n(A)$$
 for all measurable A with $\mu(\partial A) = 0$.



¿What is normal distribution?

• A simple and intuitive look at the normal distribution.



Central limit theorem (CLT)

- The New York Times.
- Biostatistics.



Central limit theorem (CLT)

Theorem (CLT)

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be i.i.d. random variables with $\mu := \mathbf{E}[X_1] \in \mathbb{R}$ and $\sigma^2 := \mathbf{Var}[X_1] \in]0, +\infty[$. For $n \in \mathbb{N}$, let

$$S_n := \frac{1}{\sqrt{\sigma^2 n}} \sum_{i=1}^n (X_i - \mu).$$

Then $\mathcal{N}_{0,1} = \mathbf{w} - \lim_{n \to +\infty} \mathbf{P}_{S_n}$.