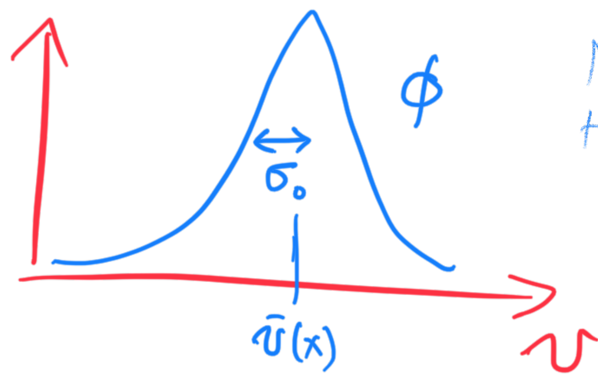
Effects of seeing on structure function

Simple case: assume uniform brightness (and widt[h]

$$I(x,v) = I_0 \phi[v; \bar{v}(x), \sigma_0]$$



Normalized to $\int_{-\infty}^{\infty} \phi \, dv = 1$

That is, the only quantity that varies with position x is the centroid velocity $\bar{v}(x)$.

In general, x is a 2d position on plane of sky.

The seeing acts on each v-slice of the velocity cube:

Even though $I_0(x)$ is constant $I(x,v)$ is not constant with x f[or] fixed v because $\bar{v}(x)$ is varying.

$$\tilde{I}(x,v) = I(x,v) \otimes K(x,s_0)$$

where $\otimes$ is convolution and $K(x,s_0)$ is the seeing profile with width s_0.

For example Gaussian: $K(x,s_0) = \frac{1}{\ldots} e^{-\frac{x^2}{2s_0^2}}$

For example, Gaussian: $K(x,s_0) = \frac{1}{\sqrt{2\pi}s_0} e$

with FWHM $\approx 2.3\, s_0$.

Two spatial points:

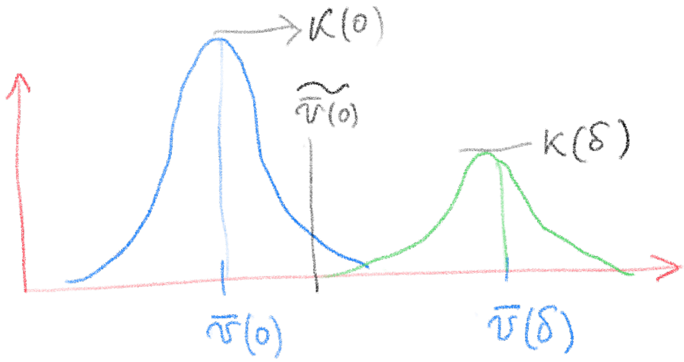
If only two points exist: $x = \{0, \delta\}$

then the convolution is just a sum:

$$\tilde{I}(0,v) = I(0,v)K(0,s_0) + I(\delta,v)K(\delta,s_0)$$

$$\tilde{I}(\delta,v) = I(\delta,v)K(0,s_0) + I(0,v)K(\delta,s_0)$$

We don't need to worry about the normalization of the convolution much, but perhaps we should also divide by $K(0,s_0) + K(\delta,s_0)$



So, the mean velocity in the blurred map is

$$\tilde{\bar{v}}(0) = \frac{K(0)\bar{v}(0) + K(\delta)\bar{v}(\delta)}{K(0) + K(\delta)}$$

$$\tilde{\bar{v}}(\delta) = \frac{K(0)\bar{v}(\delta) + K(\delta)\bar{v}(0)}{K(0) + K(\delta)}$$

The difference in velocity is

$$\Delta v = \bar{v}(0) - \bar{v}(\delta) \quad \text{in the original map}$$

$$\widetilde{\Delta v} = \tilde{\bar{v}}(0) - \tilde{\bar{v}}(\delta) \quad \text{in the blurred map}$$

$$\widetilde{\Delta v} = \frac{[K(0)-K(\delta)]\,\bar{v}(0) - [K(0)-K(\delta)]\,\bar{v}(\delta)}{K(0)+K(\delta)}$$

$$\Rightarrow \widetilde{\Delta v} = \left[\frac{K(0)-K(\delta)}{K(0)+K(\delta)}\right] \Delta v$$

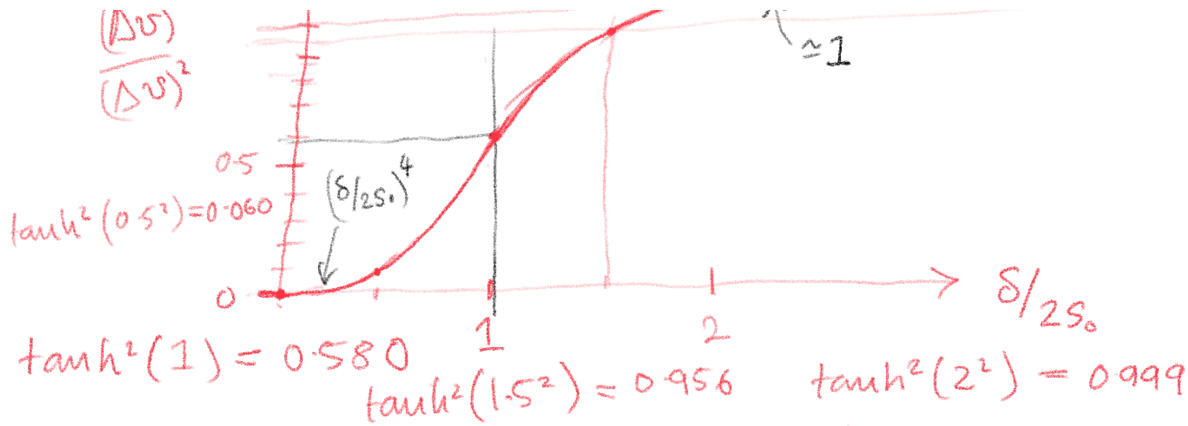
$$(\widetilde{\Delta v})^2 = \left[\frac{1 - e^{-\delta^2/2s_0^2}}{1 + e^{-\delta^2/2s_0^2}}\right]^2 (\Delta v)^2$$

or $$\boxed{(\widetilde{\Delta v})^2 = \tanh^2\left[\left(\frac{\delta}{2s_0}\right)^2\right] (\Delta v)^2}$$

Limit of small separations: $\delta/s_0 \to 0$

$$\Rightarrow e^{-\delta^2/2s_0^2} \simeq 1 - \frac{\delta^2}{2s_0^2}$$

$$\Rightarrow [\dots]^2 \simeq \left[\frac{1}{2}\frac{\delta^2}{2s_0^2}\right]^2 \simeq \left(\frac{\delta}{2s_0}\right)^4$$



Application to structure function.

In reality, there are more than two points

But given that $\Delta v(\Delta x)$ increases with Δx, we can assume that blurring of scales smaller than Δx has a negligible effect on $\Delta v \to \tilde{\Delta v}$

[I do not know how to prove this rigorously]

The assumption of constant I_0 can be justified as follows:

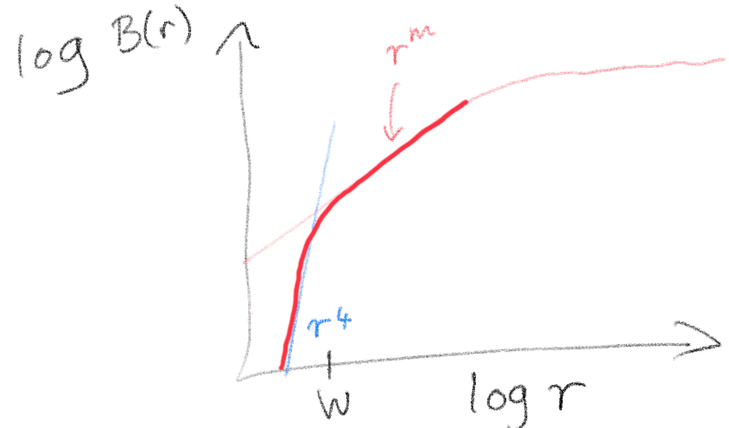
The spatial variations in I have largest amplitude for scales $\gtrsim l_0$. So long as $s_0 << l_0$ then the variations in I on scales affected by seeing should be small.

So, if we have an ideal structure function $B(r)$ then the effects of seeing will give a corrected

$$\tilde{B}(r) = \tanh^2\left[\left(\frac{r}{2s_0}\right)^2\right] B(r)$$

Note that s_0 is RMS seeing width, so FWHM is $W = 2(2\ln 2)^{1/2} s_0 \approx 2.355\, s_0$, so $r \simeq 0.85\, r$

$2S_0$ W



The following bit doesn't really go anywhere . . .

Some of our sources may not satisfy $S_0 \ll t_0$, so it is worth thinking what the effect of I variations would be: regions where I is large would have $\bar{v}$ less affected by seeing than in the constant I case, but low-I regions would have $\bar{v}$ more affected. With luck, the net change would be small.

For instance, in the 2-point calculation, assume that

$$I_0(\delta) = A\, I_0(0)$$

$$\Rightarrow \tilde{\bar{v}}(0) = \frac{K(0)\,\bar{v}(0) + A\,K(\delta)\,\bar{v}(\delta)}{K(0) + A\,K(\delta)}$$

$$\tilde{\bar{v}}(\delta) = \frac{AK(0)\,\bar{v}(\delta) + K(\delta)\,\bar{v}(0)}{AK(0) + K(\delta)}$$

$$= \frac{K(0)\,\bar{v}(\delta) + A^{-1}K(\delta)\,\bar{v}(0)}{K(0) + A^{-1}K(\delta)}$$

So $\Delta\tilde{v} = \tilde{\bar{v}}(0) - \tilde{\bar{v}}(\delta)$

Define $R(\delta) \equiv \dfrac{K(0) + A\,K(\delta)}{K(0) + A^{-1}K(\delta)}$

$$\tilde{\bar{v}}(\delta) = \frac{R(\delta)\,K(0)\,\bar{v}(\delta) + A^{-1}R(\delta)\,K(\delta)\,\bar{v}(0)}{K(0) + A\,K(\delta)}$$

Now we have the same denominator, so we can subtract:

Now we

$$\Delta\hat{v} = \frac{K(0)\,\bar{v}(0) + A\,K(\delta)\,\bar{v}(\delta) - R(\delta)\,K(0)\,\bar{v}(\delta) - A^{-1}R(\delta)\,K(\delta)\,\bar{v}(0)}{K(0) + A\,K(\delta)}$$

$$= \frac{\left[K(0) - A^{-1}R(\delta)\,K(\delta)\right]\bar{v}(0) - \left[R(\delta)\,K(0) - A\,K(\delta)\right]\bar{v}(\delta)}{K(0) + A\,K(\delta)}$$

Unfortunately, the two terms [...] are not the same, so we cannot put $\Delta\hat{v} \propto \Delta v$

What happens when $A \to \infty$?

In that case: $\left.\begin{array}{l} \hat{v}(0) \to \bar{v}(\delta) \\ \hat{v}(\delta) \to \bar{v}(\delta) \end{array}\right\} \Delta\hat{v} \to 0$

This doesn't help much ...