Midterm Exam

Subject: Probability Theory **Period: 2016.2**

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1. Let X and Y be independent real random variables with $X \sim \exp(\alpha)$ and $Y \sim \exp(\beta)$. Show that
$$\mathbf{P}[X < Y] = \frac{\alpha}{\alpha + \beta}.$$

2. Let $A_1, A_2, \ldots$ be independent events. Let X be a real random variable which is $\sigma(A_n, A_{n+1} \ldots)$-measurable for each $n \in \mathbb{N}$. Prove that X is deterministic.

3. Let $A_1, A_2, \ldots$ be independent events, let $x \in \mathbb{R}$, and let
$$S_x = \left(\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{A_i} \leq x \right).$$
Prove that $\mathbf{P}(S_x)$ must equal 0 or 1.

4. Let X_1, X_2, X_3 be independent real random variables. Prove that X_1, X_2, X_3 are deterministic if $X_1 + X_2 + X_3$ is.

5. (Box–Muller method) Let U and V be independent and uniformly distributed on $[0, 1]$. Define
$$X := \sqrt{-2\log(U)} \cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2\log(U)} \sin(2\pi V).$$
Show that X and Y are independent with distribution $\mathcal{N}_{0,1}$.

6. Toss a coin with $\mathbf{P}[\text{Heads}] = p$ repeatedly. Let A_k be the event that k or more consecutive heads occurs amongst the tosses numbered $2^k, 2^k + 1, \ldots, 2^{k+1} - 1$. Show that
$$\mathbf{P}[A_k \quad \text{i.o.}] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2}. \end{cases}$$

7. Let $X_1, \ldots, X_n$ be independent and symmetric real random variables and let $S = X_1 + \cdots + X_n$. Show that
    1. for all $x \in \mathbb{R}$, $\mathbf{P}[S \geq x] = \mathbf{P}[S \leq -x]$,
    2. is the conclusion necessarily true without the assumption of independence?

Thursday, October 13