Chapter 1

Independence

We enter the realm of probability theory at this point, where we define independence of events and random variables. In the following, $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space and the sets $A \in \mathcal{A}$ are the events. As soon as constructing probability spaces has become routine, the concrete probability space will lose its importance and it will be only the random variables that will interest us.

1.3 Kolmogorov's 0-1 Law

With the Borel–Cantelli lemma, we have seen a first 0–1 law for independent events. We now come to another 0–1 law for independent events and for independent σ-algebras. To this end, we first introduce the notion of the tail σ-algebra.

Definition 1.44 (Tail σ-algebra)**.** Let I be a countably infinite index set and let $(\mathcal{A}_i)_{i\in I}$ be a family of σ-algebras on Ω. Then

$$\mathcal{T}((\mathcal{A}_i)_{i\in I}) := \bigcap_{J\subset I,\, J \text{ finite}} \sigma(\mathcal{A}_i \;;\; i \in I \setminus J)$$

is called the **tail σ-algebra of the σ-algebras** $(\mathcal{A}_i)_{i\in I}$. If $(A_i)_{i\in I}$ is a family of events of $\mathcal{A}$, then we define the **tail σ-algebra of the events** $(A_i)_{i\in I}$ as the tail σ-algebra of the σ-algebras $(\sigma(A_i))_{i\in I}$ and denote it by

$$\mathcal{T}((A_i)_{i\in I}).$$

Finally, if $(X_i)_{i\in I}$ is a family of random variables all defined on Ω, then we define the **tail σ-algebra of the random variables** $(X_i)_{i\in I}$ as the tail σ-algebra of the σ-algebras $(\sigma(X_i))_{i\in I}$ and denote it by

$$\mathcal{T}((X_i)_{i\in I}).$$

□

Theorem 1.45. Let I be a countably infinite index set and let $(\mathcal{A}_i)_{i\in I}$ be a family of σ-algebras on Ω. Let $J_1, J_2, \dots$ be finite sets with $J_n \uparrow I$. Then

$$\mathcal{T}((\mathcal{A}_i)_{i\in I}) = \bigcap_{n\in\mathbb{N}} \sigma(\mathcal{A}_i\ ;\ i \in I \setminus J_n).$$

In the particular case $I = \mathbb{N}$, this reads

$$\mathcal{T}((\mathcal{A}_n)_{n\in\mathbb{N}}) = \bigcap_{n\in\mathbb{N}} \sigma(\mathcal{A}_m\ ;\ m \geq n).$$

Remark 1.46. The name "tail σ-algebra" is due to the interpretation of $I = \mathbb{N}$ as a set of times. As is made clear in the theorem, any event in $\mathcal{T}$ does not depend on the first finitely many time points. □

Remark 1.47. Maybe at first glance it is not evident that there are any interesting events in the tail σ-algebra at all. It might not even be clear that we do not have $\mathcal{T} = \{\emptyset, \Omega\}$. Hence we now present simple examples of tail events and tail σ-algebra measurable random variables. □

Example 1.48.

(i) Let $A_1, A_2, \dots$ be events. Then the events A_* and A^* are in $\mathcal{T}((A_n)_{n\in\mathbb{N}})$.

(ii) Let $X_1, X_2, \dots$ be $\overline{\mathbb{R}}$-valued random variables. Then the maps X_* and X^* are $\mathcal{T}((X_n)_{n\in\mathbb{N}})$-measurable.

(iii) Let $X_1, X_2, \dots$ be real random variables. Then the *Cesàro limits*

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$$

are $\mathcal{T}((X_n)_{n\in\mathbb{N}})$-measurable. □

(*Proof of theorem 1.45.*) *Exercise.* □

Theorem 1.49 (Kolmogorov's 0–1 law)**.** Let I be a countably infinite index set and let $(\mathcal{A}_i)_{i\in I}$ be an independent family of σ-algebras on Ω. Then the tail σ-algebra is $\mathbf{P}$-trivial, that is,

$$\mathbf{P}[A] \in \{0,1\} \quad \text{for any } A \in \mathcal{T}((\mathcal{A}_i)_{i\in I}).$$

Example 1.50.

(i) Let $A_1, A_2, \dots$ be independent events. Then $\mathbf{P}[A_*] \in \{0,1\}$ and $\mathbf{P}[A^*] \in \{0,1\}$.

(ii) Let $X_1, X_2, \dots$ be independent $\overline{\mathbb{R}}$-valued random variables. Then the maps X_* and X^* are deterministic.

(iii) Let $X_1, X_2, \dots$ be independent real random variables. Then the *Cesàro limits*

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{and} \quad \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i$$

are deterministic. $\square$

(*Proof of theorem 1.49.*) *Exercise.* $\square$