# 14 Weak Law of Large Numbers

**Property 14.1.** Let $X$ be a real random variable and let $f : [0, +\infty[ \to [0, +\infty[$ be monotene increasing. Then for any $\epsilon > 0$ with $f(\epsilon) > 0$, the Markov's inequality holds,

$$\mathbf{P}[\,|X| \geq \epsilon\,] \leq \frac{\mathbf{E}[f(|X|)]}{f(\epsilon)}.$$

In the special case $f(x) = x^2$, we get

$$\mathbf{P}[\,|X| \geq \epsilon\,] \leq \frac{\mathbf{E}[X^2]}{\epsilon^2}.$$

In particular, if $X \in \mathcal{L}^2(\mathbf{P})$, the Chebyshev's inequality holds:

$$\mathbf{P}[\,|X - \mathbf{E}[X]| \geq \epsilon\,] \leq \frac{\mathbf{Var}[X]}{\epsilon^2}.$$

*Proof. Exercise.* □

**Definition 14.2.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables in $\mathcal{L}^1(\mathbf{P})$ and let $\tilde{S}_n := \sum_{i=1}^n (X_i - \mathbf{E}[X_i])$. We say that $(X_n)_{n\in\mathbb{N}}$ fulfills the weak law of large numbers if

$$\lim_{n\to+\infty} \mathbf{P}\left[\left|\frac{1}{n}\tilde{S}_n\right| > \epsilon\right] = 0 \qquad \text{for any } \epsilon > 0.$$

And we say that $(X_n)_{n\in\mathbb{N}}$ fulfills the strong law of large numbers if

$$\mathbf{P}\left[\limsup_{n\to+\infty}\left|\frac{1}{n}\tilde{S}_n\right| = 0\right] = 1.$$

□

**Remark 14.3.** The strong law of large numbers implies the weak law.

*Proof. Exercise.* □

**Theorem 14.4.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of uncorrelated real random variables in $\mathcal{L}^2(\mathbf{P})$ with $V := \sup_{n\in\mathbb{N}} \mathbf{Var}[X_n] < +\infty$. Then $(X_n)_{n\in\mathbb{N}}$ fulfills the weak law of large numbers. More precisely, for any $\epsilon > 0$, we have

$$\mathbf{P}\left[\left|\frac{1}{n}\tilde{S}_n\right| > \epsilon\right] \leq \frac{V}{\epsilon^2 n} \qquad \text{for all } n \in \mathbb{N}.$$

*Proof. Exercise.* □

**Example 14.5.** we present a probabilistic proof of the Weierstraß's approximation theorem

*Proof. Exercise.* □

**Exercise 14.1.** Let $S_n$ be the number of successes in $n$ Bernoulli trials with probability $p$ for success on each trial.

(i) Show, using Chebyshevs inequality, that for any $\epsilon > 0$

$$\mathbf{P}\left[\left|\frac{S_n}{n} - p\right| \geq \epsilon\right] \leq \frac{p(1-p)}{n\epsilon^2}.$$

(ii) Find the maximum possible value for $p(1-p)$ if $0 < p < 1$. Using this result show that for any $\epsilon > 0$

$$\mathbf{P}\left[\left|\frac{S_n}{n} - p\right| \geq \epsilon\right] \leq \frac{1}{4n\epsilon^2}.$$

**Exercise 14.2.** A fair coin is tossed 100 times. The expected number of heads is 50, and the standard deviation for the number of heads is $(100 \cdot 1/2 \cdot 1/2)^{1/2} = 5$. What does Chebyshevs inequality tell you about the probability that the number of heads that turn up deviates from the expected number 50 by three or more standard deviations, i.e., by at least 15?

**Exercise 14.3.** A fair coin is tossed a large number of times. Does the Weak Law of Large Numbers assure us that, if $n$ is large enough, with probability greater than 0.99 the number of heads that turn up will not deviate from $n/2$ by more than 100?

**Exercise 14.4.** Let $X$ be a real random variable with $\mathbf{P}[X \in \mathbb{N}_0] = 1$ and has $\mathbf{E}[X] = \mathbf{Var}[X] = 1$. Show that, for any integer $k$,

$$\mathbf{P}[X \geq k+1] \leq \frac{1}{k^2}.$$

**Exercise 14.5.** We have two coins: one is a fair coin and the other is a coin that produces heads with probability 3/4. One of the two coins is picked at random, and this coin is tossed $n$ times. Let $S_n$ be the number of heads that turns up in these $n$ tosses.

(i) Does the Weak Law of Large Numbers allow us to predict the proportion of heads that will turn up in the long run?

(ii) After we have observed a large number of tosses, can we tell which coin was chosen?

(iii) How many tosses suffice to make us 95 percent sure?

**Exercise 14.6.** Let $X_1, \ldots, X_n$ be independent real random variables and let $S_n$ be their sum. Let $M_n = \mathbf{E}[X_1] + \ldots + \mathbf{E}[X_n]$ and assume that $\mathbf{Var}[X_i] < R$ for all $i \in \mathbb{N}$. Prove that, for any $\epsilon > 0$,

$$\lim_{n \to +\infty} \mathbf{P}\left[\left|\frac{S_n}{n} - \frac{M_n}{n}\right| < \epsilon\right] = 1.$$

**Exercise 14.7.** The Chela beer company runs a fleet of trucks along the 300 kilometers road from Huancayork to Callao, and maintains a garage halfway in between. Each of the trucks is apt to break down at a point $X$ miles from Huancayork, where $X$ is a real random variable uniformly distributed over $[0, 300]$.

(i) Find a lower bound for the probability $\mathbf{P}[|X - 150| \leq 10]$.

(ii) Suppose that in one bad week, 20 trucks break down. Find a lower bound for the probability $\mathbf{P}[|A_{20} - 150|] \leq 10]$, where $A_{20}$ is the average of the distances from Huancayork at the time of breakdown.

**Exercise 14.8** (Bernstein-Chernov bound)**.** Let $n \in \mathbb{N}$ and $p_1, \ldots, p_n \in [0, 1]$. Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \mathrm{Ber}_{p_i}$ for any $i = 1, \ldots, n$. Define $S_n := X_1 + \ldots + X_n$ and $m := \mathbf{E}[S_n]$. Show that for any $\delta > 0$:

$$\mathbf{P}[S_n \geq (1+\delta)m] \leq \left( \frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^m$$

and

$$\mathbf{P}[S_n \leq (1-\delta)m] \leq e^{-\frac{\delta^2 m}{2}}.$$