Probability Theory
School of Mathematics
Faculty of Science
National University of Engineering

Midterm Exam

Topic: Independence **Period: 2016.1**

Throughout this exam $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space (Ω is a nonempty set); $U, V, X, Y, X_1, X_2, \ldots :$ $(\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be real random variables and $A_1, A_2, \ldots \in \mathcal{A}$.

1. [4 pts.] Let $X_1, X_2, \ldots$ be i.i.d with $X_1 \sim \mathrm{Rad}_{1/2}$, i.e., $\mathbf{P}[X_1 = -1] = \mathbf{P}[X_1 = 1] = 1/2$, and let $S_n = X_1 + \ldots + X_n$ for any $n \in \mathbb{N}$. Show that $\mathbf{P}[S^* = \infty] = 1$.
2. [4 pts.] Let $A_1, A_2, \ldots$ be independent. If Y is measurable with respect to $\sigma(A_n, A_{n+1} \ldots,)$ for each $n \in \mathbb{N}$, prove that Y is deterministic.
3. [4 pts.] Let $A_1, A_2, \ldots$ be independent, let $x \in \mathbb{R}$, and let $S_x = \left[\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{A_i} \leq x\right]$. Prove that $\mathbf{P}[S_x]$ must equal 0 or 1.
4. [4 pts.] Prove that X and X are independent iff X is deterministic.
5. [4 pts.] (Box–Muller method) Let U and V be independent and uniformly distributed on $[0, 1]$. Define
$$X := \sqrt{-2\log(U)}\cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2\log(U)}\sin(2\pi V).$$
Show that X and Y are independent with distribution $\mathcal{N}_{0,1}$.
6. [4 pts.] Toss a coin with $\mathbf{P}[\text{Heads}] = p$ repeatedly. Let A_k be the event that k or more consecutive heads occurs amongst the tosses numbered $2^k, 2^k + 1, \ldots, 2^{k+1} - 1$. Show that
$$\mathbf{P}[A_k \quad \text{i.o.}] = \begin{cases} 1 & \text{if } p \geq \frac{1}{2} \\ 0 & \text{if } p < \frac{1}{2}. \end{cases}$$
7. [4 pts.] Three players, A, B, and C, take turns to roll a die; they do this in the order ABCABCA...
   (a) Show that the probability that, of the three players, A is the first to throw a 6, B the second, and C the third, is 216/1001.
   (b) Show that the probability that the first 6 to appear is thrown by A, the second 6 to appear is thrown by B, and the third 6 to appear is thrown by C, is 46656/753571.
8. [4 pts.] Let $X_1, \ldots, X_n$, be independent and symmetric and let $S = X_1 + \ldots + X_n$. Show that
   (a) for all $x \in \mathbb{R}$, $\mathbf{P}[S \geq x] = \mathbf{P}[S \leq -x]$,
   (b) Is the conclusion necessarily true without the assumption of independence?

Wednesday, May 18