3 Kolmogorov's 0-1 law

With the Borel–Cantelli lemma, we have seen a first 0–1 law for independent events. We now come to another 0–1 law for independent events and for independent σ-algebras. To this end, we first introduce the notion of the tail σ-algebra.

Definition 3.1 (Tail σ-algebra)**.**

(i) Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be σ-algebras on Ω. Then

$$\mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \ldots) := \bigcap_{J \subset \mathbb{N} \text{ finite}} \sigma(\mathcal{A}_i \ ; \ i \in \mathbb{N} \setminus J) \tag{3.1}$$

is called the **tail σ-algebra of the σ-algebras** $\mathcal{A}_1, \mathcal{A}_2, \ldots$

(ii) If $A_1, A_2, \ldots$ are events of $\mathcal{A}$, then we define the **tail σ-algebra of the events** $A_1, A_2, \ldots$ as the tail σ-algebra of the σ-algebras $\sigma(A_1), \sigma(A_2), \ldots$ and denote it by

$$\mathcal{T}(A_1, A_2, \ldots).$$

(iii) Finally, if $X_1, X_2, \ldots$ are random variables all defined on Ω, then we define the **tail σ-algebra of the random variables** $X_1, X_2, \ldots$ as the tail σ-algebra of the σ-algebras $\sigma(X_1), \sigma(X_2), \ldots$ and denote it by

$$\mathcal{T}(X_1, X_2, \ldots).$$

Theorem 3.2. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be σ-algebras on Ω. Then

$$\mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \ldots) = \bigcap_{k \in \mathbb{N}} \sigma(\mathcal{A}_i \ ; \ i = k, k+1, \ldots). \tag{3.2}$$

Remark 3.3. The name "tail σ-algebra" is due to the interpretation of $\mathbb{N}$ as a set of times. As is made clear in the theorem, any event in $\mathcal{T}$ does not depend on the first finitely many time points.

Remark 3.4. Maybe at first glance it is not evident that there are any interesting events in the tail σ-algebra at all. It might not even be clear that we do not have $\mathcal{T} = \{\emptyset, \Omega\}$. Hence we now present simple examples of tail events and tail σ-algebra measurable random variables.

Example 3.5.

(i) Let $A_1, A_2, \ldots$ be events. Then the events A_* and A^* are in $\mathcal{T}(A_1, A_2, \ldots)$. Indeed, let $k \in \mathbb{N}$. Since

$$\bigcap_{i \geq m} A_i \in \sigma(\sigma(A_i) \ ; \ i = k, k+1, \ldots)$$

for any $m = k, k+1, \ldots$ and

$$A_* = \bigcup_{m \geq k} \bigcap_{i \geq m} A_i,$$

$A_* \in \sigma(\sigma(A_i) \ ; \ i = k, k+1, \ldots)$. Then, $A_* \in \mathcal{T}(A_1, A_2, \ldots)$. Analogously, it holds that $A^* \in \mathcal{T}(A_1, A_2, \ldots)$.

(ii) Let $X_1, X_2, \ldots$ be $\overline{\mathbb{R}}$-valued random variables. Then the maps X_* and X^* are $\mathcal{T}(X_1, X_2, \ldots)$-measurable. Indeed, let $k \in \mathbb{N}$. Since

$$\inf_{i \geq m} X_i \text{ is } \sigma(\sigma(X_i) \ ; \ i = k, k+1, \ldots)\text{-measurable}$$

for any $m = k, k+1, \ldots$ and

$$X_* = \sup_{m \geq k} \inf_{i \geq m} X_i,$$

X_* is $\sigma(\sigma(X_i) \ ; \ i = k, k+1, \ldots)$-measurable. Then, X_* is $\mathcal{T}(X_1, X_2, \ldots)$-measurable. Analogously, it holds that X^* is $\mathcal{T}(X_1, X_2, \ldots)$-measurable.

(iii) Let $X_1, X_2, \ldots$ be real random variables. Then the *Cesàro limits*:

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$$

are $\mathcal{T}(X_1, X_2, \ldots)$-measurable. Indeed, let $k \in \mathbb{N}$. Since,

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \liminf_{n \to +\infty} \frac{1}{n} \sum_{i=k}^{n} X_i,$$

by the example above,

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \text{ is } \sigma(\sigma(X_i) \ ; \ i = k, k+1, \ldots)\text{-measurable.}$$

Then,

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$$

is $\mathcal{T}(X_1, X_2, \ldots)$-measurable. Analogously, it holds that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$$

is $\mathcal{T}(X_1, X_2, \ldots)$.

Proof of theorem 3.2. Let $\mathcal{B}$ and $\mathcal{C}$ be the σ-algebras defined by equations 3.1 and 3.2, respectively. Let $k \in \mathbb{N}$; then

$$\mathcal{B} \subset \sigma(\mathcal{A}_i \ ; \ i = k, k+1, \ldots).$$

Thus, $\mathcal{B} \subset \mathcal{C}$. Let $J \subset \mathbb{N}$ be finite; then

$$\mathcal{C} \subset \sigma(\mathcal{A}_i \ ; \ i = k, k+1, \ldots) \subset \sigma(\mathcal{A}_i \ ; \ i \in \mathbb{N} \setminus J)$$

for some $k \in \mathbb{N}$. Hence, $\mathcal{C} \subset \mathcal{B}$. $\square$

Theorem 3.6 (Kolmogorov's 0–1 law)**.** Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be independent σ-algebras on Ω. Then the tail σ-algebra of $\mathcal{A}_1, \mathcal{A}_2, \ldots$ is $\mathbf{P}$-trivial, that is,

$$A \in \mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \ldots) \quad \Rightarrow \quad \mathbf{P}(A) \in \{0, 1\}.$$

Example 3.7.

