Correlation length

assume $C(r) = 2^{-(r/r_0)^m}$

Use definition $\ell_c^2 = \frac{\pi}{2} \int_0^\infty r\, C(r)\, dr$

$\ln(2^x) = x \ln 2 \Rightarrow 2^x = e^{x \ln 2}$

$\Rightarrow \quad C(r) = e^{-\ln 2 \cdot (r/r_0)^m}$

So, put $\quad u = \ln 2\, (r/r_0)^m$.

or $\quad r = \left(\frac{u}{\ln 2}\right)^{1/m} r_0 = \frac{r_0}{(\ln 2)^{1/m}} u^{1/m}$

$\Rightarrow \quad dr = \frac{r_0}{m (\ln 2)^{1/m}} u^{\frac{1}{m}-1}$

$\Rightarrow \quad \ell_c^2 = \frac{\pi}{2} \frac{r_0^2}{m (\ln 2)^{2/m}} \int_0^\infty u^{\frac{2}{m}-1} e^{-u}\, du$

According to Wolfram, $\int_0^\infty x^a e^{-x} dx$ is $\Gamma(a+1)$ for $\mathrm{Re}(a) > -1$

This means that

$$\ell_c = \left(\frac{\pi}{2m}\right)^{1/2} \frac{r_0}{(\ln 2)^{1/m}} \left[\Gamma\left(\frac{2}{m}\right)\right]^{1/2}$$

We can do the integral scale in a similar way: