Probability Theory
School of Mathematics
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List 2

Topic: Independent random variables **Period: 2016.1**

Throughout this list $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set) and I is an index set.

1. Let X and Y be independent real random variables with $X \sim \exp_\alpha$ and $Y \sim \exp_\beta$ for certain $\alpha, \beta > 0$. Show that
$$\mathbf{P}[X < Y] = \frac{\alpha}{\alpha + \beta}.$$

2. (Box–Muller method) Let U and V be independent real random variables that are uniformly distributed on $[0, 1]$. Define
$$X := \sqrt{2\log(U)}\cos(2\pi V) \quad \text{and} \quad Y := \sqrt{2\log(U)}\sin(2\pi V).$$
Show that X and Y are independent with distribution $\mathcal{N}_{0,1}$.

3. Let X, Y be independent real random variables taking values in $\mathbb{N}$ with
$$\mathbf{P}[X = k] = \mathbf{P}[Y = k] = \frac{1}{2^k},$$
for every $k = 1, 2, \ldots$. Find the following probabilities:
   (a) $\mathbf{P}[\min(X, Y) \leq k]$.
   (b) $\mathbf{P}[X = Y]$.
   (c) $\mathbf{P}[X < Y]$.
   (d) $\mathbf{P}[X \text{ divides } Y]$.
   (e) $\mathbf{P}[X \geq kY]$, for every $k = 1, 2, \ldots$.

4. Let X be a real random variable with $X \sim \mathcal{U}_{[-1,1]}$. Find the density of $Y = X^k$ for every $k = 1, 2, \ldots$

5. Let X be a real random variable with $X \sim \mathcal{U}_{]-\pi,\pi[}$. Find the density of $Y = a\tan(X)$ with $a > 0$.

6. Let X be a real random variable and let f be a density function of X. Show that X has symmetric distribution iff f is an even function.

7. Let F be a distribution function that is continuous and is such that the inverse function F^{-1} exists. Let U be real random variable with $U \sim \mathcal{U}_{]0,1[}$. Show that $X = F^{-1}(U)$ has distribution function F.

8. Let F be a continuous distribution function and let U be real random variable with $U \sim \mathcal{U}_{]0,1[}$. Define $G(u) = \inf\{x \, ; \, F(x) \geq u\}$. Prove that $G(U)$ has distribution function F.

9. Let (X, Y) be a random variable uniformly ditributed on the unit ball, i.e.,

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $R = \sqrt{X^2 + Y^2}$.

10. Let (X, Y) be a random variable with joint density f. Find the density of $Z = X + Y$.

11. Let X and Y be independent real random variables and let g and h be injective and differentiable real functions on $\mathbb{R}$. Find a formula for the joint density of (Z, W) where $Z = g(X)$ and $W = h(Y)$.

12. Let X and Y be independent real random variables with distribution $\mathcal{N}_{0,\sigma^2}$. Let

$$Z = \sqrt{X^2 + Y^2} \quad \text{and} \quad W = \arctan\left(\frac{X}{Y}\right), \; -\pi/2 < W \leq \pi/2.$$

Show that

(a) Z has Rayleigh distribution.
(b) W is uniformly distributed on $]-\pi/2, \pi/2[$.
(c) Z and W are independent.

13. Let $X_1, \ldots, X_n$ be real random variables. Define

$$\begin{aligned} Y_1 &= \text{smallest of } X_1, \ldots, X_n \\ Y_2 &= \text{second smallest of } X_1, \ldots, X_n \\ &\vdots \\ Y_n &= \text{largest of } X_1, \ldots, X_n \end{aligned}$$

Then $Y_1, \ldots, Y_n$ are also real random variables and $Y_1 \leq \ldots \leq Y_n$. They are called the order statistics of $(X_1, \ldots, X_n)$ and are usually denoted

$$Y_k = X_{(k)}.$$

If $X_1, \ldots, X_n$ are i.i.d. with joint density f, then show that the joint density of the order statistics is given by

$$f_{(X_{(1)},\ldots,X_{(n)})}(y_1, \ldots, y_n) = \begin{cases} n! \prod_{i=1}^n f(y_i) & \text{if } y_1 < y_2 < \ldots < y_n \\ 0 & \text{otherwise.} \end{cases}$$

14. Let $X_1, \ldots, X_n$ be real random variables i.i.d. with distribution $\mathcal{U}_{]0,a[}$. Prove that

$$f_{(X_{(1)},\ldots,X_{(n)})}(y_1, \ldots, y_n) = \begin{cases} \frac{n!}{a^n} & \text{if } y_1 < y_2 < \ldots < y_n \\ 0 & \text{otherwise.} \end{cases}$$