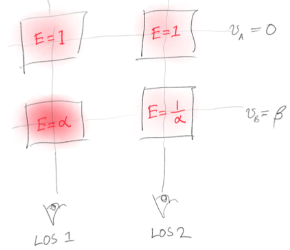
So this time we have more variation in the emissivity — up and down wrt default value.



Emissivity Statistics:

Mean: $\langle E \rangle = (1 + \alpha + 1 + \alpha^{-1})/4$

$= \frac{1}{4}(2 + \alpha + \alpha^{-1})$

Variance: $\mathrm{Var}(E) = \frac{1}{4} \times \{$

$2\left[1 - \frac{1}{4}(2+\alpha+\alpha^{-1})\right]^2$

$+ \left[\alpha - \frac{1}{4}(2+\alpha+\alpha^{-1})\right]^2$

$+ \left[\alpha^{-1} - \frac{1}{4}(2+\alpha+\alpha^{-1})\right]^2 \}$

Alternatively, use $\mathrm{Var}(E) = \langle E^2 \rangle - \langle E \rangle^2$

$\langle E^2 \rangle = \frac{1}{4}(2 + \alpha^2 + \alpha^{-2})$

$\langle E \rangle^2 = \frac{1}{16}(2+\alpha+\alpha^{-1})^2 = \frac{1}{16}\left(4 + 4\alpha + 4\alpha^{-1} + (\alpha+\alpha^{-1})^2\right)$

where $(\alpha+\alpha^{-1})^2 = \alpha^2 + 2 + \alpha^{-2}$

$\Rightarrow \langle E \rangle^2 = \frac{1}{16}\left[6 + 4\alpha + 4\alpha^{-1} + \alpha^2 + \alpha^{-2}\right]$

$\Rightarrow \mathrm{Var}(E) = \frac{1}{16}\left\{8 + 4\alpha^2 + 4\alpha^{-2} - 6 - 4\alpha - 4\alpha^{-1} - \alpha^2 - \alpha^{-2}\right\}$

$= \frac{1}{16}\left\{2 + 3\alpha^2 + 3\alpha^{-2} - 4\alpha - 4\alpha^{-2}\right\}$

So, if $\alpha \to \infty$ then $\langle E \rangle \to \alpha/4$

$\mathrm{Var}(E) \to 3\alpha^2/16$

$\mathrm{std}(E) \to \sqrt{3}\alpha/4$

$\sigma_E \equiv \mathrm{std}(E)/\langle E \rangle \to \sqrt{3}$

So this is much better — we can have a relative width of the distribution as high as $\sqrt{3} \simeq 1.73$

Going back to $\langle E \rangle$ and $\langle E^2 \rangle$ we can write $\langle E^2 \rangle = \frac{1}{4}(\alpha+\alpha^{-1})^2 = \frac{\gamma^2}{4}$ where $\gamma \equiv \alpha + \alpha^{-1}$

whereas $\langle E \rangle = \frac{1}{4}(2+\gamma)$, $\langle E \rangle^2 = \frac{1}{16}(4 + 4\gamma + \gamma^2)$

$\Rightarrow \mathrm{Var}(E) = \langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{4}\gamma^2 - \frac{1}{4} - \frac{1}{4}\gamma - \frac{1}{16}\gamma^2$

$= \frac{3}{16}\gamma^2 - \frac{1}{4}(1+\gamma)$

$\Rightarrow \mathrm{std}(E) = \frac{1}{4}\left[3\gamma^2 - 4(1+\gamma)\right]^{1/2}$

so $\sigma_E \equiv \frac{\mathrm{std}(E)}{\langle E \rangle} = \frac{\left[3\gamma^2 - 4(1+\gamma)\right]^{1/2}}{2+\gamma}$

What happens when fluctuations are small:

$\alpha = 1 + x$ with $x \ll 1$:

$\gamma = \alpha + \alpha^{-1} = 1 + x + 1 - x \simeq 2 + O(x^2)$

So that is OK, we will get cancellations and $\sigma(E) = [O(x^2)]^{1/2}$
$= O(|x|)$

Velocity stats should be the same as calculated previously.

For LOS 1, it is identical calculation:

$\bar{v}_1 = \frac{\alpha\beta}{1+\alpha}$ $\sigma_{los,1} = \beta\frac{\alpha^{1/2}}{1+\alpha}$

For LOS 2, it is same but with emissivities all smaller by a factor α, which makes no difference

$\bar{v}_2 = \frac{\beta}{1+\alpha}$ $\sigma_{los,2} = \beta\frac{\alpha^{1/2}}{1+\alpha}$

Plane-of-sky velocity variations, also same:

$\langle v \rangle_{pos} = \frac{\beta}{2}$

$\mathrm{Var}_{pos}(\bar{v}) = \langle v^2 \rangle_{pos} - \langle v \rangle_{pos}^2$

$\langle v^2 \rangle_{pos} = \frac{1}{2(1+\alpha)^2}\left\{\alpha^2\beta^2 + \beta^2\right\} = \frac{\beta^2(1+\alpha^2)}{2(1+\alpha)^2}$

$\langle v \rangle_{pos}^2 = \frac{\beta^2}{4}$

$\Rightarrow \mathrm{Var}_{pos}(\bar{v}_{los}) = \frac{\beta^2}{4}\left\{\frac{2(1+\alpha^2)}{(1+\alpha)^2} - 1\right\}$

$= \frac{\beta^2}{4(1+\alpha)^2}\left\{2 + 2\alpha^2 - (1+\alpha)^2\right\}$

$\boxed{(1+\alpha)^2 = 1 + 2\alpha + \alpha^2}$

$= \frac{\beta^2}{4(1+\alpha)^2}\left\{1 + \alpha^2 - 2\alpha\right\}$

$= \frac{\beta^2(1-\alpha)^2}{4(1+\alpha)^2}$

$\Rightarrow \sigma_{pos} = \frac{\beta}{2}\frac{|\alpha - 1|}{\alpha + 1}$

So, again we have

$\frac{\sigma_{pos}}{\sigma_{los}} = \frac{|\alpha - 1|}{2\alpha^{1/2}}$