UNIVERSIDAD NACIONAL DE INGENIERIA | FACULTAD DE CIENCIAS

Week 10: The Central limit theorem (CLT)

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The space $\mathcal{L}^p(\mu)$

Definition

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space (Ω is a nonempty set) and $\overline{\mathbb{R}} := [-\infty, +\infty]$. For measurable $f : \Omega \to \overline{\mathbb{R}}$, define

$$|f|_p := \begin{cases} \left(\int |f|^p d\mu\right)^{1/p} & \text{if } p \in [1, +\infty[\\ \inf\{K \geq 0 \; ; \; \mu(|f| > K) = 0\} & \text{if } p = +\infty. \end{cases}$$

Further, for any $p \in [1, +\infty]$, define the vector space

$$\mathcal{L}^p(\mu) := \{f : \Omega \to \overline{\mathbb{R}} \; ; \; f \text{ is measurable and } |f|_p < +\infty\}.$$

$\square$

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The space $\mathcal{L}^p(\mu)$

Property

$(\mathcal{L}^p(\mu), |\ |_p)$ is a seminormed vector space. $\square$

Moments

Definition

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space (Ω is a nonempty set) and let $X : \Omega \to \mathbb{R}$ be a random variable.

1. If $X \in \mathcal{L}^1(\mathbf{P})$, then X is called **integrable** and we call

$$\mathbf{E}[X] := \int X d\mathbf{P}$$

   the **expectation** or **mean** of X. If $\mathbf{E}[X] = 0$, then X is called **centered**.

Moments

Definition

2. If $n \in \mathbb{N}$ and $X \in \mathcal{L}^n(\mathbf{P})$, then the quantities

$$m_k := \mathbf{E}[X^k], \text{ for any } k = 1, \dots, n,$$

are called the kth **moments** of X.

3. If $X \in \mathcal{L}^2(\mathbf{P})$, then X is called **square integrable** and

$$\mathbf{Var}[X] := \mathbf{E}[X^2] - \mathbf{E}[X]^2$$

is the **variance** of X. The number

$$\sigma := \sqrt{\mathbf{Var}[X]}$$

is called the **standard deviation** of X.

$\square$

Polish spaces

Definition

A topological space (E, τ) is called a **Polish space** if it is separable and if there exists a complete metric that induces the topology τ. $\square$

Example

Examples of Polish spaces are the Euclidean spaces $\mathbb{R}^n$ and the space $C([0,1])$ of continuous functions $[0,1] \to \mathbb{R}$, equipped with the supremum norm. In practice, all spaces that are of importance in probability theory are Polish spaces. $\square$

Spaces of measures on E

Definition

In the following, let (E, τ) be a Polish space with Borel σ-algebra $\mathcal{E} = \mathcal{B}(E) := \sigma(\tau)$. We introduce the following spaces of measures on E:

$$\mathcal{M}_f(E) := \{\text{finite measures on } (E, \mathcal{E})\},$$
$$\mathcal{M}_{\leq 1}(E) := \{\mu \in \mathcal{M}_f(E)\ ;\ \mu(E) \leq 1\}.$$

The elements of $\mathcal{M}_{\leq 1}(E)$ are called **sub-probability measures** on E. $\square$

Spaces of continuous functions

Definition

We agree on the following notation for spaces of continuous functions:

$$C(E) := \{f : E \to \mathbb{R} \text{ is continuous}\},$$
$$C_b(E) := \{f \in C(E) \text{ is bounded}\}.$$

Unless otherwise stated, the vector spaces $C(E)$ and $C_b(E)$ are equipped with the supremum norm. $\square$

Weak convergence

Definition

Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_f(E)$. We say that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ, and we write $\mu = w - \lim_{n\to+\infty} \mu_n$, if for all $f \in C_b(E)$:

$$\int f\, d\mu = \lim_{n\to+\infty} \int f\, d\mu_n.$$

. $\square$

Portmanteau theorem

Theorem

Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_{\leq 1}(E)$. The following are equivalent:

1. $\mu = w - \lim_{n\to+\infty} \mu_n$.
2. $\mu(A) = \lim_{n\to+\infty} \mu_n(A)$ for all measurable A with $\mu(\partial A) = 0$.

□

¿What is normal distribution?

- A simple and intuitive look at the normal distribution.

Central limit theorem (CLT)

- The New York Times.
- Biostatistics.

Central limit theorem (CLT)

Theorem (CLT)

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be i.i.d. random variables with $\mu := \mathbf{E}[X_1] \in \mathbb{R}$ and $\sigma^2 := \mathbf{Var}[X_1] \in]0, +\infty[$. For $n \in \mathbb{N}$, let

$$S_n := \frac{1}{\sqrt{\sigma^2 n}} \sum_{i=1}^{n} (X_i - \mu).$$

Then $\mathcal{N}_{0,1} = w - \lim_{n \to +\infty} \mathbf{P}_{S_n}$. $\square$