Probability Theory
School of Mathematics
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Test 1

Topic: Independence of events, classes of events and random variables Period: 2016.1

Throughout this list $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set) and I is an index set.

1. [4 pts.] Let $(\mathcal{E}_i)_{i\in I}$ be a collection of σ-algebras over Ω with $\mathcal{E}_i \subset \mathcal{A}$ for every i. Let $\mathcal{E}$ be a σ-algebra over Ω with $\mathcal{E} \subset \mathcal{A}$. Prove that if for every $J \subset I$ finite $\mathcal{E}$ and $\sigma(\mathcal{E}_i\,; i \in J)$ are independent, then $\mathcal{E}$ and $\sigma(\mathcal{E}_i\,; i \in I)$ are independent.

2. [4 pts.] Let $X_1, X_2, \dots$ be a sequence of random variables. Prove that if for every $k \geq 1$ $\sigma(X_{k+1})$ and $\sigma(X_1, X_2, \dots, X_k)$ are independent, then $(X_n)_{n\in\mathbb{N}}$ is independent.

3. [4 pts.] Let $X_1, X_2, \dots$ be a sequence of (real) random variables i.i.d. with $X_n \sim \exp(1)$ for every $n \in \mathbb{N}$. Show that
$$\mathbf{P}\left[X_n > \ln(n) \quad \text{i.o.}\right] = 1$$
and
$$\mathbf{P}\left[X_n > 2\ln(n) \quad \text{i.o.}\right] = 0.$$

4. [4 pts.] Let X be a (real) random variable. We say that X is **deterministic** if there exists $c \in \mathbb{R}$ such that $\mathbf{P}[X = c] = 1$. Prove that X and X are independent iff X is deterministic.

5. [4 pts.] Toss a coin with $\mathbf{P}[\text{Heads}] = p$ repeatedly. Let A_k be the event that k or more consecutive heads occurs amongst the tosses numbered $2^k, 2^k + 1, \dots, 2^{k+1} - 1$. Show that
$$\mathbf{P}[A_k \quad \text{i.o.}] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2}. \end{cases}$$

April 13, 2016