School of Mathematics
Faculty of Science
National University of Engineering

Substitute Exam

Subject: Probability Theory **Period: 2016.1**

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1. [6 pts.] Let X, Y be independent (real) random variables. Prove that if for some $c \in \mathbb{R}$, $\mathbf{P}[X + Y = c] = 1$, then X and Y are constant random variables.

1. [6 pts.] Let $X_1, \ldots, X_n$ be real random variables. Define

$$\begin{aligned} Y_1 &= \text{smallest of } X_1, \ldots, X_n \\ Y_2 &= \text{second smallest of } X_1, \ldots, X_n \\ &\vdots \\ Y_n &= \text{largest of } X_1, \ldots, X_n \end{aligned}$$

Then $Y_1, \ldots, Y_n$ are also real random variables and $Y_1 \leq \ldots \leq Y_n$. They are called the order statistics of $(X_1, \ldots, X_n)$ and are usually denoted

$$Y_k = X_{(k)}.$$

If $X_1, \ldots, X_n$ are i.i.d. with joint density f, then show that the joint density of the order statistics is given by

$$f_{(X_{(1)}, \ldots, X_{(n)})}(y_1, \ldots, y_n) = \begin{cases} n! \prod_{i=1}^{n} f(y_i) & \text{if } y_1 < y_2 < \ldots < y_n \\ 0 & \text{otherwise.} \end{cases}$$

3. [7 pts.] Let $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in [0, 1]$. Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \text{Ber}_{p_i}$ for any $i = 1, \ldots, n$. Define $S_n := X_1 + \ldots + X_n$ and $m := \mathbf{E}[S_n]$. Show that for any $\delta > 0$:

$$\mathbf{P}[S_n \geq (1 + \delta)m] \leq \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}} \right)^m$$

and

$$\mathbf{P}[S_n \leq (1 - \delta)m] \leq e^{-\frac{\delta^2 m}{2}}.$$

Saturday, July 9