School of Mathematics
Faculty of Science
National University of Engineering

Final Exam

Subject: Probability Theory **Period: 2016.2**

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1. If the plus and minus signs in $\sum_{n=1}^{\infty} \pm n^{-1}$ are determined by successive tosses of a fair coin, prove that the resulting series converges almost surely.

2. eLet $X_1, X_2, \ldots$ be real random variables i.i.d. that are not integrable. Prove that

$$\limsup_{n\to\infty} n^{-1}|X_1 + \cdots + X_n| \xrightarrow{\text{a.s.}} \infty.$$

   (Hint: show that $\sum_{n=1}^{\infty} \mathbf{P}(|X_n| > n) = \infty$ and apply the Borel-Cantelli lemma.)

3. Let $X_1, X_2, \ldots$ be i.i.d. real random variables with

$$\frac{X_1 + \ldots + X_n}{n} \xrightarrow{\text{a.s.}} Y$$

   for some random variable Y. Show that $X_1 \in \mathcal{L}^1(\mathbf{P})$ and $Y = \mathbf{E}[X_1]$ a.s. (Hint: first show that

$$\mathbf{P}(|X_n| > n \text{ for infinitely many } n) = 0 \quad \Leftrightarrow \quad X_1 \in \mathcal{L}^1(\mathbf{P}).)$$

4. A collection or "population" of N objects (such as mice, grains of sand, etc.) may be considered as a sample space in which each object has probability N^{-1}. Let X be a random variable on this space (a numerical characteristic of the objects such as mass, diameter, etc.) with mean m and variance v. In statistics one is interested in determining m and v by taking a sequence of random samples from the population and measuring X for each sample, thus obtaining a sequence (X_n) of numbers that are values of independent random variables with the same distribution as X. The nth sample mean is $M_n = n^{-1} \sum_{i=1}^{n} X_n$ and the nth sample variance is $V_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i - M_i)^2$.

   (a) Show that $\mathbf{E}[M_n] = m$, $\mathbf{E}[V_n] = v$, and $M_n \xrightarrow{\text{a.s.}} m$ and $V_n \xrightarrow{\text{a.s.}} v$.

   (b) Can you see why one uses $(n-1)^{-1}$ instead of $n-1$ in the definition of V_n?

5. Let $X_1, X_2, \ldots$ be i.i.d real random variables with $X_1 \sim \mathrm{Rad}_{1/2}$, i.e., $\mathbf{P}(X_1 = -1) = \mathbf{P}(X_1 = 1) = 1/2$, and let $S_n = X_1 + \ldots + X_n$. Show that $\mathbf{P}(S^* = \infty) = 1$.

6. Let $X_1, X_2, \ldots$ be i.i.d Poisson random variables with parameter $\lambda = 1$. Let $S_n = X_1 + \cdots + X_n$. Show that

$$\lim_{n \to \infty} \frac{S_n - n}{\sqrt{n}} = X,$$

where X is a standard normal random variable.

7. Show that

$$\lim_{n \to \infty} e^{-n} \left(\sum_{k=0}^{n} \frac{n^k}{k!} \right) = \frac{1}{2}.$$

(Hint: use exercise 6.)

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