Probability Theory
School of Mathematics
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Test 3

Topic: Kolmogorov's 0-1 law **Period: 2016.1**

Throughout this test $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set); $Y, X_1, X_2, \ldots : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be real random variables and $A_1, A_2, \ldots \in \mathcal{A}$.

1. [5 pts.] Let $X_1, X_2, \ldots$ be i.i.d with $X_1 \sim_{1/2}$, i.e., $\mathbf{P}[X_1 = -1] = \mathbf{P}[X_1 = 1] = 1/2$, and let $S_n = X_1 + \ldots + X_n$ for any $n \in \mathbb{N}$. Show that $\mathbf{P}[S^* = \infty] = 1$.

2. [5 pts.] Let $A_1, A_2, \ldots$ be independent. If Y is measurable with respect to $\sigma(A_n, A_{n+1} \ldots,)$ for each $n \in \mathbb{N}$, prove that Y is deterministic.

3. [5 pts.] Consider infinite, independent, fair coin tossing, and let H_n be the event that the nth coin is heads. Determine the following probabilities.
   (a) $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+9} \quad \text{i.o.}]$.
   (b) $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{2^n} \quad \text{i.o.}]$.

4. [5 pts.] Let $A_1, A_2, \ldots$ be independent, let $x \in \mathbb{R}$, and let
$$S_x = \left[\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{A_i} \leq x \right].$$
   Prove that $\mathbf{P}[S_x]$ must equal 0 or 1.

Wednesday, May 11