(i) Let $A_1, A_2, \ldots$ be independent events. Then $\mathbf{P}(A_*), \mathbf{P}(A^*) \in \{0, 1\}$. Indeed, $A_*, A^* \in \mathcal{T}(A_1, A_2, \ldots)$.

(ii) Let $X_1, X_2, \ldots$ be independent $\overline{\mathbb{R}}$-valued random variables. Then the maps X_* and X^* are deterministic. Indeed, since X^* is $\mathcal{T}(X_1, X_2, \ldots)$-measurable,

$$\mathbf{P}(X^* \leq x) \in \{0, 1\} \quad \text{for every } x \in \mathbb{R}.$$

Let

$$x^* := \inf\{x \in \mathbb{R} \ ; \ \mathbf{P}(X^* \leq x) = 1\}.$$

Then we have three cases:

If $x^* = -\infty$, then

$$\mathbf{P}(X^* = -\infty) = \lim_{n \to +\infty} \mathbf{P}(X^* \leq -n) = 1.$$

If $x^* \in \mathbb{R}$, then

$$\mathbf{P}(X^* \leq x^*) = \lim_{n \to +\infty} \mathbf{P}\left(X^* \leq x^* + \frac{1}{n}\right) = 1,$$

and

$$\mathbf{P}(X^* < x^*) = \lim_{n \to +\infty} \mathbf{P}\left(X^* \leq x^* - \frac{1}{n}\right) = 0.$$

If $x^* = +\infty$, then

$$\mathbf{P}(X^* < +\infty) = \lim_{n \to +\infty} \mathbf{P}(X^* \leq n) = 0.$$

Since the conclusion that X_* is deterministic only depends from the fact that X_* is $\mathcal{T}(X_1, X_2, \ldots)$-measurable, we can also conclude that X^* is deterministic.

(iii) Let $X_1, X_2, \ldots$ be independent real random variables. Then the *Cesàro limits*

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$$

are deterministic. Indeed, the Cesàro limits are $\mathcal{T}(X_1, X_2, \ldots)$-measurable.

Proof of theorem 3.6. It is enough to prove that $\mathcal{T}$ and $\mathcal{T}$ are independent. Indeed, let $n \in \mathbb{N}$. Then, by virtue of theorem **??**,

$$\mathcal{C}_n := \sigma(\mathcal{A}_i\ ;\ i = 1, \ldots, n) \quad \text{and} \quad \sigma(\mathcal{A}_i\ ;\ i = n+1, n+2, \ldots)$$

are independent. Since

$$\mathcal{T} := \mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \ldots) \subset \sigma(\mathcal{A}_i\ ;\ i = n+1, n+2, \ldots),$$

$\mathcal{C}_n$ and $\mathcal{T}$ are independent. Hence $\mathcal{C}$ and $\mathcal{T}$ are independent, where $\mathcal{C}_n \uparrow \mathcal{C}$. Then, because $\mathcal{C}$ is a π-system and generates

$$\mathcal{C}_\infty := \sigma(\mathcal{A}_i\ ;\ i = 1, 2, \ldots),$$

$\mathcal{C}_\infty$ and $\mathcal{T}$ are independent. Finally, the assertion follows since $\mathcal{T} \subset \mathcal{C}_\infty$. □

Exercise 3.1. Let $X_1, X_2, \ldots$ be independent real random variables. Then $\mathbf{P}(\lim_{n\to+\infty} X_n \text{ exists})$ must be 0 or 1.

Exercise 3.2. Let $X_1, X_2, \ldots$ be independent real random variables with $X_n \sim \mathcal{U}_{\{1,\ldots,n\}}$. Compute $\mathbf{P}(X_n = 5 \quad \text{i.o.})$.

Exercise 3.3. Let $A_1, A_2, \ldots$ be events such that

1. $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are independents whenever $i_{j+1} \geq i_j + 2$ for $j = 1, \ldots, k-1$, and
2. $\sum_{n=1}^{+\infty} \mathbf{P}(A_n) = +\infty$.

Prove that $\mathbf{P}(A^*) = 1$.

Exercise 3.4. Consider infinite, independent, fair coin tossing, and let H_n be the event that the nth coin is heads. Determine the following probabilities.

1. $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{n+9} \quad \text{i.o.}]$.
2. $\mathbf{P}[H_{n+1} \cap H_{n+2} \cap \ldots \cap H_{2^n} \quad \text{i.o.}]$.

Exercise 3.5. Let $A_1, A_2, \ldots$ be independent events, let $x \in \mathbb{R}$, and let

$$S_x = \left(\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{A_i} \leq x \right).$$

Prove that $\mathbf{P}(S_x)$ must equal 0 or 1.

Exercise 3.6. Let $A_1, A_2, \ldots$ be independent events. Let X be a real random variable which is $\sigma(A_n, A_{n+1} \ldots)$-measurable for each $n \in \mathbb{N}$. Prove that X is deterministic.