Probability Theory
School of Mathematics
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Test 2

Topic: Independent random variables **Period: 2016.1**

Throughout this test $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set) and $X, Y, U, V : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are random variables.

1. [5 pts.] Let X and Y be independent with $X \sim \exp_\alpha$ and $Y \sim \exp_\beta$ for certain $\alpha, \beta > 0$. Show that
$$\mathbf{P}[X < Y] = \frac{\alpha}{\alpha + \beta}.$$

2. [5 pts.] (Box–Muller method) Let U and V be i.i.d. with $U \sim \mathcal{N}_{[0,1]}$. Define
$$X := \sqrt{2\log(U)}\cos(2\pi V) \quad \text{and} \quad Y := \sqrt{2\log(U)}\sin(2\pi V).$$
Show that X and Y are independent with distribution $\mathcal{N}_{0,1}$.

3. [5 pts.] Let (X, Y) be uniformly ditributed on the unit ball, i.e.,
$$f_{(X,Y)}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
Find the distribution of $R = \sqrt{X^2 + Y^2}$.

4. [5 pts.] Let X and Y be i.d.d. with $X \sim \mathcal{N}_{0,\sigma^2}$. Let
$$Z = \sqrt{X^2 + Y^2} \quad \text{and} \quad W = \arctan\left(\frac{X}{Y}\right).$$
Show that
   (a) Z has Rayleigh distribution.
   (b) W is uniformly distributed on $]-\pi/2, \pi/2[$.
   (c) Z and W are independent.

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